Topological properties of (tall) monotone complexity one spaces

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Based on:

- "*On topological properties of positive complexity one spaces*", S. and Sepe, *Transformation Groups* **9** (2020).
- "*Tall and monotone complexity one spaces of dimension six*", Charton, PhD Thesis, Cologne 2021.
- "*Compact monotone tall complexity one T-spaces*" Charton, S. and Sepe, arXiv:2307.04198 [math.SG].

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Henceforth consider *positive monotone symplectic manifolds*

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(Example: $\dim_{\mathbb{C}}(Y) = 1 \implies Td(Y) = \frac{c_1}{2}[Y]$,

$\dim_{\mathbb{C}}(Y) = 2 \implies Td(Y) = \frac{c_1^2 + c_2}{12}[Y]$,

$\dim_{\mathbb{C}}(Y) = 3 \implies Td(Y) = \frac{c_1 c_2}{24}[Y]$)

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What if one assumes that (M, ω) has symmetries?

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Assume $T \circlearrowleft (M, \omega)$ is *Hamiltonian*:

$\exists \ \psi: (M, \omega) \to Lie(T)^*$ (*moment map*) s.t.

- ψ is T-invariant
- $\forall \xi \in Lie(T)$

$$d\langle \psi, \xi \rangle = -\iota_{X_\xi} \omega$$

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$$\psi_2 \circ \Psi = a \circ \psi_1$$

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- $\exists$ (equivariant) symplectomorphism?
- Finitely many examples in each dimension? (Modulo equivalence)

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Remark: GKM action $\implies$ the torus acting is T^2

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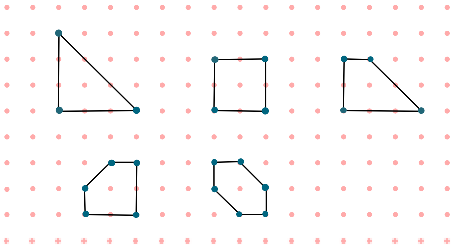

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To prove (b): prove that $\psi^{-1}(v)$ is a sphere.

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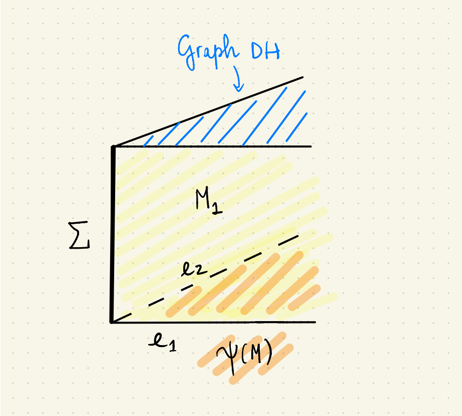
- $N_\Sigma = N_1 \oplus \cdots \oplus N_{n-1}$
- $M_i := \psi^{-1}(e_i)$: compact symplectic 4-dimensional submanifold with a Hamiltonian S^1 action, $\Sigma \subset M_i$, for all $i = 1, \ldots, n-1$
- Normal bundle to Σ in M_i is N_i



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$$DH(x) = \int_\Sigma \omega - c_1(N_i)[\Sigma](x - v_{\min})$$

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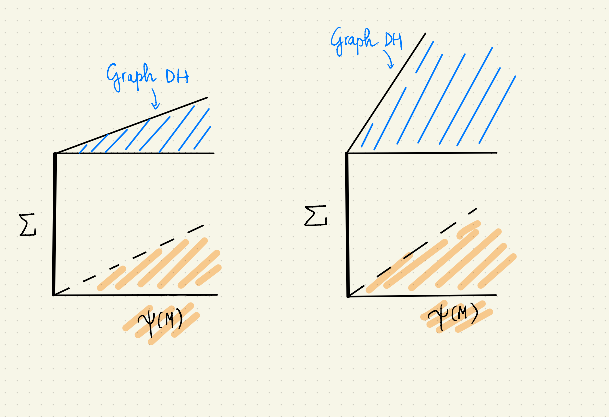

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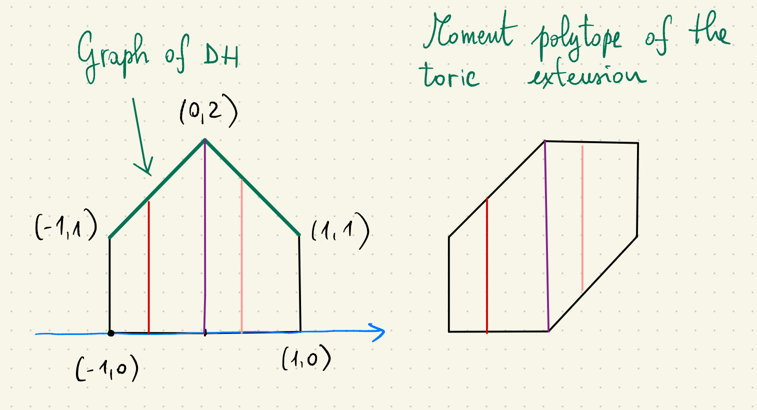
- Prove that each of those "comes" from a toric one

Example:


Graph of DH
(0,2)
(-1,1)
(1,1)
(-1,0)
(1,0)
Moment polytope of the toric extension

THANK YOU!