Random holomorphic sections and Berezin-Toeplitz operators

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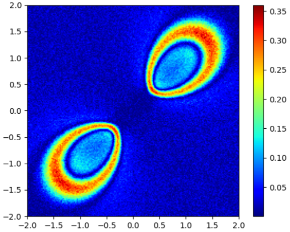
Quantization in geometry - Cologne



Section 1

Introduction: zeros of random polynomials

Random homogeneous polynomials

A broadly studied topic: zeros (real, complex) of random polynomials of one or several variables. Dates back at least to the 1930s (Bloch-Pólya 1931, Littlewood-Offord 1938) and popularized by Kac (1943). Here we look at the following example:

- $\mathbb{C}_k^{\text{hom}}[z_0, z_1]$: degree k homogeneous polynomials in two complex variables.
- Choose $P_k \in \mathbb{C}_k^{\text{hom}}[z_0, z_1]$ "at random".
- Zeros of P_k:

$$Z_{P_k} = \{[z_0 : z_1] \mid P_k(z_0, z_1) = 0\} \subset \mathbb{CP}^1$$

  discrete set of points.
- How are the elements of Z_{P_k} distributed (for large k)?

"At random"

- $\mathcal{B}_k = (e_{0,k}, \dots e_{k,k})$ basis of $\mathbb{C}_k^{\text{hom}}[z_0, z_1]$. Take $P_k = \sum_{\ell=0}^{k} \alpha_{\ell,k} e_{\ell,k}$ where $\alpha_{\ell,k}$ are random coefficients. Choice of basis $\mathcal{B}_k$ and probability distribution of $\alpha_{\ell,k}$?
- Fix an inner product on $\mathbb{C}_k^{\text{hom}}[z_0, z_1]$ and choose an orthonormal $\mathcal{B}_k$.
- Choose the $\alpha_{\ell,k}$ to be i.i.d. Gaussian.
- Here we choose

$$\langle P, Q \rangle_k = \int_{\mathbb{C}} \frac{P(z,1)\overline{Q(z,1)}}{(1+|z|^2)^{k+2}} |dz \wedge d\bar{z}|.$$

- A choice of orthonormal basis: $e_{\ell,k} = \sqrt{\frac{(k+1)\binom{k}{\ell}}{2\pi}} z_0^{\ell} z_1^{k-\ell}$; $\alpha_{\ell,k} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d.
- Equivalently: probability measure μ_k on $\mathbb{C}_k^{\text{hom}}[z_0, z_1]$ given by $\mathrm{d}\mu_k(P) = \frac{1}{\pi^{k+1}} e^{-\|P\|_k^2} \mathrm{d}P$. Here $\mathrm{d}P$: Lebesgue measure induced by $\langle \cdot, \cdot \rangle_k$.
- Later: these choices are important.

Results

- Fubini-Study measure on $\mathbb{CP}^1$: ω_{FS} unique measure invariant under the natural action of $U(2)$ and such that $\omega_{\mathrm{FS}}(\mathbb{CP}^1) = 2\pi$.
- Equivalently: multiple of the standard measure on S^2 when identifying $\mathbb{CP}^1 \simeq S^2$.
- ω_{FS} symplectic (Kähler) form. In local coordinate $z = \frac{z_0}{z_1}$ on $\{[z_0 : z_1] \mid z_1 \neq 0\}$, $\omega_{\mathrm{FS}} = i\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2}$.
- $U \subset \mathbb{CP}^1$ measurable; we are interested in $\mathbb{E}[\#(Z_{P_k} \cap U)]$.
- Result (Bogomolny-Bohigas-Leboeuf 1996): $\mathbb{E}[\#(Z_{P_k} \cap U)] = \frac{k}{2\pi}\omega_{\mathrm{FS}}(U)$.
- Exact because of symmetries, but keep in mind: $\frac{1}{k}\mathbb{E}[\#(Z_{P_k} \cap U)] \underset{k \to +\infty}{\longrightarrow} \frac{\omega_{\mathrm{FS}}(U)}{2\pi}$.

General idea

- Choose $P_k \in \mathbb{C}_k^{\text{hom}}[z_0, z_1]$ at random as before.
- Apply a (well-chosen, see later) differential operator $T_k : \mathbb{C}_k^{\text{hom}}[z_0, z_1] \to \mathbb{C}_k^{\text{hom}}[z_0, z_1]$ to P_k.
- Zeros of $T_k P_k$: $Z_{T_k P_k} = \{[z_0 : z_1] \mid (T_k P_k)(z_0, z_1) = 0\} \subset \mathbb{CP}^1$.
- Question: behavior of distribution of elements of $Z_{T_k P_k}$? Differences with previous case $T_k = \mathrm{Id}$? Deduce information on T_k from this distribution?
- Example (see next two slides): $T_k = \frac{1}{k+2}\left(2 z_0 \frac{\partial}{\partial z_0} - k \text{ Id}\right)$; then $T_k e_{\ell,k} = \frac{2\ell - k}{k+2} e_{\ell,k}$. So $T_k P_k = \sum_{\ell=0}^{k} \frac{2\ell-k}{k+2} \alpha_{\ell,k} e_{\ell,k}$.

Example (first sample)

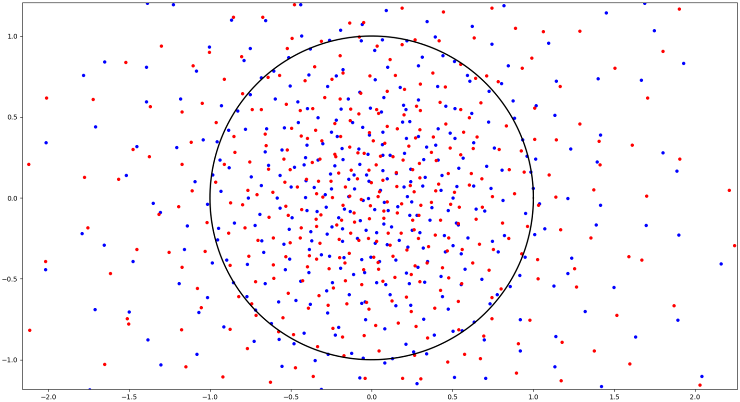


Figure: Red: zeros of P_k. Blue: zeros of $T_k P_k$. $k = 500$.

Example (second sample)

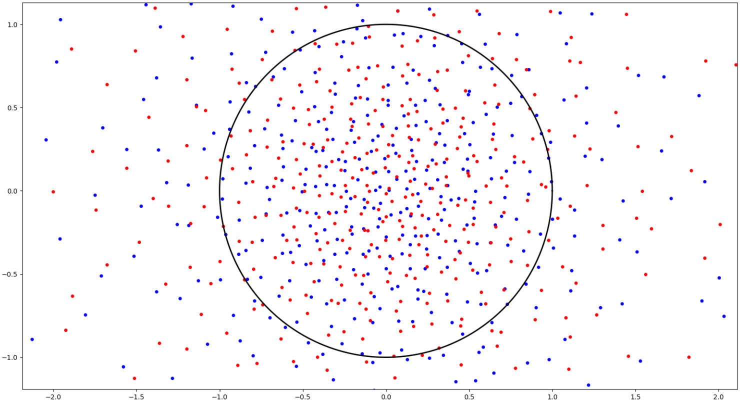


Figure: Red: zeros of P_k. Blue: zeros of $T_k P_k$. $k = 500$.

Remarks about higher dimension ($n \geq 2$)

- $\mathbb{C}_k^{\text{hom}}[z_0, \ldots, z_n]$: degree k homogeneous polynomials in $n+1$ variables.
- Choose $P_k \in \mathbb{C}_k^{\text{hom}}[z_0, \ldots, z_n]$ at random as before.
- Zeros of P_k: $Z_{P_k} = \{[z_0 : \ldots : z_n] \mid P_k(z_0, \ldots, z_n) = 0\} \subset \mathbb{CP}^n$ submanifold of complex dimension $n-1$ with probability 1.
- Study Z_{P_k} (or $Z_{T_k P_k}$)?
- Still call Z_{P_k} the integration current associated with Z_{P_k}. If Z_{P_k} regular

$$\forall \varphi \in \Omega^{n-1,n-1}(\mathbb{CP}^n) \qquad \langle Z_{P_k}, \varphi \rangle := \int_{Z_{P_k}} \varphi.$$

- In local coordinates: $\varphi = \displaystyle\sum_{I,J \subset \{1,\ldots,n\}} a_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_{n-1}} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_{n-1}}.$
- Convergence in the sense of currents: convergence of $\langle Z_{P_k}, \varphi \rangle$ for all $\varphi \in \Omega^{n-1,n-1}(\mathbb{CP}^n)$. And $\langle \mathbb{E}(Z_{P_k}), \varphi \rangle := \mathbb{E}(\langle Z_{P_k}, \varphi \rangle)$.

Section 2

Quantization and random sections

Quantization and semiclassical limit

- Classical (Hamiltonian) mechanics: state space (M, ω) symplectic manifold, observable $f \in \mathcal{C}^\infty(M, \mathbb{R})$.
- Quantum mechanics: state space $\mathcal{H}_\hbar$ Hilbert, observable $T_\hbar$ self-adjoint operator on $\mathcal{H}_\hbar$.
- (Approximate) Dirac quantization conditions: $f \mapsto T_\hbar(f)$ linear, $1 \mapsto \mathrm{Id}_{\mathcal{H}_\hbar}$, $[T_\hbar(f), T_\hbar(g)] = -i\hbar T_\hbar(\{f, g\}) + O(\hbar^2)$.
- Standard example: $(M, \omega) = (\mathbb{R}^{2n}, \mathrm{d}\xi \wedge \mathrm{d}x)$, $f(x, \xi) = \|\xi\|^2 + V(x)$; $\mathcal{H}_\hbar = L^2(\mathbb{R}^n)$, $T_\hbar = -\hbar^2 \Delta + V$.
- Here: setting of compact (M, ω). Geometric quantization and Berezin-Toeplitz operators.

Geometric quantization

- (M, ω) compact, Kähler: M complex, ω symplectic and compatibility between these two structures. In particular, Riemannian metric g_ω on M.
- Assume also that there exists a prequantum line bundle $(L, \nabla, h) \to M$: complex line bundle, holomorphic, h Hermitian metric $\rightsquigarrow$ Chern connection ∇, ask that it satisfies $\mathrm{curv}(\nabla) = -i\omega$.
- Hilbert spaces $\mathcal{H}_k = H^0(M, L^{\otimes k})$, $k \geq 1$ integer: holomorphic sections of $L^{\otimes k} \to M$. With inner product $\langle \cdot, \cdot \rangle_k$ (see next slide).
- $\dim \mathcal{H}_k < +\infty$, can be estimated thanks to Riemann-Roch-Hirzebruch (not necessary for this story).
- Example: $(M, \omega) = (\mathbb{CP}^1, \omega_{\mathsf{FS}})$. $L = \mathcal{O}(1) = \mathcal{O}(-1)^*$ with $\mathcal{O}(-1) = \{([u], v) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid v \in \mathbb{C}u\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$ tautological bundle and h induced by the standard metric on $\mathbb{C}^2$. Then $\mathcal{H}_k \simeq \mathbb{C}_k^{\mathrm{hom}}[z_0, z_1]$ with $\langle \cdot, \cdot \rangle_k$ shown earlier.

Random holomorphic sections

- h induces a Hermitian metric h_k on $L^{\otimes k}$.
- Volume form $\mu = \frac{\omega^{\wedge n}}{n!}$.
- $\langle s, t \rangle_k = \int_M h_k(s, t) \, \mathrm{d}\mu$ inner product on $H^0(M, L^{\otimes k})$.
- $N_k = \dim H^0(M, L^{\otimes k})$. $e_{1,k}, \ldots, e_{N_k,k}$ orthonormal basis of $H^0(M, L^{\otimes k})$.
- Random holomorphic section $s_k = \sum_{\ell=1}^{N_k} \alpha_{\ell,k} e_{\ell,k}$ with $\alpha_{\ell,k} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ i.i.d.
- $Z_{s_k} = \{m \in M \mid s_k(m) = 0\}$.

Theorem (Shiffman-Zelditch 1999)

$$\frac{1}{k}\mathbb{E}[Z_{s_k}] \underset{k \to +\infty}{\longrightarrow} \frac{\omega}{2\pi}$$

in the sense of currents.

Moreover, $\mathbb{E}[Z_{s_k}] - \frac{k\omega}{2\pi}$ is of order $O(k^{-1})$.

Berezin-Toeplitz operators

- $\Pi_k : L^2(M, L^{\otimes k}) \to H^0(M, L^{\otimes k})$ orthogonal projector.
- $f \in \mathcal{C}^\infty(M, \mathbb{R}) \rightsquigarrow T_k = T_k(f) = \Pi_k f : H^0(M, L^{\otimes k}) \to H^0(M, L^{\otimes k})$.
- $Z_{T_k s_k} = \{m \in M \mid (T_k s_k)(m) = 0\} \subset M$, s_k random holomorphic section as before.
- What can we say about $\mathbb{E}[Z_{T_k s_k}]$?
- Idea: $T_k s_k \approx f s_k$ so one can expect $f^{-1}(0)$ to play a part.
- Motivation: recover some information about $f^{-1}(0)$ from $\mathbb{E}[Z_{T_k s_k}]$?
- Example: $(M, \omega) = (\mathbb{CP}^1, \omega_{\mathrm{FS}})$, f = height function on $S^2 \simeq \mathbb{CP}^1 \rightsquigarrow$ $T_k(f) = \frac{1}{k+2}\left(2z_0 \frac{\partial}{\partial z_0} - k \,\mathrm{Id}\right)$ on $\mathbb{C}_k^{\mathrm{hom}}[z_0, z_1]$ and $f^{-1}(0)$ = equator. Recall previous samples.

First results

Theorem (Ancona-LF 2022)

Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ *be such that* 0 *is a regular value of* f. *Let* $T_k = T_k(f)$. *Then*

$$\frac{1}{k}\mathbb{E}[Z_{T_k s_k}] \underset{k\to+\infty}{\longrightarrow} \frac{\omega}{2\pi}$$

in the sense of currents. Moreover,

$$\mathbb{E}[Z_{T_k s_k}] - \frac{k\omega}{2\pi} \underset{k\to+\infty}{\longrightarrow} \frac{i}{2\pi}\partial\bar{\partial}\log f^2$$

in the sense of currents.

Remarks:

- The result of Shiffman-Zelditch discussed earlier corresponds to $f = 1$.
- Here the remainder is not of order $O(k^{-1})$ anymore.
- Part played by $f^{-1}(0)$.

Analysis at scale $k^{-\frac{1}{2}}$

Theorem (Ancona-LF 2022)

Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ be such that 0 is a regular value of f. Let $T_k = T_k(f)$. Let $x \in M$ and let φ be a $(n-1, n-1)$-form on M. Then for every $R > 0$,

$$\int_{B(x, \frac{R}{\sqrt{k}})} \left(\mathbb{E}[Z_{T_k s_k}] - \frac{k}{2\pi}\omega \right) \wedge \varphi = \begin{cases} k^{-n+1} \frac{F_\varphi(x)}{\pi |\mathrm{d}f(x)|^2_\omega} C_n(R) + O(k^{-n+\frac{1}{2}}) \text{ if } x \in f^{-1}(0), \\ k^{-n} \frac{R^{2n} L_\varphi(x) \mathrm{Vol}(B_{\mathbb{R}^{2n}}(0,1))}{2\pi} + O(k^{-n-\frac{1}{2}}) \text{ if } x \notin f^{-1}(0). \end{cases}$$

- $B(x, \frac{R}{\sqrt{k}})$: geodesic ball centered at x and of radius $\frac{R}{\sqrt{k}}$.
- F_φ and L_φ defined by $i\partial f \wedge \bar{\partial} f \wedge \varphi = F_\varphi \frac{\omega^n}{n!}$ and $i\partial\bar{\partial} \log f^2 \wedge \varphi = L_\varphi \frac{\omega^n}{n!}$.
- $C_n(R) > 0$ fully explicit constant:

$$C_n(R) = \frac{2^n \pi^n (n-1)!}{(2n-2)!} \left(\sum_{\ell=0}^{n-1} \binom{n - \frac{3}{2}}{\ell} 2^\ell R^{2\ell} - (1 + 2R^2)^{n-\frac{3}{2}} \right)$$

- The zeros of $T_k s_k$ have a slightly higher concentration near the zero set of f.

Case $n = 1$, $\varphi = 1$

Theorem (Ancona-LF 2022)

Assume $\dim_{\mathbb{C}} M = 1$ *and let* $f \in \mathcal{C}^\infty(M, \mathbb{R})$ *be such that* 0 *is a regular value of* f*. Let* $T_k = T_k(f)$*. Let* $x \in M$*. Then for every* $R > 0$*,*

$$\mathbb{E}\left[\#\left(Z_{T_k s_k} \cap B\left(x, \frac{R}{\sqrt{k}}\right)\right)\right] - \frac{k}{2\pi}\int_{B(x, \frac{R}{\sqrt{k}})} \omega = \begin{cases} \frac{C_1(R)}{2\pi} + O(k^{-\frac{1}{2}}) \text{ if } x \in f^{-1}(0), \\ k^{-1}\frac{R^2 L_1(x)}{2} + O(k^{-\frac{3}{2}}) \text{ if } x \notin f^{-1}(0). \end{cases}$$

- $B(x, \frac{R}{\sqrt{k}})$: geodesic ball centered at x and of radius $\frac{R}{\sqrt{k}}$.
- L_1: function defined by $i\partial\bar{\partial}\log f^2 = L_1 \omega$.
- $C_1(R) = 2\pi\left(1 - \frac{1}{\sqrt{1+2R^2}}\right)$.

Examples

- We illustrate the result on $\mathbb{E}\left[\#\left(Z_{T_k s}\cap B(x,\frac{R}{\sqrt{k}})\right)\right]-\frac{k}{2\pi}\int_{B(x,\frac{R}{\sqrt{k}})}\omega$ for the example of $\mathbb{CP}^1$.
- On $(\mathbb{CP}^1,\omega_{\mathrm{FS}})$, one computes $\int_{B(x,\frac{R}{\sqrt{k}})}\omega_{\mathrm{FS}}=2\pi\left(1-\frac{1}{1+\tan^2(\frac{R}{\sqrt{k}})}\right)$.
- N: number of samples.
- $s_k^{(m)}$: one sample of the random holomorphic section s_k.
- Use sample mean:

$$\mathcal{E}(x,R,k,N)=\frac{1}{N}\sum_{m=1}^{N}\#\left(Z_{T_k s_k^{(m)}}\cap B(x,\frac{R}{\sqrt{k}})\right)-k\left(1-\frac{1}{1+\tan^2(\frac{R}{\sqrt{k}})}\right).$$

For fixed k, converges to the quantity in the first item as $N\to+\infty$.

First example: height function on S^2 ($x \in f^{-1}(0)$)

$T_k = \frac{1}{k+2}\left(2z_0\frac{\partial}{\partial z_0} - k\ \mathrm{Id}\right)$ sur $\mathbb{C}_k^{\mathrm{hom}}[z_0, z_1]$.

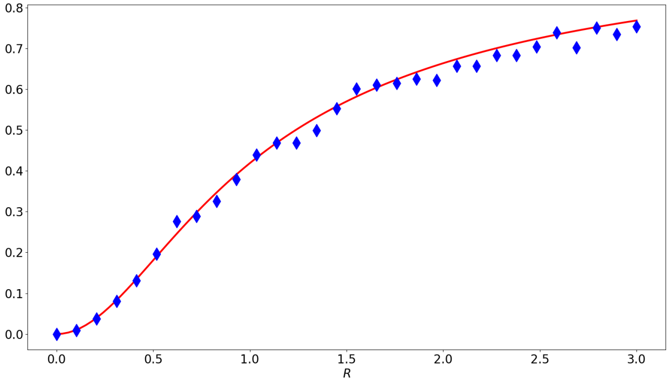


Figure: Blue diamonds: numerical values of $\mathcal{E}(x, R, k, N)$ for $x = (1, 0, 0)$, $k = 400$, $N = 1000$ and various values of R. Red curve: graph of $\frac{C_1}{2\pi} : R \mapsto 1 - \frac{1}{\sqrt{1+2R^2}}$.

First example: height function on S^2 ($x \notin f^{-1}(0)$)

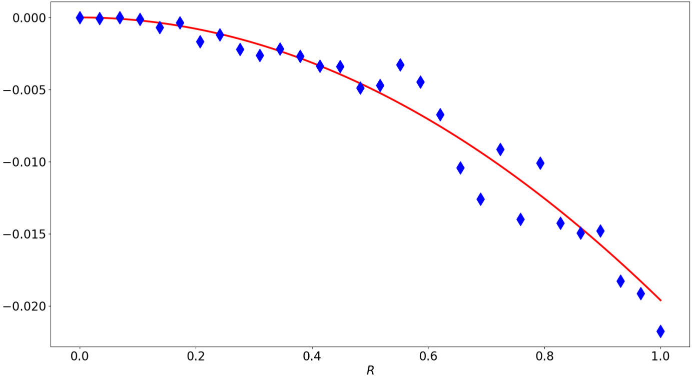


Figure: Blue diamonds: numerical values of $\mathcal{E}(\pi_N^{-1}(z), R, k, N)$ for $z = 0$, $k = 100$, $N = 100000$ and various values of R. Red curve: graph of $R \mapsto k^{-1}\frac{R^2 L_1(\pi_N^{-1}(z))}{2} = -\frac{2k^{-1}R^2(1+|z|^4)}{(|z|^2-1)^2}$ for these values of k and z.

Second example: $f_\lambda = x_1 x_2 - \lambda$ on S^2

$M = S^2$ with coordinates (x_1, x_2, x_3). $T_k = T_k(x_1 x_2 - \lambda)$ with λ regular value of $x_1 x_2$.

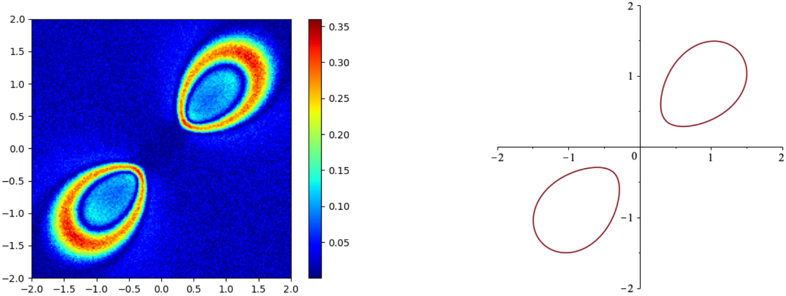


Figure: Recovering $f_\lambda^{-1}(0)$ for $f_\lambda = x_1 x_2 - \lambda$ on S^2, with $\lambda = \frac{1}{3}$. Left: values of $|\mathcal{E}(z, R, k, N)|$ for $R = \frac{1}{\sqrt{2}}$, $k = 100$, $N = 1000$, and z on a 200×200 discretizing $\{|\Re(z)|, |\Im(z)| \leq 2\}$. Right: $f_\lambda^{-1}(0)$ for $\lambda = \frac{1}{3}$.

Section 3

Comments and ideas of proof

Remarques

- More general results: replace $T_k(f)$ with $T_k(f + k^{-1} f_1 + k^{-2} f_2 + \ldots)$.
- Statements about the forms $\Phi_{T_k}^* \omega_{\mathrm{FS}}$ with Φ_{T_k} the "Kodaira embedding" induced by T_k:

$$\Phi_{T_k} : M \dashrightarrow \mathbb{CP}^{N_k - 1}, \qquad m \mapsto [(T_k e_{1,k})(m) : \ldots : (T_k e_{N_k,k})(m)]$$

  with $(e_{\ell,k})_{1 \leq \ell \leq N_k}$ orthonormal basis of $H^0(M, L^{\otimes k})$.
- Relationship between $\Phi_{T_k}^* \omega_{\mathrm{FS}}$ and $\mathbb{E}[Z_{T_k s}]$ (Poincaré-Lelong formula).
- Similar results on higher concentration of zeros near $f^{-1}(0)$ (Drewitz-Liu-Marinescu 2023): non-compact setting, $f \geq 0$, $\Delta f \neq 0$ on $f^{-1}(0)$.

Sketch of proof

- $2\pi\mathbb{E}[Z_{T_k s_k}] - k\omega = i\partial\bar{\partial}\log B_k$ with $B_k : M \to \mathbb{R}$ restriction to the diagonal of the Schwartz kernel of $T_k^* T_k$ (standard computation + Poincaré-Lelong formula).
- $B_k(x) = \left(\frac{k}{2\pi}\right)^n \left(f(x)^2 + k^{-1} b_1(x) + O(k^{-2})\right)$.
- $b_1 = 2f\Delta f + \frac{r}{2} f^2 + \frac{1}{2}|\mathrm{d}f|_\omega^2$.
- r: scalar curvature (hence depends on M).
- If $f(x) \neq 0$, the leading term in $B_k(x)$ is given by $f(x)^2$.
- If $f(x) = 0$:
  - Explicit computation of $i\partial\bar{\partial}\log(f^2 + k^{-1}b_1) \rightsquigarrow$ $\int_{B(x,\frac{R}{\sqrt{k}})} \frac{k^{-1}|\mathrm{d}f|_\omega^2 - 2f^2}{(2f^2 + k^{-1}|\mathrm{d}f|_\omega^2)^2} \partial f \wedge \bar{\partial} f \wedge \varphi$.
  - Normal coordinates + Hadamard's lemma $\rightsquigarrow \frac{k^{-n+1} F_{\varphi(x)}}{|\mathrm{d}f(x)|_\omega^2} \int_{B_{\mathbb{R}^{2n}}(0,R)} \frac{1-2t_1^2}{(1+2t_1^2)^2} \mathrm{d}\lambda(t)$.
  - Computation with hypergeometric functions (using identities between those).