Quantizing singular symplectic manifolds

Quantization in Geometry, University of Cologne

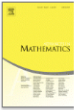
Eva Miranda

UPC and CRM

Geodesics on pseudo-Riemannian manifolds

- Given a pseudo-Riemannian manifold (M, g), let $\mathcal{L}$ be the space of all oriented non-parametrized geodesics.
- $\mathcal{L}$ splits as $\mathcal{L}_\pm$, the space of space-like $(g(\dot\gamma, \dot\gamma) > 0)$- and time-like $(g(\dot\gamma, \dot\gamma) < 0)$ geodesics, and $\mathcal{L}_0$, the space of light-like geodesics ($g(\dot\gamma, \dot\gamma) = 0)$.
- $\mathcal{L}_\pm$ is even dimensional and symplectic.
- $\mathcal{L}_0$ can be seen as the common boundary of $\mathcal{L}_\pm$ and has an induced contact structure.
- In dimension 2, this structure is indeed Poisson.

A discovery by Khesin and Tabachnikov



ELSEVIER

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Pseudo-Riemannian geodesics and billiards

Boris Khesin [a], Serge Tabachnikov [b]

Theorem 2.1 *The manifolds* $\mathcal{L}_\pm$ *carry symplectic structures obtained from* T^*M *by Hamiltonian reduction on the level hypersurfaces* $H = \pm 1$*. The manifold* $\mathcal{L}_0$ *carries a contact structure whose symplectization is the Hamiltonian reduction of the symplectic structure in* T^*M *(without the zero section) on the level hypersurface* $H = 0$*.*

The 2-dimensional case

Example 2.2 Let us compute the area form on the space of lines in the Lorentz plane with the metric $ds^2 = dxdy$. A vector (a, b) is orthogonal to $(a, -b)$. Let $D(a, b) = (b, a)$ be the linear operator identifying vectors and covectors via the metric.

The light-like lines are horizontal or vertical, the space-like have positive and the time-like negative slopes. Each space $\mathcal{L}_+$ and $\mathcal{L}_-$ has two components. To fix ideas, consider space-like lines having the direction in the first coordinate quadrant. Write the unit directing vector of a line as (e^{-u}, e^u), $u \in \mathbf{R}$. Drop the perpendicular $r(e^{-u}, -e^u)$, $r \in \mathbf{R}$, to the line from the origin. Then (u, r) are coordinates in $\mathcal{L}_+$. Similarly one introduces coordinates in $\mathcal{L}_-$.

Lemma 2.3 *(cf.[4, 6]) The area form ω on $\mathcal{L}_+$ is equal to $2du \wedge dr$, and to $-2du \wedge dr$ on $\mathcal{L}_-$.*

The 2-dimensional case

Now consider a neighborhood of a light-like line among all lines, that is, a neighborhood of a point in $\mathcal{L}_0$ regarded as a boundary submanifold between $\mathcal{L}_+$ and $\mathcal{L}_-$. Look at the variation $\xi_\varepsilon = (1, \varepsilon)$ of the horizontal (light-like) direction $\xi_0 = (1, 0)$, and regard (ε, r) as the coordinates in this neighborhood. For $\varepsilon > 0$ the corresponding half-neighborhood lies in $\mathcal{L}_+$, while the coordinates u and ε in this half-neighborhood are related as follows. Equating the slope of $(1, \varepsilon)$ to the slope of (e^{-u}, e^u) we obtain the relation $\varepsilon = e^{2u}$ or $u = \frac{1}{2} \ln \varepsilon$. Then the symplectic structure $\omega = 2du \wedge dr = d \ln \varepsilon \wedge dr = \frac{1}{\varepsilon} d\varepsilon \wedge dr$. One sees that $\omega \to \infty$ as $\varepsilon \to 0$. The Poisson structure, inverse to ω, is given by the bivector field $\varepsilon \frac{\partial}{\partial \varepsilon} \wedge \frac{\partial}{\partial r}$ and it extends smoothly across the border $\varepsilon = 0$.

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

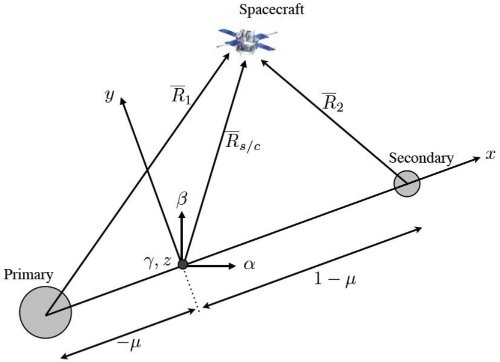


Figure: Circular 3-body problem

Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+$, can be then extended to infinity $(x = 0)$.
- The symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$. for $x > 0$

Why singular?

1. The 2-dimensional space of geodesics on a pseudoriemannian surface is singular.
2. Some non-compact symplectic manifolds can be compactified as singular symplectic manifolds.
3. Singular forms appear after regularization transforms in celestial mechanics and sigma coordinates in Painlevé equations.
4. They model certain manifolds with boundary.
5. Why not?

Zooming in...

But sometimes it is good to zoom out...

Zooming out...to gain perspective

Poisson, 1809

MÉMOIRE

Sur la Variation des Constantes arbitraires dans les questions de Mécanique,

Lu à l'Institut le 16 Octobre 1809;

Par M. POISSON.

constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de a et b, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi}-\frac{da}{ds}\cdot\frac{db}{d\varphi}+\frac{db}{du}\cdot\frac{da}{d\psi}-\frac{da}{du}\cdot\frac{db}{d\psi}+\frac{db}{dv}\cdot\frac{da}{d\theta}-\frac{da}{dv}\cdot\frac{db}{d\theta}=(b, a).$$

Figure: Poisson bracket

Singular symplectic manifolds as Poisson manifolds

The local model

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

does not define a smooth form but its dual defines a smooth Poisson structure!

$$\mathbf{\Pi} = \mathbf{x_1^m}\frac{\partial}{\partial \mathbf{x_1}} \wedge \frac{\partial}{\partial \mathbf{y_1}} + \sum_{\mathbf{i \geq 2}}^{\mathbf{n}} \frac{\partial}{\partial \mathbf{x_i}} \wedge \frac{\partial}{\partial \mathbf{y_i}}$$

The structure Π is a bivector field which satisfies the integrability equation $[\Pi, \Pi] = 0$. The Poisson bracket associated to Π is given by the equation

$$\{f, g\} := \Pi(df, dg)$$

The local Poisson case. Splitting Theorem.

The local structure for Poisson manifolds is given by the following:

Theorem (Weinstein)

Let (M^n, Π) *be a smooth Poisson manifold and let* p *be a point of* M *of rank* $2k$*, then there is a smooth local coordinate system* $(x_1, y_1, \ldots, x_{2k}, y_{2k}, z_1, \ldots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} *vanish at the origin.*

b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

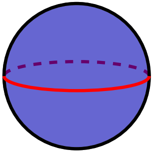
Theorem

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \dots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Examples

- A Radko surface.



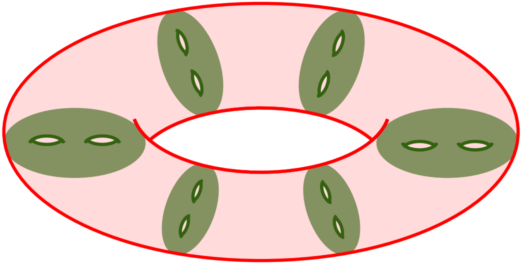
- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

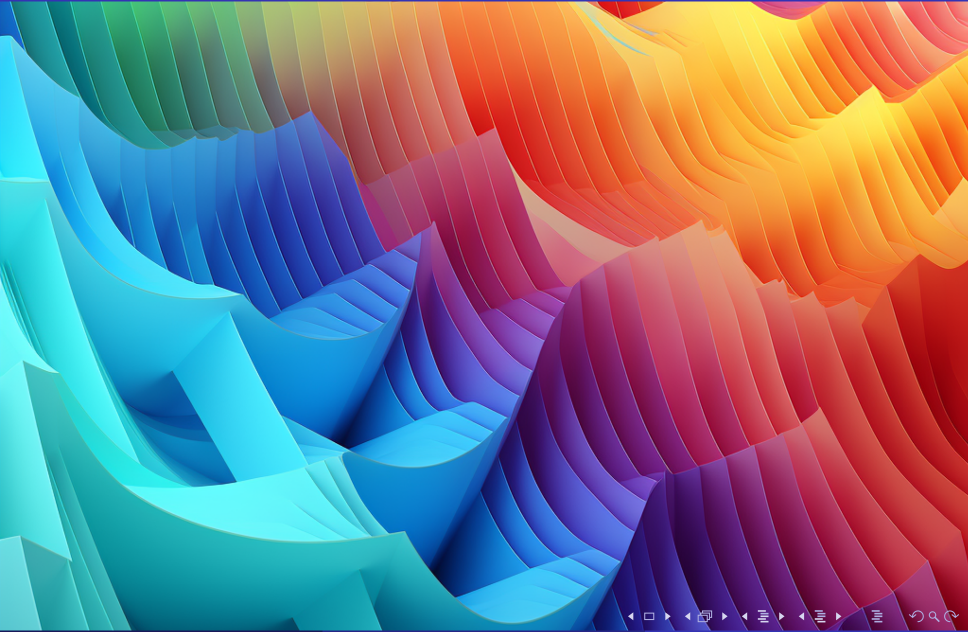
Poisson Geometry of the critical hypersurface

This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field v transverse to the symplectic foliation induced on Z (modular vector field).
3. (Guillemin-M. Pires) Z is a mapping torus with glueing diffeomorphism the flow of v.



Quantization of Poisson manifolds?



Quantization of Poisson manifolds

Theorem (Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer, Fedosov)

Given any symplectic manifold (M, ω), there exists a formal associative deformation $*_h$ of the product on $C^\infty(M)$ such that

$$f *_h g = fg + h\{f, g\} + O(h^2), \tag{1}$$

which is essentially unique.

In the Poisson case there is the following result.

Theorem (Kontsevich)

Given any Poisson manifold $(M, \{\ ,\ \})$, there exists a formal associative deformation of $C^\infty(M)$ satisfying equation (1).

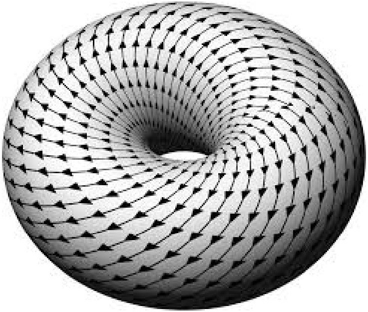
For b-Poisson manifolds, this deformation can be understood in terms of an enlarged De Rham complex.

Time to change my glasses

Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d: {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

Key point: A b-form of degree k decomposes as:

$$\omega = \alpha \wedge \frac{dz}{z} + \beta, \quad \alpha \in \Omega^{k-1}(M),\ \beta \in \Omega^k(M) \quad d\omega := d\alpha \wedge \frac{dz}{z} + d\beta.$$

- This defines the b-cohomology groups. (Mazzeo-Melrose)

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2. It is also a Poisson structure!.
- This dual point of view, allows us to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path methods.

Symplectic manifolds with boundary as Poisson manifolds

- Consider formal deformation quantization of manifolds with boundary à la Fedosov.
- These symplectic manifolds with boundary have local normal form of type (b-symplectic):

$$\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i$$

Theorem (Nest-Tsygan)

Equivalence classes of star products on a b-symplectic manifold are in one-to-one correspondence with elements in

$$^{b}H^2(M, \mathbb{C}[\hbar]) \simeq H^2(M, \mathbb{C}[\hbar]) \oplus H^1(\partial M, \mathbb{C}[\hbar]).$$

Deformation quantization of E-manifolds

- The b-tangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.
- An important class is that of b^m-tangent bundle defined as the bundle whose sections are given by vector which are tangent to an hypersurface to order m.
- Deformation quantization of E-manifolds:

Theorem (Nest-Tsygan)

The set of isomorphism classes of E-deformations is in bijective correspondence with the space

$$\frac{1}{i\hbar}\omega + \ {}^{E}H^2(M, \mathcal{C}[[\hbar]])$$

Poisson cohomology and deformation quantization

Theorem (Guillemin-M.-Pires, Marcut-Osorno)

$$^{b}H^*(M) \cong H^*_{\Pi}(M)$$

So in particular; $H^*_{\Pi}(M) \simeq H^*(M) \oplus H^{*-1}(Z)$. What about other E-symplectic manifolds? $^{bm}H^*(M) \ncong H^*_{\Pi}(M)$

Theorem (Nicolás)

$$\mathrm{H}^k_{\Pi}(M) \simeq \mathrm{H}^k(M) \oplus \mathrm{H}^{k-1}(Z) \oplus \left(\mathrm{H}^{k-1}_{\Pi}(\mathcal{F}_Z)\right)^{m-1} \oplus \left(\frac{\mathrm{H}^{k-1}(Z)}{\alpha \wedge \mathrm{H}^{k-1}_{\Pi}(\mathcal{F})}\right)^{m-1}$$

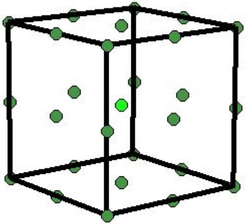
What about Geometric quantization of Poisson manifolds?

Geometric quantization of b-manifolds?

- By extending the admissible Hamiltonian functions with b-functions, we can consider toric actions on b-symplectic manifolds with n-dimensional orbits (also along the critical set).
- Their orbits would be an example of *"Lagrangian"* submanifold $\rightsquigarrow$ polarization.
- There is a analogue of Delzant theorem for b-toric actions.
- Lagrangian orbits (*polarization*) can be read as points on the image polytope.

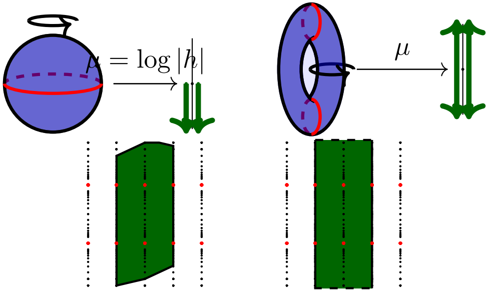
In symplectic geometry we can quantize counting Bohr-Sommerfeld leaves.

BS leaves of the polarization correspond to the **integer points in the interior of the Delzant polytope (Guillemin-Sternberg)**.



Bohr-Sommerfeld leaves are the integer points in the Delzant polytope.

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* close to the critical hypersurface.



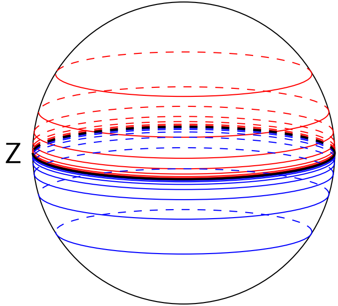
Guillemin-M.-Pires-Scott

There is a one-to-one correspondence between b-toric manifolds and b-Delzant polytopes and toric b-manifolds are either:

- ${}^b\mathbb{T}^2 \times X$ (X a toric symplectic manifold of dimension $(2n-2)$).
- obtained from ${}^b\mathbb{S}^2 \times X$ via symplectic cutting.

We can use the polytopes to quantize in the b-case too!

$$({}^bS^2, Z = \{h = 0\}, \omega = \frac{1}{h} dh \wedge d\theta, \mu = -\log|h|)$$



The b-sphere contains as many Bohr-Sommerfeld leaves on the northern hemisphere (in red) as on the southern hemisphere (in blue).

Formal quantization (Meinrenken, Paradan, Weitsman)

1. (M, ω) compact symplectic manifold and $(\mathbb{L}, \nabla)$ line bundle with connection of curvature ω.
2. By twisting the spin-$\mathbb{C}$ Dirac operator on M by $\mathbb{L}$ we obtain an elliptic operator $\bar{\partial}_{\mathbb{L}}$.

Since M is compact, $\bar{\partial}_{\mathbb{L}}$ is Fredholm, and we define the geometric quantization $Q(M)$ by

$$Q(M) = \mathrm{ind}(\bar{\partial}_{\mathbb{L}})$$

as a virtual vector space.

What if M is non-compact?

- Describe a method for quantizing non-compact prequantizable Hamiltonian T-manifolds based upon the "quantization commutes with reduction" principle.
- Important assumption: The moment map ϕ is proper.
- Apply this method to b-symplectic manifolds.

Assume M is non-compact but ϕ proper:

- Let $\mathbb{Z}_T \in \mathfrak{t}^*$ be the weight lattice of T and α a regular value of the moment map.
- Let V be an infinite-dimensional virtual T-module with finite multiplicities. We say $V = Q(M)$, formal quantization if for any compact Hamiltonian T-space N with integral symplectic form, we have
$$(V \otimes Q(N))^T = Q((M \times N)//_0 T). \tag{2}$$
In other words, denote by $M_\alpha = \phi^{-1}(\alpha)/T$ the reduced space, $[Q, R] = 0$ implies $Q(M)_\alpha = Q(M_\alpha)$ where $Q(M)_\alpha$ is the α-weight space of $Q(M) \rightsquigarrow Q(M) = \bigoplus_\alpha Q(M_\alpha)\alpha$

Theorem (Braverman-Paradan)

$Q(M) = ind(\overline{\partial})$

Formal quantization of b-symplectic manifolds

A b-symplectic manifold is prequantizable if:

- $M \setminus Z$ is prequantizable
- The cohomology classes given under the Mazzeo-Melrose isomorphism applied to $[\omega]$ are integral.

Theorem (Guillemin-M.-Weitsman)

- $Q(M)$ *exists.*
- $Q(M)$ *is finite-dimensional.*

Idea of proof

$Q(M) = Q(M_+) \bigoplus Q(M_-)$

and an ϵ-neighborhood of Z does not contribute to quantization.

Braverman, Loizides, and Song

$Q(M) = ind(D_{APS})$

with D_{APS} the Dolbeault-Dirac operator endowed with the Atiyah-Patodi-Singer boundary conditions.

Bohr-Sommerfeld leaves and formal geometric quantization

Assume that there is a T-action with non-vanishing modular weight,

$$Q(M) = \oplus_\alpha \epsilon_\alpha Q(M//_\alpha T)\alpha,$$

For the toric case, the quotient $M//_\alpha T$ is a point. This proves,

Theorem (Mir, M., Weitsman)

*Let (M^{2n}, Z, ω, μ) be a b-symplectic toric manifold Then, the formal geometric quantization of M coincides with the count of its Bohr-Sommerfeld leaves **with sign** (Bohr-Sommerfeld quantization).*

Geometric quantization of symplectic toric manifolds

Geometric quantization of b-symplectic toric manifolds

Count of the integer points in the image of the moment map

Count **with sign** of the integer points in the image of the moment map

Formal geometric quantization of symplectic toric manifolds

Formal geometric quantization of b-symplectic toric manifolds

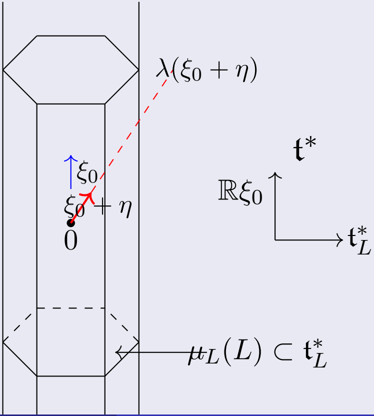
What about quantization of b^m-symplectic manifolds?

Theorem (Guillemin, M., Weitsman)

1. *If m is odd, $Q(M)$ is a finite dimensional virtual T-module.*
2. *If m is even, there exists a weight $\xi \in \mathfrak{t}^*$, integers $c_\pm$, and $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, and $\eta \in \mathfrak{t}^*$ is a weight of T,*

$$\dim Q(M)^{\lambda\eta} = \begin{cases} 0 & if \quad \eta \neq \pm\xi \\ c_\pm & if \quad \eta = \pm\xi \end{cases}$$

(In fact, $c_\pm = \epsilon_\pm \dim Q(M)^{\pm\lambda\xi}$, where $\epsilon_\pm \in \{\pm 1\}$, for any λ sufficiently large.)

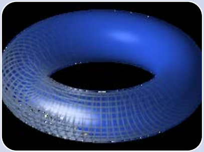


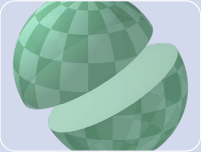
(Singular) symplectic manifolds

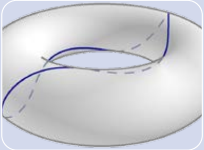
- b^m -Symplectic
- Symplectic
- Folded symplectic

(Singular) symplectic manifolds

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Theorem (M.-Weitsman)

For any b^m-manifold endowed with a T-action with non-vanishing modular weight,

$$Q(M) = ind(D).$$

Outside the b-box

To b or not to b...

Work in progress, M-Nest

Semisimple linearizable Poisson structures have associated E-symplectic manifolds.
This association is done via a *desingularization process* gives a hierarchy of E-symplectic manifolds.

What comes next?

- Understand Poisson Geometry through E-glasses.
- Compare Deformation quantization of E-symplectic manifolds and Poisson manifolds through the desingularization scheme.
- Dream: Understand Geometric Quantization of Poisson manifolds.