Geometric quantization of Lagrangian torus fibrations

and: a fact about integral-integral affine geometry

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Motivation: "invariance of polarization" phenomenon in geometric quantization.

affine maps $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$x \longmapsto Ax + b$$

	A	b
	$GL(n, \mathbb{R})$	$\mathbb{R}^n$
integral affine:	$GL(n, \mathbb{Z})$	$\mathbb{R}^n$
integral-integral affine:	$GL(n, \mathbb{Z})$	$\mathbb{Z}^n$

B integral-integral affine mfld:

$:\Leftrightarrow$ mfld + maximal $\underbrace{\text{integral-integral affine atlas.}}_{\text{transition maps are locally integral-integral affine}}$

$\Rightarrow$

- flat torsion free connection on TB
- integral lattice $\Lambda \subset TB$

  period lattice $\Lambda^* \subset T^*B$

  $\Rightarrow$ measure on B
- integral points $B_{\mathbb{Z}} \subset B$

Fact:

B cpct $\Rightarrow$ $|B_{\mathbb{Z}}| = vol(B)$

Examples: $B = \mathbb{R}^n/\Gamma$, $\Gamma \curvearrowright \mathbb{R}^n$

B	n	Γ generated by...
torus	$n \geq 1$	$x_j \mapsto x_j + 1$ for $j = 1, \dots, n$
Klein bottle	$n = 2$	$x_1 \mapsto x_1 + 1$, $(x_1, x_2) \mapsto (-x_1, x_2 + 1)$
Kodaira-Thurston-like	$n \geq 3$	$(x_1, x_2, x_3) \mapsto (x_1 + x_2, x_2, x_3 + 1)$ $x_j \mapsto x_j + 1$ for $j \neq 3$

Question:

B ~~integral~~ integral affine, cpct $\overset{?}{\Longrightarrow}$ $B \cong \mathbb{R}^n/\Gamma$

<u>Proper Lagrangian fibration</u> :

$$\begin{array}{c}(M,\omega)\\ \pi \downarrow \\ B\end{array}$$

<u>Arnold - Liouville charts</u> :

semilocal coordinates:

$$\begin{array}{ccc} M \supset \pi^{-1}(U) & (x_1, \cdots, x_n, t_1, \cdots, t_n) & t_j \bmod 1 \\ \downarrow & \downarrow & \omega = \sum_{j=1}^{n} dx_j \wedge dt_j \\ B \supset U & (x_1, \cdots, x_n) & \end{array}$$

$\Rightarrow$ integral affine str on B

Prequantize :

$$\begin{array}{c} (L, \langle\,\rangle, \nabla) \\ \downarrow \\ (M, \omega) \\ \downarrow \\ B \end{array} \qquad \text{curvature } \nabla = \omega$$

Enhanced Arnold Liouville charts :

Arnold-Liouville charts + local trivializations of L

s.t. $\nabla = d + i \sum_j x_j \, dt_j$

$\Rightarrow$ integral-integral affine str. on B

$$M_0 := T^*B/\Lambda^* \qquad \text{Lagrangian torus fibration}$$

$$\begin{array}{c} \downarrow \\ B \end{array}$$

$$M^\vee := TB/\Lambda \qquad \text{semi-local coordinates :}$$

$$\begin{array}{c} \pi \downarrow \\ B \end{array}$$

$$\underbrace{x_1, \dots, x_n}_{\substack{\text{integral-integral} \\ \text{local coordinates} \\ \text{on } B}}, \quad \underbrace{y_1, \dots, y_n}_{\substack{\text{mod } 1 \\ \text{adapted fibre coordinates}}}$$

Local torus $\mathbb{T}^n$ action on $M^\vee$

up to $\mathrm{Aut}(\mathbb{T}^n)$ on chart overlaps

$\alpha \in \Omega^k(W)$, $\quad W = \pi^{-1}(U) \underset{\text{open}}{\subset} M^\nu$

"α is $\mathbb{T}^n$-invariant"

$:\Leftrightarrow$ locally $\alpha = g^*\alpha \quad \forall g \in \mathbb{T}^n$

is well defined.

$\Rightarrow$ Complex $(\Omega^\bullet_{\text{inv}}(M^\nu), d)$

$\Rightarrow$ Cohomology $H^\bullet_{\text{inv}}(M^\nu)$

$$\Omega^\bullet_{\text{inv}}(M^\nu) \xrightarrow{i} \Omega^\bullet(M^\nu)$$

$\Rightarrow$ i_*: $H^\bullet_{\text{inv}}(M^\nu) \longrightarrow H^\bullet(M^\nu)$

Lemma i_* is an isomorphism.

pf for global $\mathbb{T}^n$ action:

averaging $\quad \bar{\alpha} := \int_{g \in \mathbb{T}^n} (g^* \alpha) \, dg$

$d\bar{\alpha} = \overline{d\alpha}.$

$\mathbb{T}^n$ connected $\Rightarrow \overset{\forall g}{[g^*\alpha] = [\alpha]} \Rightarrow [\bar{\alpha}] = [\alpha]$

$\therefore \quad H^{\circ}_{inv} \to H^{\circ}$ is onto.

If α is invariant and $\alpha = d\beta$.

Then also $\alpha = d\bar{\beta}$.

$\therefore \quad H^{\circ}_{inv} \longrightarrow H^{\circ}$ is one-to-one.

pf for local $\mathbb{T}^n$-action: induction on size of covering by open sets w/ $\mathbb{T}^n$ actions.

$l=1$: Case of global T^n action.

$l \rightsquigarrow l+1$: Mayer Vietoris + five lemma:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_{inv}\left(\bigcup_{i=1}^{l+1} W_i\right) \arrow[r] \arrow[d, "i_*"] & H^k_{inv}\left(\bigcup_{i=1}^{l} W_i\right) \oplus H^k_{inv}(W_{l+1}) \arrow[r] \arrow[d, "i_* \oplus i_*"] & H^k_{inv}\left(\left(\bigcup_{i=1}^{l} W_i\right) \cap W_{l+1}\right) \arrow[r] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r] & H^k\left(\bigcup_{i=1}^{l+1} W_i\right) \arrow[r] & H^k\left(\bigcup_{i=1}^{l} W_i\right) \oplus H^k(W_{l+1}) \arrow[r] & H^k\left(\left(\bigcup_{i=1}^{l} W_i\right) \cap W_{l+1}\right) \arrow[r] & \cdots
\end{tikzcd}$$

$\blacksquare$

Note. $av_* \circ i_* =$ Identity on $H^\bullet_{inv}$

& i_* is an isomorphism

$\Rightarrow$ also $i_* \circ av_* =$ Identity on $H^\bullet$

B integral affine, $\Lambda \subset TB$ integral lattice.

$M^\vee = TB/\Lambda$. $\quad x_1, \dots, x_n, \underbrace{y_1, \dots, y_n}_{\text{mod } 1}$ semi-local adapted coordinates.

$\downarrow$

B

$Z_0 := \{y_j = 0\}$ zero section.

Assume B is oriented. & $x_1, \dots, x_n$ are compatible with the orientation.

Thm.

- $dy_1 \wedge \dots \wedge dy_n$ is independent of the choice of adapted coordinates
- $[dy_1 \wedge \dots \wedge dy_n] = P.D.(Z_0)$

pf. Take any coh. class $[\alpha] \in H^n$.

WLG α is inv't.

Locally $\alpha = a(x)\, dx_1 \wedge \dots \wedge dx_n$.

+ terms with dy_j s

$$\int_{Z_0} \alpha = \int_{Z_0} a(x)\, dx_1 \wedge \dots \wedge dx_n$$

$$\underset{\text{Fubini}}{=} \int_{M^r} a(x)\, dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n$$

$$= \int_{M^r} \alpha \wedge (dy_1 \wedge \dots \wedge dy_n).$$

$Z_0 := \{y_j = 0\} \subset M^\vee$ zero section

$D := \{y_j = x_j \bmod 1\} \subset M^\vee$ diagonal section.

$|Z_0 \cap D| = |B_{\mathbb{Z}}|$

pf that $|B_{\mathbb{Z}}| = vol(B)$:

$$|B_{\mathbb{Z}}| = |D \cap Z_0| = \left| \int_D [dy_1 \wedge \dots \wedge dy_n] \right|$$

$$= \left| \int_B \underbrace{\sigma^* dy_1 \wedge \dots \wedge dy_n}_{= dx_1 \wedge \dots \wedge dx_n} \right|$$

$\sigma: x_j \mapsto (x_j, x_j \bmod 1)$

$$= vol(B).$$

Bohr-Sommerfeld quantization

$$\begin{array}{c} L \\ \downarrow \\ M \\ \pi \downarrow \\ B \end{array}$$

$x \in B$ Bohr-Sommerfeld

$:\Longleftrightarrow$ $\left(L|_{\pi^{-1}(x)}, \nabla\right)$ is trivial

$\Longleftrightarrow$ $x \in B_{\mathbb{Z}}$.

The Hilbert space: $\mathcal{H}_{BS} := \bigoplus_{\text{Bohr-Sommerfeld}} \mathbb{C}$.

Dirac-Dolbeault quantization:

The Hilbert space: $\mathcal{H}$ s.t.

$$\dim \mathcal{H} = RR(M,\omega) := \int_M e^{\omega}\ \mathrm{Todd}(TM, J).$$

cptble a.c. str.

The RHS:

$$e^{\omega} = 1 + \omega + \frac{\omega^2}{2!} + \cdots$$

$$\int_{M^n} e^{\omega} = \int_M \frac{\omega^n}{n!} \qquad \text{Liouville volume}$$

$$\mathrm{Todd}(x) := \prod_j \frac{x_j}{1-e^{-x_j}} \quad \rightsquigarrow \quad \mathrm{Todd}: \mathrm{Lie}(U(n)) \to \mathbb{R}$$

$$\underset{\text{Chern Weil}}{\rightsquigarrow} \quad \mathrm{Todd}: \left\{ \begin{matrix} \text{cplx v. bundles} \\ \text{over } M \end{matrix} \right\} \longrightarrow H^{\bullet}(M,\mathbb{R})$$

The LHS: e.g., can take

$$\mathcal{H} = \Gamma_{hol}(L)$$

or $$\mathcal{H} = \Sigma (-1)^i H^i(\mathcal{O}_L)$$

or $$\mathcal{H} = \ker \bar{\partial}_L - \operatorname{coker} \bar{\partial}_L$$

Theorem:

Bohr sommerfeld set.

$$RR(M, \omega) = |B_{\mathbb{Z}}|$$

interpretation: "invariance of polarization"

Proof

$$\begin{array}{c} L \\ \downarrow \\ M \\ \pi \downarrow \\ B \end{array}$$

$V := \ker \pi_* \subset TM$

$$0 \to V \to TM \to \pi^* TB \to 0$$

vector bundles over M

$$V \cong \pi^* T^* B$$

$$V \otimes \mathbb{C} \cong (TM, J) \quad \text{via} \quad u + iv \longmapsto u + Jv$$

$$u, v \in V_y \qquad y \in M$$

$$\Rightarrow \quad (TM, J) \cong \pi^* (\underbrace{T^* B \otimes \mathbb{C}}_{\text{flat}})$$

$\Rightarrow$ (TM, J) is flat. $\Rightarrow$ $\mathrm{Todd}(TM, J) = 1$

$$\Rightarrow \quad RR(M, \omega) = \int_M e^{\omega} \underbrace{1}_{\mathrm{Todd}}$$

$= $ Liouville volume of M

$\overset{\text{Fubini}}{=}$ integral affine volume of B

$= |B_{\mathbb{Z}}|$

By the fact from integral-integral geometry