Random eigenstates of the Quantum Cat Map

Stéphane Nonnenmacher + Nir Schwartz

Institut de Mathématiques d'Orsay, Université Paris-Saclay

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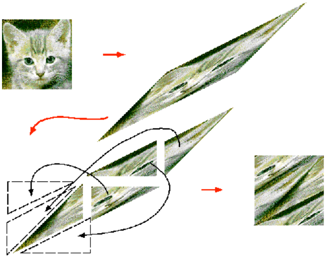
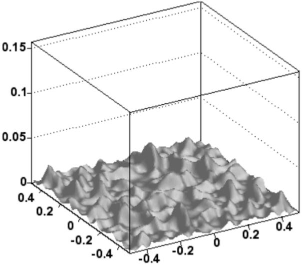


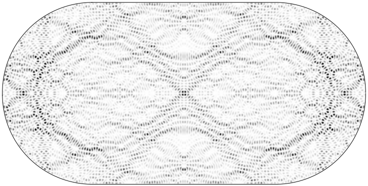
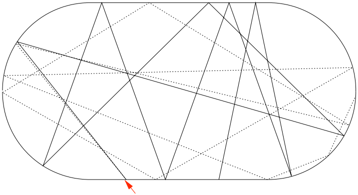
Cat map ©L.Poon



A random state on $\mathbb{T}^2$ ©F.Faure

A question in Quantum Chaos

How do the eigenmodes of classically chaotic systems look like?

$-\Delta_\Omega \psi_n = \lambda_n^2 \psi_n$ Helmholtz equation: λ_n = eigenfrequency

Examples:
– eigenmode of the (Dirichlet) Laplacian on a chaotic Euclidean billiards (figure ©Arnd Bäcker)
– Laplacian on a compact manifold of negative curvature (M, g).

It is assumed that the billiard flow / geodesic flow enjoys chaotic dynamical properties : ergodicity, mixing, exponential instability of the trajectories.

The strongest form of chaos is satisfied by Anosov flows (e.g. on manifolds of negative curvature).

Macroscopic properties of chaotic modes: Quantum Ergodicity

The connection beween wave dynamics and ray dynamics can be realized in the high frequency ($\equiv$ semiclassical) regime $\lambda_n \gg 1$.

- **Quantum ergodicity** [SCHNIRELMAN, ZELDITCH, COLIN DE VERDIÈRE, ZELDITCH-ZWORSKI,..] Assume the billiard / geodesic flow on $S^*\Omega$ is ergodic. Then, there is a density-1 subsequence $\mathcal{S} \subset \mathbb{N}$ such that,

$$\text{for any open } \omega \subset \Omega, \qquad \int_\omega |\psi_n(x)|^2 \, dL(x) \xrightarrow{\mathcal{S} \ni n \to \infty} \frac{\mathrm{Vol}(\omega)}{\mathrm{Vol}(\Omega)}$$

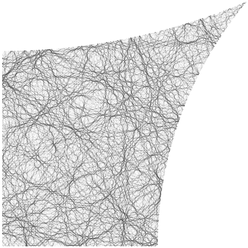
Almost all the eigenmodes are asymptotically equidistributed over Ω (at the *macroscopic* scale).

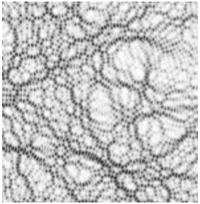
Equidistibution as well on phase space $S^*\Omega$:
$\langle \psi_n, \mathrm{Op}(a)\psi_n \rangle \to \frac{\int_{S^*\Omega} a(x,\xi)\, dx d\xi}{\mathrm{Vol}(S^*\Omega)}$, for $a(x,\xi)$ a 0-homogeneous function.

- If the flow is Anosov: small scale QE [HAN, HEZARI-RIVIÈRE] Equidistribution in discs of sizes $\sim (\log \lambda_n)^{-\alpha}$.

- [RUDNICK-SARNAK]: **Quantum Unique Ergodicity conjecture** on manifolds of negative curvature: all eigenmodes equidistribute when $n \to \infty$.

Microscopic properties of chaotic modes: a Random Wave Model





Eigenmode of a 2D Sinai billiard ©Alex Barnett, and a zoom.

Statistical properties at microscopic scale (wavelength $\sim \lambda_n^{-1}$) ?

- Random Wave model [BERRY'77] ($d = 2$): random combination of many plane waves of same frequency λ_n but arbitrary directions $\xi_j \in \{|\xi| = 1\}$:

$\psi_{RW,\lambda}(x) = \mathrm{Re} \sum_{j=1}^{J(\lambda)} a_j e^{i\lambda\xi_j \cdot x}$, with a_j iid $\sim \mathcal{N}_{\mathbb{C}}(0,1)$.

$\rightsquigarrow$ Monochromatic Gaussian random field on $\mathbb{R}^2$: $\mathbb{E}(\psi_{RW,\lambda}(x)) = 0$, correlations $\mathbb{E}(\psi_{RW,\lambda}(x)\psi_{RW,\lambda}(x + y/\lambda)) = J_0(|y|)$ at scale λ^{-1}.

$\rightsquigarrow$ the *value distribution* of $\psi_{RW,\lambda}(x)$ is Gaussian

- RW Conjecture [BERRY'77]: the local statistical properties of the eigenmodes ψ_n should converge to those of ψ_{RW,λ_n} when $n \to \infty$.

Random quasimodes of the Laplacian (1)

$\ominus$ The RW conjecture for Laplacian eigenstates of chaotic billiards / manifolds remains wide open.

$\oplus$ Weaker ambition [ZELDITCH'09]: on (M, g), take random linear combinations of eigenstates of Δ in spectral windows $I_\lambda := [\lambda, \lambda + W]$. Random state in the spectral space $\mathcal{V}_{I_\lambda}$. Random quasimode.

$$\Phi_{I_\lambda} = \sum_{\lambda_n \in I_\lambda} a_n \psi_n, \qquad a_n \text{ random i.i.d. Gaussian}.$$

Alternatively: randomly choose a state in the unit sphere of $\mathcal{V}_{I_\lambda}$ (Haar meas.).

[BURQ-LEBEAU'11,MAPLES'13,ZELDITCH'14] Consider spectral windows I_λ of widths $W = W(\lambda) \to \infty$. Then a *generalized Weyl's law* holds on the $\mathcal{V}_{I_\lambda}$:

$$\frac{\mathrm{Tr}(\mathrm{Op}(a)\Pi_{I_\lambda})}{\mathrm{Tr}(\Pi_{I_\lambda})} \xrightarrow{\lambda \to \infty} \frac{1}{\mathrm{Vol}(S^*M)} \int_{S^*M} a(x, \xi)\, dx d\xi$$

$\rightsquigarrow$ probabilistic estimates show that random quasimodes in $\mathcal{V}_{I_\lambda}$ almost surely equidistribute when $\lambda \to \infty$.

Random quasimodes of the Laplacian (2)

[Burq-Lebeau'11,Maples'13,Zelditch'14] Consider spectral windows I_λ of widths $W(\lambda) \gg 1$. *Generalized Weyl's law* holds on the $\mathcal{V}_{I_\lambda}$:

$$\frac{\mathrm{Tr}(\mathrm{Op}(a)\Pi_{I_\lambda})}{\mathrm{Tr}(\Pi_{I_\lambda})} \xrightarrow{\lambda\to\infty} \frac{1}{\mathrm{Vol}(S^*M)} \int_{S^*M} a(x,\xi)\, dx d\xi$$

$\rightsquigarrow$ random quasimodes in windows I_λ a. s. satisfy QUE when $\lambda \to \infty$.

$\ominus$ This result holds on any (M, g), no use of chaotic dynamics.
What is special about chaotic manifolds?

• On a surface (M, g) of negative curvature, the remainder in Weyl's law is $\mathcal{O}(\lambda/\log\lambda)$ [Bérard'77]

$\rightsquigarrow$ The generalized Weyl's law holds on thinner windows of widths $W(\lambda) \gg 1/\log\lambda$ [Keeler'19, Canzani-Galkowski'20]

$\rightsquigarrow$ Random quasimodes in windows I_λ of widths $W(\lambda) \gg 1/\log\lambda$ will a.s. satisfy QUE.

Random eigenstates in presence of large spectral multiplicities

In case of manifolds featuring spectral multiplicities, the space $\mathcal{V}_{I_\lambda}$ may reduce to a single eigenspace, and hence Φ_{I_λ} to a random eigenstate.

Two well-known (non chaotic) surfaces:

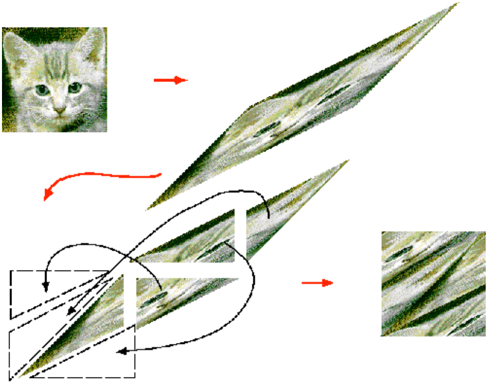
- Round sphere $\mathbb{S}^2$ [ZELDITCH'92,VANDERKAM,NAZAROV-SODIN]: eigenspace $\mathcal{V}_\ell$ of dimension $2\ell+1 \asymp \lambda = \sqrt{\ell(\ell+2)}$.
$\rightsquigarrow \Phi_\ell$ random spherical harmonic in $\mathcal{V}_\ell$. Convergence to the RW local statistics when $\ell \to \infty$.

- Square torus $\mathbb{T}^2$. Spectral multiplicities $\sim \log \lambda$, strong arithmetic dependence ($\lambda_n^2 = k_1^2 + k_2^2$, $k_i \in \mathbb{Z}$).
Random eigenstates = Arithmetic Random Waves.
For a generic subsequence (λ_{n_k}), random eigenstates Φ_{n_k} asymp. enjoy the same local statistics as Berry's RW. [KRISHNAPUR-KURLBERG-WIGMAN'13]

- On a surface of negative curvature, Weyl's law bounds multiplicities by $\mathcal{O}(\lambda/\log\lambda)$.
Yet, no surface of negative curvature is known to enjoy such high multiplicities when $\lambda \to \infty$.

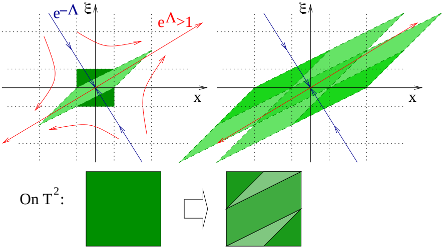
Now, let us present our toy model of quantized chaotic dynamics:

the Quantum Cat Map



The Quantum Cat Map

Discrete time dynamics: symplectic map on the compact phase space $\mathbb{T}^2$.



$S \in SL(2,\mathbb{Z})$ acts on $\mathbb{T}^2$.
If $\mathrm{Tr}(S) > 2$, the matrix is hyperbolic (eigenvalues $e^{\pm\Lambda}$).
Anosov diffeomorphism.

"Arnold's Cat Map"

Quantization of S: [HANNAY-BERRY'80]

- The linear map $S : \mathbb{R}^2_{x,\xi} \to \mathbb{R}^2_{x,\xi}$ can be quantized into a metaplectic operator $M_\hbar(S)$ acting on $\mathcal{S}'(\mathbb{R}_x)$ ($\hbar \in \mathbb{R}^*_+$ arbitrary semiclassical parameter).

- How to "project" on $\mathbb{T}^2_{x,\xi} = \mathbb{R}^2/\mathbb{Z}^2$? Consider distributions $\psi \in \mathcal{S}'(\mathbb{R})$ such that ψ and its $\hbar$-Fourier transform $\mathcal{F}_\hbar\psi$ are $\mathbb{Z}$-periodic. Nontrivial iff $\hbar = (2\pi N)^{-1}$ for $N \in \mathbb{N}^*$: N-dimensional space of distributions $\mathcal{H}_N$, spanned by $\{\delta_{\frac{j}{N}+\mathbb{Z}}, j = 1,\ldots,N\}$. Set a Hermitian structure on $\mathcal{H}_N$.

$\rightsquigarrow$ Ladder of quantum spaces $(\mathcal{H}_N)_{N\geq 1}$ assoc. with the torus phase space.

- If $S \in SL_\theta(2,\mathbb{Z})$ ($ac \equiv bd \equiv 0$ mod 2), then $M_\hbar(S)$ preserves $\mathcal{H}_N$, acting through a unitary operator $U_N(S)$. **Quantum cat map** = $(U_N(S))_{N\geq 1}$.

The Quantum Cat Map through Toeplitz quantization

Remark that $\mathbb{T}^2 \equiv \mathbb{T}_{\mathbb{C}} = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Hermitian holomorphic line bundle $(L, h) \to \mathbb{T}^2$: each section $\phi \in C^\infty(\mathbb{T}_{\mathbb{C}}, L)$ can be identified with a function $\tilde{\phi} : \mathbb{C} \to \mathbb{C}$ with quasiperiodicity properties:

$$\tilde{\phi}(z+1) = e^{\pi(1/2+z)}\,\tilde{\phi}(z), \quad \tilde{\phi}(z+i) = e^{\pi(1/2-iz)}\,\tilde{\phi}(z)\,;$$

Hermitian metric $\langle \phi(z), \phi(z) \rangle_h = |\tilde{\phi}(z)|^2\, e^{-\pi|z|^2}$.

- $\forall N \in \mathbb{N}^*$, the space H_N of holomorphic sections $\mathbb{T}_{\mathbb{C}} \to L^{\otimes N}$ is the N-dimensional space of theta functions of degree N on $\mathbb{T}_{\mathbb{C}}$.

The Bargmann transform unitarily maps $\mathcal{H}_N$ to H_N: $\psi \in \mathcal{H}_N \mapsto \mathcal{B}_N\psi \in H_N$.

$z \mapsto \langle \mathcal{B}_N\psi(z), \mathcal{B}_N\psi(z) \rangle_{h^N}$ is called the Husimi function of $\psi \in \mathcal{H}_N$.

- [ZELDITCH'97] the action of $U_N(S)$ on $\mathcal{H}_N$ is equivalent with the following action on H_N:

$$\phi \in H_N \mapsto c_S\, \Pi_N(\phi \circ S^{-1}), \quad \text{for some explicit constant } c_S > 0.$$

Here Π_N is the Bergman projector on H_N. Toeplitz quantization of S.

Quantum Ergodicity of the Quantum Cat Map

Instead of a single operator Δ_g quantizing a geodesic flow, we now have a ladder of unitary operators $(U_N(S))_{N\geq 1}$ (quantum propagators), quantizing the Anosov diffeomorphism S.

Semiclassical limit: $N \to \infty$.

Unitarity $\Longrightarrow$ each $U_N(S)$ admits an o.n.b of eigenstates $(\psi_j^{(N)})_{j=1,\ldots,N}$.

Each $\psi_j^{(N)}$ is a N-dimensional vector in the basis $(\delta_{\frac{j}{N}+\mathbb{Z}})_{j=1,\ldots,N}$ of $\mathcal{H}_N$.

- To test the localization properties of $\psi_j^{(N)}$ on the phase space $\mathbb{T}^2$, use (Weyl) quantization: to each test function $f \in C^\infty(\mathbb{T}^2)$ corresponds an operator $\mathrm{Op}_N(f)$ acting on $\mathcal{H}_N$.

Quantum Ergodicity [Bouzouina-DeBièvre'96, Han'20]: there is a density 1 subsequence $\mathcal{S}$ s.t.

$$\forall f \in C^\infty(\mathbb{T}^2, \mathbb{C}), \quad \langle \psi_j^{(N)}, \mathrm{Op}_N(f)\, \psi_j^{(N)} \rangle \xrightarrow{\mathcal{S} \ni (N,j) \to \infty} \int_{\mathbb{T}^2} f(x,\xi)\, dx d\xi$$

- [Degli Esposti-Graffi-Isola'93, Kurlberg-Rudnick'00,'01] construct "Hecke" (arithmetic) eigenstates of $U_N(S)$, which satisfy *QUE*.

Quantum periods of the quantum cat map

- U_N is a (proj.) representation of $SL_\theta(2,\mathbb{Z})$: $U_N(S)\,U_N(S') = e^{i\varphi_N} U_N(SS')$
- $U_N(S)$ only depends on $S \bmod 2N$

$\Longrightarrow$ quantum periodicity: $\forall N \geq 1$, there is an integer $P_N > 1$ s.t.

$$(U_N(S))^{P_N} = e^{i\theta_N}\, Id_N \quad \text{for some } \theta_N \in [0, 2\pi)$$

$\Longrightarrow$ eigenphases $\alpha_{\ell,N} = \frac{2\pi(\ell+\theta_N)}{P_N}$, $\ell = 1\ldots, P_N$.

$\Longrightarrow$ explicit expression for the projector on the eigenspace $\mathcal{V}_{\ell,N} \subset \mathcal{H}_N$:

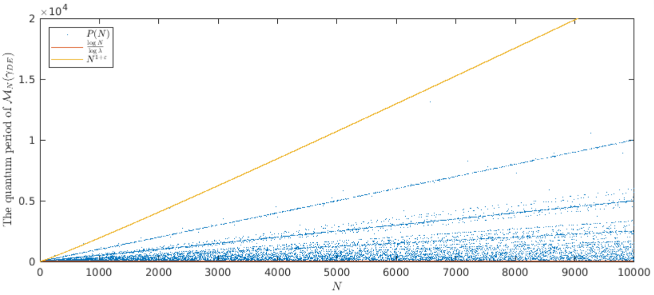
$$\Pi_{\ell,N} = \frac{1}{P_N} \sum_{k=0}^{P_N-1} e^{-ik\alpha_{\ell,N}} (U_N(S))^k\,, \qquad \ell = 1, \ldots, P_N\,.$$

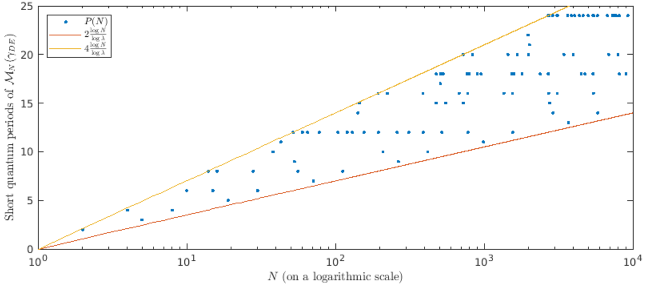
The quantum periods P_N are essentially those of ($S^k \bmod 2N$), they depend erratically of N, within the range

$$\frac{2\log N}{\Lambda} - C \leq P_N \leq N \log N \qquad \text{[Keating'91, Kurlberg'03]}$$

There exists a (scarce) subsequence $\mathcal{S}_{short}$ such that $P_N = \frac{2\log N}{\Lambda} + \mathcal{O}(1)$: short (minimal) periods. [Bonechi-De Bièvre'03]

Quantum periods : numerics





Top: quantum periods of $U_N(S_0)$, for

$$S_0 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

(linear-linear scale).

Bottom: the rare short periods (log-linear scale).

Local statistics of Quantum Cat eigenstates

Each eigenstate $\psi_j^{(N)}$ is a (normalized) vector in $\mathcal{H}_N \equiv \mathbb{C}^N$.

Inspired by Berry's Random Wave conjecture for the Laplacian eigenmodes, we are interested in the *statistical* properties of our eigenvectors $\psi_j^{(N)}$.

What model to replace Berry's Random Waves?

- Take the random vector on the unit sphere of $\mathcal{H}_N$ (Haar measure).
- Up to normalization, equivalent with random Gaussian vector:

$$\Phi_{Gauss}^{(N)} = \sum_{\ell=0}^{N-1} a_j \, \delta_{\frac{j}{N}+\mathbb{Z}}, \qquad a_j \ i.i.d. \sim \mathcal{N}_{\mathbb{C}}(0,1).$$

equivalently $\Phi_{Gauss}^{(N)} \sim \mathcal{N}_{\mathbb{C}}(0, Id_N)$.

$\ominus$ [Kurlberg-Rudnick'01] For N along the sequence $\mathcal{S}_{sp}$ of "split primes", explicitly compute the "Hecke" eigenstates.

$\Longrightarrow$ when $\mathcal{S}_{sp} \ni N \to \infty$, the value distribution of the Hecke eigenstates converges to a non-Gaussian distribution.

Short quantum periods $\rightsquigarrow$ non-QUE eigenstates

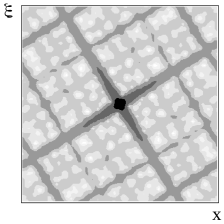
Short quantum periods $P_N \longleftrightarrow$ large spectral multiplicities

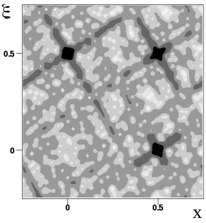
- [KURLBERG-RUDNICK'01, BOURGAIN'07]: for any $\varepsilon > 0$, consider the (dense) subsequence $\mathcal{S}_{long}$ such that $P_N \geq N^{\varepsilon}$. Then, for any choice of eigenbases, QUE holds along $\mathcal{S}_{long}$.

- Let us focus on $N \in \mathcal{S}_{short}$: $P_N = \frac{2}{\Lambda} \log N + \mathcal{O}(1)$.

$\rightarrow$ maximal spectral multiplicities $\approx \frac{N}{P_N} \asymp \frac{N}{\log N}$ (cf. Bérard's bound $\mathcal{O}(\frac{\lambda}{\log \lambda})$).

[FAURE-N-DEBIÈVRE'03] constructed eigenstates which do not equidistribute as $N \to \infty$: failure of QUE. Trick: project on $\mathcal{V}_{\ell,N}$ a coherent state centered on a periodic point (x_0, ξ_0) of S: $\psi_{\ell,N} := \Pi_{\ell,N}\, \eta_{(x_0,\xi_0)}$.





Husimi functions for two eigenmodes of $U_N(S_0)$ for $N \in \mathcal{S}_{short}$.
They are partially concentrated on a periodic orbit of S_0.

(The modes differ by the choice of $\eta_{(x_0,\xi_0)}$).

Short quantum periods: random eigenstates of $U_N(S)$

The Cat Map model allows us to conjugate classical chaos with large spectral multiplicities $\rightsquigarrow$ random eigenstates are "very random".

- Take $N \in \mathcal{S}_{short}$, choose on each eigenspace $\mathcal{V}_{\ell,N}$ a random o.n.b. $\mathcal{O}_{\ell,N}$.

$\rightsquigarrow \mathcal{O} = (\bigoplus_{\ell=1}^{P_N} \mathcal{O}_{\ell,N})_{N \in \mathcal{S}_{short}}$ sequence of random eigenbases.

Theorem (N-Schwartz'22)

Almost surely, the sequence of eigenbases $\mathcal{O}$ satisfies QUE, down to the scale $N^{-1/4+\varepsilon}$.

- For $N \in \mathcal{S}_{short}$, choose a random eigenstate $\Phi_{\ell,N} \in \mathcal{V}_{\ell,N}$, normalized by $\|\Phi_{\ell,N}\| = \sqrt{N}$.

Theorem (N-Schwartz'22)

Fix $r \in \mathbb{N}^*$. For each $N \in \mathcal{S}_{short}$, choose a r-uplet $(i_1, \ldots, i_r) \subset [1, N]$. Then, when $N \to \infty$, the random r-uplets $(\Phi_{\ell,N}(i_1), \ldots, \Phi_{\ell,N}(i_r))$ converge in law to the r-uplet $(\Phi^{(N)}_{Gauss}(i_1), \ldots, \Phi^{(N)}_{Gauss}(i_r)) \sim \mathcal{N}_{\mathbb{C}}(0,1)^r$.

As a consequence, the value distribution of the random eigenvectors $\Phi_{\ell,N}$ a.s. converge to the Gaussian distribution.

$\oplus$ First example of chaotic eigenstates verifying universal local statistics.

Elements of proof: almost sure QUE

Following the proofs of [ZELDITCH,BURQ-LEBEAU], the main step is to prove a generalized Weyl's law on the eigenspaces $\mathcal{V}_{\ell,N}$:

$$\forall f \in C^\infty(\mathbb{T}^2), \quad \frac{1}{\dim \mathcal{V}_{\ell,N}} \operatorname{Tr}\left(\Pi_{\ell,N} \operatorname{Op}_N(f)\right) = \int_{\mathbb{T}^2} f(x,\xi)\, dx d\xi + o_f(1)_{N\to\infty}.$$

- We use the explicit representation of the projector:

$$\Pi_{\ell,N} = \frac{1}{P_N} \sum_{k=-P_N/2}^{P_N/2-1} e^{-ik\alpha_{\ell,N}} (U_N)^k, \quad \text{and the Fourier expansion} \quad f = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{m}}\, e_{\boldsymbol{m}}.$$

- Exact (dressed) Gutzwiller's trace formula (generalizes [KEATING'91]):

$$\operatorname{Tr}\left((U_N)^k \operatorname{Op}_N(e_{\boldsymbol{m}})\right) = \sum_{\rho\in\mathrm{Fix}(S^k)} \frac{1}{\sqrt{|\det(S^k - I)|}} e^{i\phi(\rho;k;\boldsymbol{m};N)}$$

$\#\mathrm{Fix}(S^k) \sim C e^{\Lambda|k|}$, while $\sqrt{|\det(S^k - I)|} \sim c' e^{\Lambda|k|/2}$, so we get the bound

$$\operatorname{Tr}\left((U_N)^k \operatorname{Op}_N(e_{\boldsymbol{m}})\right) = \mathcal{O}(e^{\Lambda|k|/2}) \quad \text{for} \quad k \neq 0$$

(notice the bound is indep. of N)

Elements of proof: almost sure QUE (2)

$$\mathrm{Tr}\left((U_N)^k \,\mathrm{Op}_N(e_{\boldsymbol{m}})\right) = \mathcal{O}(e^{\Lambda|k|/2}) \quad \text{for} \quad 0 < |k| \leq P_N/2$$

$$\mathrm{Tr}\left((U_N)^0 \,\mathrm{Op}_N(e_{\boldsymbol{m}})\right) = \begin{cases} N, & \boldsymbol{m} \equiv 0 \bmod N, \\ 0, & \text{otherwise} \end{cases}$$

Maximal time in the sum: $P_N/2 = \frac{\log N}{\Lambda} + \mathcal{O}(1)$

$$\Longrightarrow d_{\ell,N} := \mathrm{Tr}\,\Pi_{\ell,N} = \frac{N}{P_N} + \mathcal{O}(N^{1/2}), \qquad \mathrm{Tr}(\Pi_{\ell,N} e_{\boldsymbol{m}}) = \mathcal{O}(N^{1/2}), \ \forall \boldsymbol{m} \not\equiv 0 \bmod N$$

Summing over Fourier modes, we obtain the generalized Weyl's law:

$$\frac{1}{d_{\ell,N}} \mathrm{Tr}\left(\Pi_{\ell,N} \,\mathrm{Op}_N(f)\right) = \int_{\mathbb{T}^2} f(x,\xi)\, dx d\xi + \mathcal{O}_f(N^{-1/2+\varepsilon}).$$

- Use large deviation estimates for indiv. $\Phi_{\ell,N}$ [Rudelson-Vershynin'13]:

$$\forall t > 0, \quad \mathbb{P}\left(\left| \langle \Phi_{\ell,N}, \mathrm{Op}_N(f) \Phi_{\ell,N} \rangle - \frac{1}{d_{\ell,N}} \mathrm{Tr}\left(\Pi_{\ell,N} \,\mathrm{Op}_N(f)\right) \right| > t \right) \leq \exp(-C d_{\ell,N}\, t^2)$$

$t = N^{-\alpha}$ controls the deviations from $\int_{\mathbb{T}^2} f(x,\xi)\, dx d\xi$ with high proba. □

Elements of proof: local statistics of $\Phi_{j,N}$

Recall that $\Phi_{\ell,N}$ is a (Haar)-random eigenstate in $\mathcal{V}_{\ell,N}$, of norm $\|\Phi_{\ell,N}\| = \sqrt{N}$.

- Observation: since the dimension $d_{\ell,N} >> 1$, $\Phi_{\ell,N}$ statistically resembles the **Gaussian** random eigenstate

$$\Psi_{\ell,N} = \sqrt{P_N}\,\Pi_{\ell,N}\,\Phi^{(N)}_{Gauss}\,,$$

which has covariance matrix $\Sigma_{\ell,N} = P_N\,\Pi_{\ell,N}$ and typical norm $\sqrt{N} + \mathcal{O}(1)$.

- Computation of $\Sigma_{\ell,N} = I_N + \sum_{1\leq|k|\leq P_N/2} e^{-ik\alpha_{\ell,N}}\,(U_N)^k$.

We use explicit expressions for the matrices $(U_N)^k = U_N(S^k)$ [HANNAY-BERRY'80], and the explicit construction of N along $\mathcal{S}_{short}$ [BONECHI-DE BIÈVRE'03]

$$\leadsto (U_N^k)_{ij} = \mathcal{O}(N^{-1/4}), \quad \text{uniformly for } 1 \leq |k| \leq P_N/2 - 1.$$

$$\Longrightarrow \quad \Sigma_{\ell,N} = I_N + \mathcal{O}(N^{-1/4}) \quad \text{componentwise.}$$

- Notice that I_N is the covariance matrix of $\Phi^{(N)}_{Gauss}$

$\Longrightarrow \Psi_{\ell,N}$, and then $\Phi_{\ell,N}$, have the same local statisics as $\Phi^{(N)}_{Gauss}$

□

A few remarks and questions

- By randomizing over an eigenspace of dimension $N/\log N$, it is not so surprising to obtain a universal (Gaussian) statistical behaviour, and QUE. On the contrary, Hecke eigenstates are QUE, but are not universal.

- We used some very particular properties of the Quantum Cat Map: periodiciy, maximally large spectral multiplicities.

- we had to control the propagator U_N^k up to the Ehrenfest time $\frac{\log N}{\Lambda}$: can we generalize some of the estimates to more generic quantized symplectic maps (e.g. nonlinear perturbations of the cat map)?

– [FAURE'07]: approximate Gutzwiller's trace formula for times $k \leq C \log N$.

– could we control the matrix elements of U_N^k for nonlinear maps, up to the Ehrenfest time?

- Identify other symplectic manifolds carrying chaotic symplectomorphisms, leading to new examples of quantum maps ?

EXTRA: Controlling the entries of $U_N(S)^k$ along $\mathcal{S}_{short}$

Ex: $S_0 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ gives $U_N(S_0)_{jl} = \frac{1}{\sqrt{N}} \exp(\frac{2i\pi}{N}(j^2 + l^2 - jl)) = \frac{e^{i\phi}}{\sqrt{N}}$.

Writing $S_0^k = \begin{pmatrix} \bullet & b^{(k)} \\ \bullet & \bullet \end{pmatrix}$, the nonzero entries of $U_N(S_0)^k = U_N(S_0^k)$ will be of modulus $\sqrt{\frac{\gcd(N, b^{(k)})}{N}}$.

- To estimate the gcd: use the recurrence

$$S_0^k = n_k S_0 - n_{k-1}, \quad \text{where } n_{k+1} = \mathrm{Tr}(S_0) n_k - n_{k-1}, \quad n_1 = 1,\ n_0 = 0\,,$$

to compute the entry $b^{(k)} = n_k$. Note $n_k \sim c e^{\Lambda|k|}$ when $|k| \nearrow$.

- $N \in \mathcal{S}_{short}$ is of the form $N = N_{2K+1} = n_{K+1} + n_K$, corresponding to the minimal period $P_N = 2K + 1$. So we must control $\gcd(N_{2K+1}, n_k)$ for times $1 \leq |k| \leq K$.

- Due to arithmetic properties of the n_k (and the fact that we restrict ouselves to odd periods), the largest $\gcd(N_{2K+1}, n_k)$ occurs for $|k| \approx K/2$, where it is bounded by $\mathcal{O}(e^{\Lambda K/2}) = \mathcal{O}(N^{1/2})$

$$\Longrightarrow (U_N^k)_{jl} = \mathcal{O}(N^{-1/4}), \quad \text{uniformly for all } 1 \leq |k| \leq P_N/2 - 1.$$