Bochner Laplacians and Bergman kernels for families

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Quantization in Symplectic Geometry

Bergman kernel

Let $\left(Y^n, h^{TY}\right)$ be a compact, complex Hermitian manifold.
Let $\left(E, h^E\right)$ be a holomorphic, Hermitian vector bundle.

Bergman (L^2-) projection: $\Pi_E : C^\infty(Y;E) \to \underbrace{H^0(Y;E)}_{=\text{hol. sections}}$

Bergman (Schwartz) kernel: $\Pi_E(y,y') \in C^\infty\left(Y \times Y; \pi_1^* E \otimes \pi_2^* E^*\right)$

$$\Pi_E(y,y') = \sum_{j=1}^{\dim H^0(Y;E)} \underbrace{s_j(y)}_{=\text{orth. basis}} \otimes s_j(y')^*$$

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General problem: Understand the behavior (asymptotics) of the Bergman kernel.

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Various applications to complex analysis, Kahler geometry, canonical metrics, geometric quantization ..

Bergman kernel

Classical case of asymptotics is for tensor powers.

Namely let $E = L^k$, where L is a line bundle.
And assume the Chern curvature R^L is positive,

An application/corollary of the Boutet de Monvel-Sjöstrand '75 parametrix is

$$\Pi_{L^k}(y, y) \sim k^n \left[a_0(y) + k^{-1} a_1(y) + \ldots\right], \quad \text{as } k \to \infty.$$

with $a_0 = \det\left(\frac{\dot{R}^L}{2\pi}\right)$.

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Another technique based on the use analytic localization technique of Bismut-Lebeau '91 developed in Dai-Liu-Ma & Ma-Marinescu

Spectral gap & analytic localization

Main step in analytic localization is the *spectral gap* property (cf. Kodaira, Bismut-Vasserot):

For L positive line bundle and $\Box_k : \Omega^{0,0}\left(L^k\right) \to \Omega^{0,0}\left(L^k\right)$ the Kodaira Laplacian

$\exists c > 0$ such that

$$\mathrm{Spec}\left(\Box_k\right) \subset \{0\} \cup [ck - c, \infty)$$

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Similar result on symplectic manifolds: Guillemin-Uribe '88, Ma-Marinescu '01.

Toeplitz operators

Application to geometric quantization.

Generalization of Bergman projector:

Toeplitz operator: $T_{f,E} : H^0(Y;E) \rightarrow \underbrace{H^0(Y;E)}_{=\text{hol. sections}}$,

$T_{f,E} := \Pi_E \circ f \circ \Pi_E$ *quantizes* the function $f \in C^\infty(Y)$,

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Generalized Toeplitz operator: $T_k : H^0(Y;L^k) \to H^0(Y;L^k)$,

$$T_k \sim \sum_{j=0}^{\infty} k^{-j} T_{f_j, L^k},$$

where $f_j \in C^\infty(Y), j = 0, 1 \ldots$

Toeplitz operators

Toeplitz operators provide a *geometric quantization* scheme (cf. Boutet de Monvel-Guillemin '81, Bordemann-Meinrenken-Schlichenmaier '94).

For any $f, g \in C^{\infty}(Y)$

$$\lim_{k \to \infty} \left\| T_{f,L^k} \right\| = \|f\|_{\infty}$$

$$\left[T_{f,L^k}, T_{g,L^k} \right] = \frac{i}{k} T_{\{f,g\},L^k} + O\left(k^{-2}\right)$$

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The set $\mathcal{A}_Y = \left\{ T_k \in \mathrm{End}\, H^0\left(Y; L^k\right) \text{ generalized Toeplitz operator} \right\}$ is an algebra (quantizing the algebra of smooth functions).

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The Schwartz kernel of a Toeplitz operator has a full on-diagonal expansion:

$$T_{f,L^k}(y,y) \sim k^n \left[a_0^f(y) + k^{-1} a_1^f(y) + \ldots \right].$$

Motivating question

(Semi)-Classical Bergman kernel asymptotics is for positive, line bundles

Motivating question:

Can this be generalized to
1. Semi-positive bundles (i.e. $R^E(w, \bar{w}) \geq 0$)
2. Bundles of higher rank (eg. $\mathrm{Sym}^k E$, $\mathrm{rk} E > 1$)

Semi-positive line bundles

Consider (Y^2, h^{TY}) be a Riemann surface
(L, h^L) Hermitian hol. with R^L semipositive.

Define:

$$r : Y \to \mathbb{R} \cup \{\infty\}$$
$$r_y - 2 = \underbrace{\operatorname{ord}_y \left(R^L\right)}_{=\text{order of vanishing of curvature}} .$$

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Theorem (Marinescu-S. '18)

Let (Y^2, h^{TY}) Riemann surface & R^L semi-positive of finite order vanishing order

$$\Pi_k(y,y) \sim k^{2/r_y} \left[\sum_{j=0}^{N} c_j(y)\, k^{-j/r_y} \right]$$

where $r_y - 2 = ord\left(R^L_y\right) < \infty$.

Unresolved: higher dimensional semipositive case.

Bochner Laplacian

Proof uses analytic localization technique.
Crucial again is

$$\text{Spectral gap:} \qquad \mathrm{Spec}\left(\Box_k\right) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right).$$

$$\text{Lichnerowicz:} \qquad \underbrace{2\Box_k}_{\text{Kodaira}} = \underbrace{\Delta_k}_{\text{Bochner}} + k\left[R^L\left(w, \overline{w}\right)\right], \quad \text{on } \Omega^{0,1}.$$

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Theorem (Marinescu-S. '18)

Let $\left(Y, g^{TY}\right)$ *Riemannian manifold &* $\left(L, h^L, \nabla^L\right)$ *complex Hermitian line bundle with unitary connection.*
Then

$$\underbrace{\lambda_0^k}_{smallest\ eigenvalue\ of\ \Delta_k} \sim Ck^{2/r}$$

where $r = \max_{y \in Y} r_y$.

Proved using subelliptic estimates on the unit circle.

Higher rank

General problem for higher rank (cf. Guillemin-Uribe '88).

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Let $P \to Y$ principal G-bundle with connection $A \in \Omega^1(P;\mathfrak{g})$
Let $T \subset G$, maximal torus and $\nu \in \mathfrak{t}^*$ dominant integral weight.

Associated highest weight family of hol. Hermitian bundles $\left(V^{k\nu}, h^{k\nu}, \nabla^{k\nu}\right)$, $k \in \mathbb{N}$.
(eg. $G = U(n), \nu = (1, 0, \ldots 0) \rightsquigarrow V^{k\nu} = \mathrm{Sym}^k V^{\nu}$)

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Problem:
Describe asymptotics of its Bergman kernel and spectrum Bochner/Kodaira Laplacians.

Families setting

However, *Borel-Weil-Bott:* $V^{k\nu} = H^0\left(\underbrace{G/T}_{\text{flag manifold}}\;;\;\underbrace{L^k_\nu}_{\text{homogeneous line bundle}}\right)$

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Hence more generally consider the setting of families:

Let $\left(W, h^{TW}\right) \xrightarrow{\pi} \left(Y, h^{TY}\right)$ holomorphic submersion
Let $\left(L, h^L\right) \to W$ Hermitian holomorphic

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Hence more generally consider the setting of families:

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Let $(L, h^L) \to W$ Hermitian holomorphic

(A0) Suppose R^L is fiberwise positive

$\rightsquigarrow$ Direct image bundle $\mathcal{E}_{k,y} := (R^0 \pi_* L^k)_y = H^0 \left(W_y, L^k|_{W_y} \right)$

$h^{\mathcal{E}_k} = L^2$ metric,

$\nabla^{\mathcal{E}_k} =$ Chern connection.

Families setting

Define

$$r : W \to \mathbb{R} \cup \{\infty\}$$

$$r_w - 2 = \underbrace{\mathrm{ord}_w^H \left(R^{L,H}\right)}_{=\text{horizontal order of vanishing}}$$

where horizontal curvature: $R^{L,H} = R^L\big|_{T^H W}$,

$$\mathrm{Spec}\left(\dot{R}^{L,H}\right) = \{a_1(w), \ldots, a_m(w)\}$$

(A1) Suppose r is fiberwise constant and finite

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(A1) Suppose r is fiberwise constant and finite

(A2) Suppose R^L is horizontally semipositive with *comparable eigenvalues* (i.e. $\exists c > 0$ such that

$$c^{-1} a_j(w) \leq a_k(w) \leq c a_j(w),$$

$\forall j, k \in \{1, \ldots m\}$ and $w \in W$)

Main result 1

Bergman kernel expansion for families

Theorem (Ma-Marinescu-S.)

Let $(W, h^{TW}) \xrightarrow{\pi} (Y, h^{TY})$ be holomorphic submersion of compact, complex Hermitian manifolds.
Let $(L, h^L) \rightarrow W$ Hermitian, holomorphic line bundle.

Suppose

(A0) R^L is fiberwise positive

(A1) R^L has a fiberwise constant and finite horizontal order of vanishing

(A2)R^L is horizontally semi-positive with comparable eigenvalues
Then Bergman kernel of the direct image

$$\Pi_{\mathcal{E}_k}(y, y) \sim k^{2n/r_y} \left[\sum_{j=0}^{\infty} k^{-2j/r_y} T_{g_j} \right]$$
$$\in End\ H^0 \left(W_y;\, L^k \Big|_{W_y} \right)$$

is a generalized Toeplitz operator on each fiber W_y, $y \in Y$.

Main result 2

Bochner Laplacian for families.

Theorem (Ma-Marinescu-S.)

Let $(W, h^{TW}) \xrightarrow{\pi} (Y, h^{TY})$ *be holomorphic submersion of compact complex Hermitian manifolds.*
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(A1) R^L *has a fiberwise constant and finite horizontal order of vanishing*
Then the Bochner Laplacian on the direct image satisfies

$$\underbrace{\lambda_0(k)}_{smallest\ eigenvalue\ of\ \Delta_{\mathcal{E}_k}} \sim Ck^{2/r}$$

where $r = \max_{y \in Y} r_y$.

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Special cases:

1. fibers $= pt \rightsquigarrow$ Semipositive line bundles with comparable eigenvalues
2. fibers $= G/T \rightsquigarrow$ Highest weight families $\left(V^{k\nu}, h^{k\nu}, \nabla^{k\nu}\right), k \in \mathbb{N}$.

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(Eg. 2) Let $P \to Y$ principal G-bundle with connection $A \in \Omega^1\left(P; \mathfrak{g}\right)$

Let $T \subset G$, maximal torus and $\nu \in \mathfrak{t}^*$ dominant integral weight.

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Corollary (Ma-Marinescu-S.)

The first eigenfunction/eigenvalue $\lambda_0^{k\nu}$) of the Bochner Laplacian $\Delta_{k\nu}$ satisfies

$$\lambda_0^{k\nu} \sim Ck^{2/r_\nu}$$

as $k \to \infty$.

Here; $r_{\nu,y} - 2 = ord\left(\nu.R_y^P\right)$, $\Omega^2(Y; \mathfrak{g}) \ni R_y^P$ *=principal bundle curvature*

Proof sketch

Proof uses earlier result.

Theorem (Ma-Zhang '22)

Let $(W, h^{TW}) \xrightarrow{\pi} (Y, h^{TY})$ be holomorphic submersion of compact complex Hermitian manifolds.
Let $(L, h^L) \to W$ Hermitian, holomorphic line bundle.
Suppose (A0) R^L is fiberwise positive
Then the curvature of the direct image

$$\frac{1}{k} R_y^{\mathcal{E}_k} \sim \sum_{j=0}^{\infty} k^{-j} T_{g_j},$$

$$\in End\ H^0 \left(W_y; L^k \Big|_{W_y} \right)$$

is a generalized Toeplitz operator on each fiber.
First coefficient:

$$g_0 = R^{L,H} \quad \text{(horizontal curvature).}$$

Proof sketch (of MMS)

1. Computation of first coefficient gives:

$R^{L,H}$ semipositive with comparable eigenvalues
$\implies$ same is true for $\frac{1}{k}R^{\mathcal{E}_k} = T_{R^{L,H}} + O\left(k^{-1}\right)$.

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3. In geodesic coordinates centered at $y \in Y$ and a parallel frame for direct image bundle we can write

$$\nabla^{\mathcal{E}_k} = d + \underbrace{\left[\int_0^1 d\rho\, \rho y^q R_{pq}^{\mathcal{E}_k}\left(\rho y\right)\right]}_{a_p^{\mathcal{E}_k}} dy_p$$

The Bochner, Kodaira Laplacians $\Delta_{\mathcal{E}_k}, \Box_{\mathcal{E}_k}$ are expressed in terms of $\nabla^{\mathcal{E}_k}$ and $R_y^{\mathcal{E}_k}$. Hence both are differential operators with coefficients valued in the algebra of Toeplitz operators of the fibers $\mathrm{Diff}\,(Y) \otimes \mathcal{A}_{W_y}$.

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Apply usual rescaling $y \mapsto k^{-\frac{1}{2}} y$ and local index theory for operators in $\mathrm{Diff}\left(Y\right) \otimes \mathcal{A}_{W_y}$.

Thank you.