Semiclassical Analysis, Geometric Representation and Quantum Ergodicity

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Contents

The Quantum Ergodicity 1

- The quantum ergodicity was established by Shnirelman (1974), Colin de Verdière (1985) and Zelditch (1987).
- Riemannian manifold (X, g^{TX}), (non-negative) Laplacian Δ acting on $\mathscr{C}^\infty(X)$, eigenvalues $\lim_{j\to+\infty} \lambda_j = +\infty$ and eigenfuctions

$$\Delta u_j = \lambda_j u_j, \quad \|u_j\|^2_{L^2(X)} = 1.$$

- Assumption : the geodesic flow on the unit cotangent bundle S^*X is ergodic. Example : compact hyperbolic surface $X = \Gamma\backslash\mathbb{H}^2$.

The Quantum Ergodicity 2

- We has a density one subsequence of eigenfunctions that tend to be equidistributed.
- $\mathbb{B}$ (blue points) $\subseteq \mathbb{N}^*$ is density one if

$$\lim_{\lambda \to +\infty} \frac{\{j \in \mathbb{B}, 0 \leqslant \lambda_j \leqslant \lambda\}}{\{j \in \mathbb{N}^*, 0 \leqslant \lambda_j \leqslant \lambda\}} = 1.$$

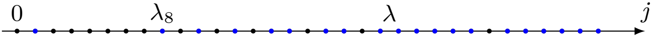

Figure – 1

- Equidistributed : for $A(x) \in \mathscr{C}^\infty(X)$,

$$\lim_{j \to +\infty, j \in \mathbb{B}} \int_X A(x) \left|u_j(x)\right|^2 dv_X(x) = \frac{1}{\mathrm{Vol}_X} \int_X A(x) dv_X(x).$$

- Born rule, for $U \subset X$,

$$P_{\text{detect the particle in } U} = \int_U \left|u_j(x)\right|^2 dv_X(x) \sim \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(M)}.$$

Flat bundles I

Differential geometric approach :

- A vector bundle (F, ∇^F) over X is flat if its curvature vanishes :

$$R^F(U, V)s = \nabla^F_U \nabla^F_V s - \nabla^F_V \nabla^F_U s - \nabla^F_{[U,V]} s = 0$$

  for $U, V \in \mathscr{C}^\infty(X, TX), s \in \mathscr{C}^\infty(X, F)$.

- A unitary flat bundle (F, ∇^F, h^F) has parallel metric $\nabla^F h^F = 0$:

$$U(h^F(s, s')) = h^F(\nabla^F_U s, s') + h^F(s, \nabla^F_U s')$$

- Ex : The Möbius band is a unitary flat bundle over $\mathbb{S}^1$. Compare with the trivial line bundle on $\mathbb{S}^1$.

Flat bundles II

Representation approach :

- Transition maps $\phi_{\alpha,\beta}$ are constant matrices.
- $\widetilde{X}$ the universal covering. For $\rho\colon \pi_1(X) \to \mathrm{U}(n)$ (called holonomy), set

$$F = \pi_1(X)\backslash(\widetilde{X} \times \mathbb{C}^n), \quad (\widetilde{x}, v) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma) \cdot v) \text{ for } \gamma \in \pi_1(X).$$

- Describe Möbius band and the trivial line bundle on $\mathbb{S}^1$ in terms of representation.

Flat bundles III

- An isomorphism

$$\mathscr{C}(X, F) \cong \mathscr{C}(\widetilde{X}, \mathbb{C}^n)^{\pi_1(X)}$$
$$s(x) \mapsto \widetilde{s}(\widetilde{x}), \quad \widetilde{s}(\gamma \cdot \widetilde{x}) = \gamma \cdot \widetilde{s}(\widetilde{x}).$$

- $(d_{\widetilde{X}}, \langle \cdot, \cdot \rangle_{\mathbb{C}^n}, \Delta_{\widetilde{X}}^{\mathbb{C}^n})$ descents to $(\nabla^F, h^F, \Delta^F)$

$$\begin{array}{ccc} s \in \mathscr{C}(X, F) & \xrightarrow{\cong} & \widetilde{s} \in \mathscr{C}(\widetilde{X}, \mathbb{C}^n)^{\pi_1(X)} \\ \Big\downarrow \Delta^F & & \Big\downarrow \Delta_{\widetilde{X}}^{\mathbb{C}^n} \\ \Delta^F s \in \mathscr{C}(X, F) & \xrightarrow{\cong} & \Delta_{\widetilde{X}}^{\mathbb{C}^n} \widetilde{s} \end{array}$$

Geometric Settings I

- A genus-2 hyperbolic surface $X = \Gamma_2 \backslash \mathbb{H}^2$ with

$$\Gamma_2 \cong \{a_1, b_1, a_2, b_2 \mid [a_1, a_2][b_1, b_2] = 1\}.$$

- A representation $\rho\colon \Gamma_2 \mapsto \mathrm{SU}(2)$. Ex : for $\theta/\pi \in \mathbb{R}$ irrational,

$$\rho(a_i) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \quad \rho(b_i) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

- The image $\rho(\Gamma_2) \subset \mathrm{SU}(2)$ is dense by the double covering

$$\mathrm{SU}(2) \to \mathrm{SU}(2)/\{\pm 1\} \cong \mathrm{SO}(3, \mathbb{R})$$

and the Euler rotation.

Geometric Settings II

- We have $\left(X = \Gamma_2\backslash\mathbb{H}^2, \rho\colon \Gamma_2 \mapsto \mathrm{SU}(2)\right)$ (a principal flat bundle).
- Need an action $\mathrm{SU}(2) \curvearrowright \mathbb{C}^n$.
- $\mathrm{SU}(2)$ acts on $\mathbb{C}^2$, gives

$$F = \Gamma_2\backslash(\mathbb{H}^2 \times \mathbb{C}^2).$$

- Also acts on $\mathrm{Sym}^p(\mathbb{C}^2)$ for $p \in \mathbb{N}^*$

$$\mathrm{Sym}^p(\mathbb{C}^2) = \left(\mathbb{C}^2\right)^{\otimes p} / \{v \otimes w \sim w \otimes v\}.$$

- We have a series of flat bundles over $\Gamma_2\backslash\mathbb{H}^2$.

$$F_p = \Gamma_2\backslash(\mathbb{H}^2 \times \mathrm{Sym}^p(\mathbb{C}^2)) \ \text{ for } p \in \mathbb{N}^*.$$

Geometric Representation I

- Laplacians Δ^{F_p} $(p \in \mathbb{N}^*)$ acting on $\mathscr{C}^\infty\big(\Gamma_2\backslash\mathbb{H}^2, \Gamma_2\backslash(\mathbb{H}^2 \times \mathrm{Sym}^p(\mathbb{C}^2))\big)$

$$\Delta^{F_p} u_{p,j} = \lambda_{p,j} u_{p,j}, \quad \|u_{p,j}\|_{L^2(X,F_p)} = 1, \quad i \in \mathbb{N}.$$

- Pauli-Schrödinger spin-$\frac{p}{2}$ Laplacian.
- Natural question : when $\lambda_{p,j}$ large, does the quantum state $u_{p,j}$ also tend to be equidistributed ?
- Need to make sense, on what ? First choice, $|u_{p,j}(x)|^2_{\mathrm{Sym}^p(\mathbb{C}^2)}\, dv_{\Gamma_2\backslash\mathbb{H}^2}(x)$,

$$A \in \mathscr{C}^\infty(\Gamma_2\backslash\mathbb{H}^2) \mapsto \int_{\Gamma_2\backslash\mathbb{H}^2} A(x)\, |u_{p,j}(x)|^2_{\mathrm{Sym}^p(\mathbb{C}^2)}\, dv_{\Gamma_2\backslash\mathbb{H}^2}(x)$$
$$\to \frac{1}{\mathrm{Vol}_{\Gamma_2\backslash\mathbb{H}^2}} \int_{\Gamma_2\backslash\mathbb{H}^2} A(x)\, dv_{\Gamma_2\backslash\mathbb{H}^2}(x)?$$

- Too coarse !

Geometric Representation II

- An isomorphism and an embedding

$$\mathrm{Sym}^p(\mathbb{C}^2) = \{a_0 z_0^p + a_1 z_0^{p-1} z_1 + \cdots + a_p {z_1}^p \mid a_0, \cdots, a_p \in \mathbb{C}\} \subset \mathscr{C}^\infty(\mathbb{C}^2).$$

- For $g \in \mathrm{SU}(2)$, $\mathscr{P} \in \mathrm{Sym}^p(\mathbb{C}^2)$, then

$$(g \cdot \mathscr{P})(z_0, z_1) = \mathscr{P}(g^{-1} \cdot (z_0, z_1)).$$

- Restricted to $\mathbb{S}^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$, $\mathscr{P}(w_1, w_2) \in \mathscr{C}^\infty(\mathbb{S}^3)$.
- For $|\lambda| = 1$, $|\mathscr{P}(\lambda z_0, \lambda z_1)|^2 = |\mathscr{P}(z_0, z_1)|^2$.
- $|\mathscr{P}(z_0, z_1)|^2$ a smooth function on $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{CP}^1 \cong \mathbb{S}^2$ (Hopf fibration)!
- The norm on $\mathrm{Sym}^p(\mathbb{C}^2)$:

$$|\mathscr{P}|^2_{\mathrm{Sym}^p(\mathbb{C}^2)} = \int_{\mathbb{CP}^1} |\mathscr{P}(z)|^2 \, dw_{\mathrm{FS}}(z).$$

Geometric Representation III

- Coarse probability measure

$$|u_{p,j}(x)|^2_{\mathrm{Sym}^p(\mathbb{C}^2)}\, dv_{\Gamma_2\backslash\mathbb{H}^2}(x) \quad \text{on } \Gamma_2\backslash\mathbb{H}^2.$$

- By the integral for $\mathscr{P} \in \mathrm{Sym}^p(\mathbb{C}^2)$:

$$|\mathscr{P}|^2_{\mathrm{Sym}^p(\mathbb{C}^2)} = \int_{\mathbb{CP}^1} |\mathscr{P}(z)|^2\, w_{\mathrm{FS}}(z), \quad \text{coarse} = \int_{\mathbb{CP}^1} \text{refined}.$$

- The Refined probability measure formally is

$$|u_{p,j}(x,z)|^2\, w_{\mathrm{FS}}(z)\, dv_{\Gamma_2\backslash\mathbb{H}^2}(x)$$

- Locally on $(\Gamma_2\backslash\mathbb{H}^2) \times \mathbb{CP}^1$, globally on

$$\Gamma_2\backslash(\mathbb{H}^2 \times \mathbb{CP}^1) = (\widetilde{x}, z) \sim (\gamma\widetilde{x}, \rho(\gamma)z), \rho\colon \Gamma_2 \to \mathrm{SU}(2).$$

The Main Theorem I

Uniform Quantum Ergodicity (2023, Ma-M.)

The quantum ergodicity holds uniformly on unitary flat bundles $\{F_p\}_{p\in\mathbb{N}^*}$.

UQE consist of two parts :

- Uniform density-1 condition.
- Uniform equidistribution theorem.

The Main Theorem II : uniform density-1 condition

- There is a two dimensional array $\mathbb{B} \subseteq \mathbb{N}^2$ that

$$\lim_{\lambda \to +\infty} \min_{p \in \mathbb{N}^*} \frac{|\{(p,j) \in \mathbb{B} \mid \lambda_{p,j} \leqslant \lambda\}|}{|\{j \in \mathbb{N} \mid \lambda_{p,j} \leqslant \lambda\}|} = 1,$$

- Compare with

$$\min_{p \in \mathbb{N}^*} \lim_{\lambda \to +\infty} \frac{|\{(p,j) \in \mathbb{B} \mid \lambda_{p,j} \leqslant \lambda\}|}{|\{j \in \mathbb{N} \mid \lambda_{p,j} \leqslant \lambda\}|} = 1.$$

The Main Theorem II : uniform density-1 condition

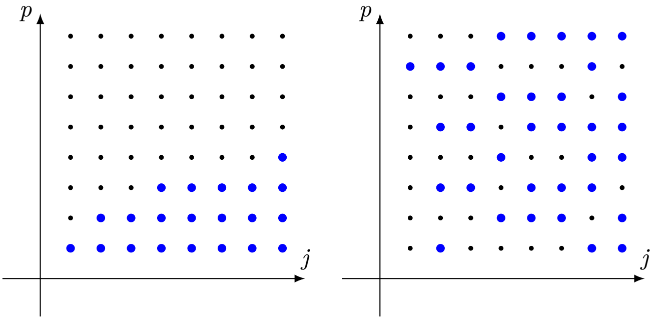

Figure – 2

The Main Theorem III : uniform equidistribution

- On the uniform density-1 set $\mathbb{B} \subseteq \mathbb{N}^2$, the equidistribution result is also uniform for $p \in \mathbb{N}^*$.
- Given $\mathscr{A} \in \mathscr{C}^\infty(\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1))$, we have

$$\lim_{\lambda \to +\infty} \sup_{(p,j) \in \mathbb{B}, \lambda_{p,j} \geqslant \lambda} \left| \int_{\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1)} \left(\mathscr{A} \left| u_{p,j} \right|^2 - \mathscr{A} \right) dv_{\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1)} \right| = 0.$$

Technique 0 : QE

- Recall the proof of QE, three key terms : classical and quantum observable spaces, a quantization procedure, classical and quantum evolutions.
- The Weyl quantization $\mathrm{Op}_h : \mathscr{C}^\infty(S^*X) \to \mathrm{End}(L^2(X))$, the geodesic flow $(g_t)_{t\in\mathbb{R}}$ and the Schrödinger propagator $\mathrm{ad}_{e^{-ith\Delta}}$.
- A commutative diagram

$$\begin{array}{ccc} a(x,\xi) & \xrightarrow{\quad \mathrm{Op}_h \quad} & \mathrm{Op}_h(a) \\ \downarrow g_t & & \downarrow \mathrm{ad}_{e^{-ith\Delta}} \\ (g_t a)(x,\xi) = a(g_t(x,\xi)) & \xrightarrow{\mathrm{Op}_h} & \mathrm{Op}_h(g_t a) \sim e^{-ith\Delta}\mathrm{Op}_h(a)e^{ith\Delta} \end{array}$$

Technique I : which flow & quantization ?

- Classical and quantum observable spaces : $\mathscr{C}^\infty(\Gamma\backslash(\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{CP}^1))$ and $\{\mathrm{End}(L^2(\Gamma\backslash\mathbb{H}^2, F_p))\}_{p\in\mathbb{N}^*}$
- Quantum evolutions : the Schrödinger propagator $\{\mathrm{ad}_{e^{-ith\Delta^{F_p}}}\}_{p\in\mathbb{N}^*}$.
- We need to find a flow $(g_t)_{t\in\mathbb{R}}$ and a quantization procedure to make the following diagram commutes

$$\begin{array}{ccc}
\mathscr{A}(x,\xi,z) & \xrightarrow{\quad Q=?\quad} & Q(\mathscr{A}) \\
\Big\downarrow g_t & & \Big\downarrow \mathrm{ad}_{e^{-ith\Delta^{F_p}}} \\
(g_t\mathscr{A})(x,\xi,z) & \xrightarrow{Q=?} & Q(g_t\mathscr{A}) \sim e^{-ith\Delta^{F_p}} Q(\mathscr{A}) e^{ith\Delta^{F_p}}
\end{array}$$

Technique II : horizontal geodesic flow

- The geodesic flow $(\widetilde{g}_t)_{t\in\mathbb{R}}$ on $\mathrm{PSL}(2,\mathbb{R})$, the unit tangent bundle of $\mathbb{H}^2$.
- This descents to the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on $\Gamma_2\backslash\mathrm{PSL}(2,\mathbb{R})$, the unit tangent bundle of $\Gamma_2\backslash\mathbb{H}^2$.
- The geodesic flow of $\Gamma_2\backslash(\mathbb{H}^2\times\mathbb{CP}^1)$ on $\Gamma_2\backslash(\mathrm{PSL}(2,\mathbb{R})\times\mathbb{CP}^1)$?
- The geodesic flow $(\widetilde{g}_t)_{t\in\mathbb{R}}$ acts on $\mathrm{PSL}(2,\mathbb{R})\times\mathbb{CP}^1$, which is $\pi_1(X)$-invariant.
- Horizontal geodesic flow : It descents to a flow $(g_t)_{t\in\mathbb{R}}$ on $\Gamma_2\backslash(\mathrm{PSL}(2,\mathbb{R})\times\mathbb{CP}^1)$.

Technique III : mixed quantization

- Go back to

$$\int_{\Gamma\backslash(\mathbb{H}^2\times\mathbb{CP}^1)} \mathscr{A}(x,z)\big|u_{p,j}(x,z)\big|^2 dv(x,z)$$

- Berezin-Toeplitz quantization $T_{\mathscr{A},p}(x)\in \mathrm{End}(F_p)|_x$, $v,w\in F_p|_x \cong \mathrm{Sym}^p(\mathbb{C}^p) = H^{(0,0)}(\mathbb{CP}^1,\mathscr{O}(p))$

$$\langle T_{\mathscr{A},p}(x)v,w\rangle_{F_p|_x} = \int_{\mathbb{CP}^1} \mathscr{A}(x,z)v(z)\overline{w(z)}\omega_{\mathbb{FS}}(z)$$

- Weyl quantization

$$\int_{\Gamma\backslash\mathbb{H}^2}\Big(\int_{\mathbb{CP}^1}\mathscr{A}(x,z)\big|u_{p,j}(x,z)\big|^2\omega_{\mathbb{FS}}(z)\Big)dv_X(x)$$

$$=\int_{\Gamma\backslash\mathbb{H}^2}\langle T_{\mathscr{A}}(x)u_{p,j}(x),u_{p,j}(x)\rangle_{F_p} dv_X(x) = \langle \mathrm{Op}_h(T_{\mathscr{A},p})u_{p,j},u_{p,j}\rangle_{L^2(\Gamma\backslash\mathbb{H}^2,F_p)}$$

Technique III : mixed quantization

- Weyl quantization governs high-frequency eigensections

$$(\mathrm{Op}_h(A)s)(x) = \frac{1}{(2\pi h)^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{\frac{i}{h}\langle x-y,\xi\rangle} A(\tfrac{x+y}{2},\xi)s(y)\,dyd\xi.$$

- Berezin-Toeplitz quantization regulates the behavior of an infinite number of linear spaces

$$T_{A,p} \colon L^2(\mathbb{CP}^1,\mathscr{O}(p)) \to H^{(0,0)}(\mathbb{CP}^1,\mathscr{O}(p)), \quad T_{A,p} = P_p A P_p.$$

- Mixed quantization simultaneous controls the high-frequency eigensections of an infinite number of bundles

$$\mathrm{Op}_h(T_{\cdot,p}) \colon \mathscr{C}^\infty(\Gamma\backslash(\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{CP}^1)) \to \mathrm{End}(L^2(\Gamma\backslash\mathbb{H}^2, F_p)).$$

A series of flat bundles $\{F_p\}_{p\in\mathbb{N}^*}$

- A compact Kähler manifold (N, J) and a positive line bundle (L, g^L) over N. These give g^{TN}, dv_N
- A holomorphic unitary action of $\pi_1(X)$ on N and this action can be lifted to L.
- A Riemannian manifold (X, g^{TX}).
- A series of flat bundles

$$\left\{F_p = \pi_1(X)\backslash\left(\widetilde{X} \times H^{(0,0)}(N, L^p)\right) \mid p \in \mathbb{N}^*\right\}$$

- Borel-Weil-Bott : irreducible representation $\rho_\tau : U \to V_\tau$,

$$V_\tau \cong H^{(0,0)}(\mathscr{O}_\tau, L_\tau), \quad V_{p\tau} \cong H^{(0,0)}(\mathscr{O}_\tau, L_\tau^p).$$

- A special case, $N = \mathbb{CP}^1, L = \mathscr{O}(1)$, $H^{(0,0)}(N, L^p) \cong \mathrm{Sym}^p(\mathbb{C}^2)$.

Main Theorem (General)

Uniform Quantum Ergodicity (Ma-M. 2023)

If the horizontal geodesic flow on $\pi_1(X)\backslash(S^*\widetilde{X}\times N)$ is ergodic, then there is a uniform density-1 two dimensional array $\mathbb{B}$. Uniformly equidistributed

$$\left\{\left|u_{p,j}(x,z)\right|^2 : (p,j)\in\mathbb{B}\right\} \quad \text{on} \quad \pi_1(X)\backslash(S^*\widetilde{X}\times N)$$

A Criterion

If X is Anosov and for all $z\in N$, the orbit $\{\pi_1(X)\cdot z\}\subseteq N$ is dense, then the horizontal geodesic flow on $\pi_1(X)\backslash(S^*\widetilde{X}\times N)$.

Analytic torsion

- In the 1930s, the Reidemeister torsion $T_{\mathrm{R}}(X, F)$, induced by Reidemeister and Franz
- Homeomorphic, not homopoty.
- Ray and Singer defined their analytic torsion $T(X, F)$ as the regularized determinant of Hodge-de Rham Laplacian, and conjectured $T_{\mathrm{R}}(X, F) = T(X, F)$.
- This conjecture was proved by Cheeger-Müller/Bismut-Zhang 1978/1992.

Asymptotic Torsion

- The asymptotics of analytic torsions of locally symmetric spaces under finite coverings, Bergeron-Venkatesh, 2010
- The asymptotics of torsions for symmetric powers of a canonical flat vector bundle on 3-dimensional compact hyperbolic manifolds, Müller, 2010
- The asymptotics of a general family of flat vector bundles $\{F_p\}_{p\in\mathbb{N}^*}$ when $p \to +\infty$, Bismut-Ma-Zhang, 2011

Comparison

- Asymptotic spectral information of flat bundles.
- Orthogonal

$$\begin{bmatrix} \text{asymptotic torsion} & \text{UQE} \\ \dim(X) = \text{odd} & \dim(X) \in \mathbb{N}^* \\ F_p \text{ very non-unitary} & F_p \text{ unitary} \end{bmatrix}$$

Uniform QUE

- Quantum unique ergodicity of Rudnick-Sarnak : can we take $\mathbb{B} = \mathbb{N}$?
- Uniform quantum unique ergodicity : can we take $\mathbb{B} = \mathbb{N}^2$?
- Do we have

$$\lim_{\lambda \to +\infty} \sup_{\lambda_{p,j} \geqslant \lambda} \left| \int_{\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1)} \left(\mathscr{A} \left| u_{p,j} \right|^2 - \mathscr{A} \right) dv_{\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1)} \right| = 0$$

for any $\mathscr{A} \in \mathscr{C}^\infty(\Gamma_2 \backslash (\mathbb{H}^2 \times \mathbb{CP}^1))$?

AQUE to AUQUE I

- AQUE : on arithmetic surfaces, (Lindenstrauss, 2006). Extra symmetries called Hecke operators.
- AUQUE (arithmetic uniform quantum unique ergodicity) ?
- $K = \mathbb{Q}[\sqrt{2}]$, $O_K = \mathbb{Z}[\sqrt{2}]$ and $\sigma\colon a + b\sqrt{2} \to a - b\sqrt{2}$
- Quadratic forms and special groups : hyperbolic

$$a(x) = x_0^2 - \sqrt{2}x_1^2 - \sqrt{2}x_2^2, \quad G = \mathrm{SO}(a(x), \mathbb{R}) \cong \mathrm{SO}(1,2,\mathbb{R}),$$

  and compact

$$b(x) = x_0^2 + \sqrt{2}x_1^2 + \sqrt{2}x_2^2, \quad U = \mathrm{SO}(b(x), \mathbb{R}) \cong \mathrm{SO}(3, \mathbb{R}).$$

- Borel density : $G_{\mathbb{Z}[\sqrt{2}]} \subset G$ discrete cocompact, and $\sigma(G_{O_K}) \subset U$ is dense.

AQUE to AUQUE II

- The arithmetic surface (orbifold) $G_{\mathbb{Z}[\sqrt{2}]}\backslash\mathbb{H}^2$.
- Selberg Lemma, pass $G_{\mathbb{Z}[\sqrt{2}]}$ to a torsion free subgroup.
- $\sigma(G_{\mathbb{Z}[\sqrt{2}]}) \subset \mathrm{SO}(3,\mathbb{R})$ has dense image.
- $\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{SU}(2)/\{\pm 1\}$, representations of $\mathrm{SO}(3,\mathbb{R})$ are $\{\mathrm{Sym}^{2p}(\mathbb{C}^2)\}_{p\in\mathbb{N}^*}$.

AUQUE (M. 2023)

The UQUE holds for

$$\Big(G_{\mathbb{Z}[\sqrt{2}]}\backslash\mathbb{H}^2,\, G_{\mathbb{Z}[\sqrt{2}]}\backslash(\mathbb{H}^2 \times \mathrm{Sym}^{2p}(\mathbb{C}^2)),\, G_{\mathbb{Z}[\sqrt{2}]}\backslash\big(\mathbb{H}^2 \times \mathbb{CP}^1\big)\Big).$$

Thank you !