Moment polytopes in real symplectic geometry

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Plan

The talk is devoted to the description of convex polyhedral cones which are associated to some representations of compact Lie groups.

- § Eigenvalues and singular values
- § Description of several geometric cones:
  - The Horn cone
  - A cone of eigenvalues
  - The singular Horn cone
- § Convexity in Hamitonian geometry
- § Convexity in *real* Hamitonian geometry: O'Shea-Sjamaar's Theorem
- § General results for the cones associated to isotropy representations of Riemannian symmetric spaces
- § Key point in the proof : Kirwan-Ness stratification and Ressayre's pairs

Eigenvalues and singular values

Let $\mathrm{e}(X) = (\mathrm{e}_1 \geq \cdots \geq \mathrm{e}_n) \in \mathbb{R}^n_+$ be the **eigenvalues** of a Hermitian (or real symmetric) $n \times n$ matrix.

Fact : two isomorphisms $\boxed{Herm(n)/U(n) \xrightarrow{\mathrm{e}} \mathbb{R}^n_+}$ and $\boxed{Sym(n)/SO(n) \xrightarrow{\mathrm{e}} \mathbb{R}^n_+}$

Let $\mathrm{s}(X) = (\mathrm{s}_1 \geq \cdots \geq \mathrm{s}_q \geq 0) \in \mathbb{R}^q_{++}$ be the **singular values** of a complex $p \times q$ matrix.

Fact : an isomorphism $\boxed{M_{p,q}(\mathbb{C})/U(p) \times U(q) \xrightarrow{\mathrm{s}} \mathbb{R}^q_{++}}$

Basic questions: what are the relations between

1. $\mathrm{e}(X)$, $\mathrm{e}(Y)$ and $\mathrm{e}(X+Y)$ for $X, Y \in Herm(n)$.
2. $\mathrm{s}(X)$, $\mathrm{s}(Y)$ and $\mathrm{s}(X+Y)$ for $X, Y \in M_{p,q}(\mathbb{C})$.
3. $\mathrm{e}(X)$ and $\mathrm{e}(\mathfrak{R}(X))$ where $\mathfrak{R}(X) \in Sym(n)$ is the real part of $X \in Herm(n)$.
4. $\mathrm{e}(X)$ and $\mathrm{s}(X_{12})$ where X_{12} is the off-diagonal bloc of $X \in Herm(n)$.
5. $\mathrm{s}(X)$, $\mathrm{s}(X_{12})$ and $\mathrm{s}(X_{21})$ for $X \in M_{n,n}(\mathbb{C})$.
6. $\mathrm{s}(X)$, $\mathrm{s}(X_{11})$ and $\mathrm{s}(X_{22})$ for $X \in M_{n,n}(\mathbb{C})$.
7. $\cdots$

The aim of this presentation is to explain the methods used to answer these kind of questions. **Keywords :** Hamiltonian action, moment map, anti-symplectic involution.

Classical geometric cone: the Horn cone

The Horn cone

$$\mathrm{Horn}(n) := \Big\{(\mathrm{e}(X), \mathrm{e}(Y), \mathrm{e}(X+Y));\ X, Y \in Herm(n)\Big\}$$

Some notations:

- $\mathbb{R}^n_+ = \{x = (x_1 \geq \cdots \geq x_n)\}$.
- $I = \{i_1 < \cdots < i_r\} \subset \mathbb{N} - \{0\} \ \rightsquigarrow\ \mu(I) = (i_r - r \geq \cdots \geq i_1 - 1 \geq 0) \in \mathbb{R}^r_+$.
- If $x \in \mathbb{R}^n$ and $I \subset \{1, \ldots, n\}$, we write $|x| = \sum_{k=1}^n x_k$ and $|x|_I = \sum_{i \in I} x_i$.

Schubert Calculus : cohomology of the Grassmannian $\mathbb{G}(r, n)$

- $\mathbb{G}(r, n) := \{E \subset \mathbb{C}^n,\ \dim E = r\}$
- $H^*(\mathbb{G}(r, n)) = \bigoplus_{I \subset [n], \sharp I = r} \mathbb{Z}\Theta_I$
- $H^{\max}(\mathbb{G}(r, n)) = \mathbb{Z}\Theta_{[r]}$
- $\Theta_I \cdot \Theta_{I^o} = \Theta_{[r]}$ when $I^o = \{n + 1 - i, i \in I\}$
- $\Theta_{I^o} \cdot \Theta_{J^o} \cdot \Theta_L = \ell\Theta_{[r]}, \ell \neq 0 \Longleftrightarrow \left(V_{\mu(I)} \otimes V_{\mu(J)} \otimes V^*_{\mu(L)}\right)^{U(n)} \neq 0$

Classical geometric cone: the Horn cone

The study of the cone $\mathrm{Horn}(n)$ started long ago: Weyl (1932), Ky Fan (1949), Lidskii (1950), Thompson-Freede (1971).

Horn conjecture (1962)

An element $(x, y, z) \in (\mathbb{R}^n_+)^3$ belongs to $\mathrm{Horn}(n)$ if and only if

- $|x| + |y| = |z|$ (trace condition)
- $|x|_I + |y|_J \geq |z|_L$ for any subsets $I, J, L \subset \{1, \ldots, n\}$ of cardinal $r < n$ satisfying :

$$\mathit{Condition}_{(I,J,L)} : \qquad (\mu(I), \mu(J), \mu(L)) \in \mathrm{Horn}(r)$$

Proof of the Horn conjecture

- Klyachko (1998): Horn conjecture holds with $\mathit{Condition}_{(I,J,L)}$ replaced by

$$\mathit{Condition}'_{(I,J,L)} : \qquad \Theta_{I^o} \cdot \Theta_{J^o} \cdot \Theta_L = \ell\Theta_{[r]}, \ell \neq 0, \quad \mathit{in} \quad H^*(\mathbb{G}(r, n))$$

- Saturation Theorem of Knutson-Tao (1999):

$$\mathit{Condition}'_{(I,J,L)} \iff \mathit{Condition}_{(I,J,L)}$$

Final improvements by Belkale (2001) and Knutson-Tao-Woodward (2004) : equations for $\ell = 1$ are sufficient and not redundant.

A cone of eigenvalues

Consider the map $\Re : Herm(n) \to Sym(n)$ which associates to a Hermitian matrix its real part. We are interested in the following cone:

$$\mathcal{E}(n) := \left\{ (\mathrm{e}(X), \mathrm{e}(\Re(X)))\,;\; X \in Herm(n) \right\}$$

First description: an application of the O'Shea-Sjamaar theorem

An element $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ belongs to $\mathcal{E}(n)$ if and only if $(x, x, 2y) \in \mathrm{Horn}(n)$.

A refinement:

Theorem (PEP, 2022)

An element $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ belongs to $\mathcal{E}(n)$ if and only if

$$|x| = |y| \qquad \text{and} \qquad |x|_I \geq |y|_J$$

holds for any subsets $I, J \subset [n]$ of cardinal $r < n$ such that $(2\mu(I), \mu(J)) \in \mathcal{E}(r)$.

Example

- $\mathcal{E}(1)$, $\mathcal{E}(2)$, $\mathcal{E}(3)$ and $\mathcal{E}(4)$ are defined by 1, 2, 7 and 16 inequalities.
- $\mathrm{Horn}(1)$, $\mathrm{Horn}(2)$, $\mathrm{Horn}(3)$ and $\mathrm{Horn}(4)$ are defined by 1, 7, 19 and 51 inequalities.

The singular Horn cone

If $p \geq q \geq 1$, the singular value map $\mathrm{s} : M_{p,q}(\mathbb{C}) \to \mathbb{R}^q_{++}$ is defined by $\mathrm{s}(A) = \sqrt{\mathrm{e}(A^*A)}$.

Singular Horn cone

$$\mathrm{Singular}(p, q) := \Big\{(\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(A+B)),\ A, B \in M_{p,q}(\mathbb{C})\Big\}$$

Map: $x \in \mathbb{R}^q \quad \longmapsto \quad \widehat{x} = (x_1, \ldots, x_q, 0, \ldots, 0, -x_q, \ldots, -x_1) \in \mathbb{R}^n$

First description: an application of the O'Shea-Sjamaar theorem

$(x, y, z) \in (\mathbb{R}^q_{++})^3$ belongs to $\mathrm{Singular}(p, q)$ if and only if $(\widehat{x}, \widehat{y}, \widehat{z}) \in \mathrm{Horn}(p+q)$.

Example

We will see that $\mathrm{Singular}(3, 3)$ is determined by **96 inequalities** whereas $\mathrm{Horn}(6)$ needs **536 inequalities.**

Some notations:

- Polarized sets $X_\bullet = X_+ \coprod X_-$ of $[q] := \{1, \ldots, q\}$
- Signature function: $\epsilon : X_\bullet \to \{\pm\}$
- Signed inequalities:

$$(\star)_{I_\bullet, J_\bullet, L_\bullet} \qquad \sum_{i \in I_\bullet} \epsilon_i \mathrm{s}_i(A) + \sum_{j \in J_\bullet} \epsilon_j \mathrm{s}_j(B) + \sum_{\ell \in L_\bullet} \epsilon_\ell \mathrm{s}_\ell(A+B) \leq 0$$

The singular Horn cone : inequalities

- To a polarized subset $X_\bullet \subset [q]$ we associate two subsets of cardinal $\sharp X_\bullet$:
  - $X_\bullet^p = X_+ \cup \{p+q+1-\ell, \ell \in X_-\} \subset [p+q]$,
  - $\widetilde{X}_\bullet^p \subset [p+q-r]$ (more complicated definition).
- Involution on $\mathbb{R}^q$: $\quad x = (x_1, \cdots, x_q) \mapsto x^* = (-x_q, \cdots, -x_1)$.

Theorem (PEP, 2021)

$\mathrm{Singular}(p,q)$ is determined by the inequalities $(\star)_{I_\bullet, J_\bullet, L_\bullet}$ where $I_\bullet, J_\bullet, L_\bullet$ satisfy the following conditions: $\sharp I_\bullet = \sharp J_\bullet = \sharp L_\bullet = r < q$ and

1. $(\mu(I_\bullet^p), \mu(J_\bullet^p), \mu(L_\bullet^p)^* + 2(p+q-r)1^r) \in \mathrm{Horn}(r)$,
2. $(\mu(\widetilde{I}_\bullet^p), \mu(\widetilde{J}_\bullet^p), \mu(\widetilde{L}_\bullet^p)^* + 2(p+q-2r)1^r) \in \mathrm{Horn}(r)$.

Why two conditions? In fact they are equivalent to the cohomological condition

$$\Theta_{I_\bullet^n} \odot \Theta_{J_\bullet^n} \odot \Theta_{L_\bullet^n} = \ell[pt], \; \ell \neq 0 \quad in \quad H^*(\mathbb{F}(r, n-r, n)),$$

where $\mathbb{F}(r, n-r, n)$ denotes the two-steps flag variety parameterizing nested sequences of linear subspaces $E \subset F \subset \mathbb{C}^n$ where $\dim E = r$ and $\dim F = n - r$.

Singular(3, 3)

$(a, b, c) \in (\mathbb{R}^3_{++})^3$ belongs to $\mathrm{Singular}(3,3)$ if and only if, modulo permutation, we have

1. 18 Weyl inequalities
   - $a_1 + b_1 \geq c_1$
   - $a_1 + b_3 \geq c_3$
   - $a_1 + b_2 \geq c_2$
   - $a_2 + b_2 \geq c_3$
2. 18 Lidskii inequalities
   - $a_1 + a_2 + b_1 + b_2 \geq c_1 + c_2$
   - $a_1 + a_2 + b_1 + b_3 \geq c_1 + c_3$
   - $a_1 + a_2 + b_2 + b_3 \geq c_2 + c_3$
   - $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 \geq c_1 + c_2 + c_3$
3. 36 signed Lidskii inequalities
   - $a_1 + a_2 + b_1 - b_2 \geq c_1 - c_2$
   - $a_1 + a_2 + b_1 - b_3 \geq c_1 - c_3$
   - $a_1 + a_2 + b_2 - b_3 \geq c_2 - c_3$
   - $a_1 + a_2 + a_3 + b_1 + b_2 - b_3 \geq c_1 + c_2 - c_3$
   - $a_1 + a_2 + a_3 + b_1 - b_2 + b_3 \geq c_1 - c_2 + c_3$
   - $a_1 + a_2 + a_3 - b_1 + b_2 + b_3 \geq -c_1 + c_2 + c_3$
4. 15 others inequalities
   - $a_1 + a_3 + b_1 + b_3 \geq c_2 + c_3$
   - $a_1 + a_3 + b_1 - b_3 \geq c_2 - c_3$
   - $(a_1 + a_2 - a_3) + (b_1 - b_2 + b_3) + (-c_1 + c_2 + c_3) \geq 0$

Convexity in Hamiltonian geometry

Kähler manifold (M, Ω) acted on by a compact Lie group U:

- Holomorphic action of $U_{\mathbb{C}} \circlearrowleft M$.
- The action $U \circlearrowleft (M, \Omega)$ is Hamiltonian, with **proper** moment map $\Phi_{\mathfrak{u}} : M \to \mathfrak{u}^*$.

Theorem (Kirwan, 1984)

$\Delta_{\mathfrak{u}}(M) = \Phi_{\mathfrak{u}}(M) \cap \mathfrak{t}^*_+$ is a closed convex locally polyhedral subset.

Basic question

Determine the inequalities defining the Kirwan polytope $\Delta_{\mathfrak{u}}(M)$.

Example

- Compact Lie groups $U \hookrightarrow \tilde{U}$ with Lie algebras $\mathfrak{u} \hookrightarrow \tilde{\mathfrak{u}}$ and projection $\pi : \tilde{\mathfrak{u}}^* \to \mathfrak{u}^*$.
- Kähler manifold: $\tilde{U}_{\mathbb{C}} \simeq T\tilde{U} \simeq T^*\tilde{U}$
- Hamiltonian action $\tilde{U} \times U \circlearrowleft \tilde{U}_{\mathbb{C}}$: $\boxed{(\tilde{g}, g) \cdot m = \tilde{g}\, m\, g^{-1}}$
- Moment map $\Phi : \tilde{U}_{\mathbb{C}} \to \tilde{\mathfrak{u}} \times \mathfrak{u}$: $\boxed{\tilde{g} e^{i\tilde{X}} \longmapsto (-\tilde{g}\tilde{X}, \pi(\tilde{X}))}$
- Kirwan polytope : $\boxed{\mathrm{Horn}(\tilde{U}, U) = \left\{(\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_+ \times \mathfrak{t}_+,\ U\xi \subset \pi(\tilde{U}\tilde{\xi})\right\}}$

Convexity in *real* Hamiltonian geometry

We suppose that (M, Ω, U, Φ) is equipped with **involutions**:

1. an involution σ on U
2. an anti-holomorphic involution τ on M such that $\tau^*(\Omega) = -\Omega$
3. compatibility conditions: $\boxed{\tau(g \cdot x) = \sigma(g) \cdot \tau(x)}$ and $\boxed{\Phi(\tau(x)) = -\sigma(\Phi(x))}$

Example ($U(n)$ with the involution $\sigma(g) = \overline{g}$)

- Any adjoint orbit $\mathcal{O}_\lambda = U(n) \cdot \mathrm{diag}(i\lambda_1, \ldots, i\lambda_n)$ is stable under $\tau(x) = -\overline{x}$.
- $(\mathcal{O}_\lambda)^\tau = i\mathcal{O}_\lambda^{\mathbb{R}}$ with $\mathcal{O}_\lambda^{\mathbb{R}} := \{X \in Sym(n), \mathrm{e}(X) = \lambda\}$.
- $\boxed{\mathcal{O}_\nu \subset \mathcal{O}_\lambda + \mathcal{O}_\mu \Longleftrightarrow \mathcal{O}_\nu^{\mathbb{R}} \subset \mathcal{O}_\lambda^{\mathbb{R}} + \mathcal{O}_\mu^{\mathbb{R}}}$

Map: $a \in \mathbb{R}^q \quad \longmapsto \quad \widehat{a} = (a_1, \ldots, a_q, 0, \ldots, 0, -a_q, \ldots, -a_1) \in \mathbb{R}^n$

Example ($U(n)$ with the involution $\sigma(g) = I_{p,q}\, g\, I_{p,q}$)

- $\mathcal{O}_\lambda$ is stable under $\tau(x) = -I_{p,q}\, x\, I_{p,q}$ if and only if $\exists a \in \mathbb{R}^q_{++}, \lambda = \widehat{a}$
- $(\mathcal{O}_{\widehat{a}})^\tau \simeq \mathcal{V}_a$ where $\mathcal{V}_a = \{X \in M_{p,q}(\mathbb{C}), \mathrm{s}(X) = a\}$
- $\boxed{\mathcal{O}_{\widehat{c}} \subset \mathcal{O}_{\widehat{a}} + \mathcal{O}_{\widehat{b}} \Longleftrightarrow \mathcal{V}_c \subset \mathcal{V}_a + \mathcal{V}_b}$

Real moment polytopes: O'Shea-Sjamaar Theorem

Involution on U

- $K := (U^\sigma)^0$ acts on $\mathfrak{p} = i\mathfrak{u}^{-\sigma}$
- σ-invariant maximal torus $T \subset U$ and $\mathfrak{t}_+ =$ Weyl chamber for U
- $\mathfrak{a} = i\mathfrak{t}^{-\sigma}$ of maximal dimension $\rightsquigarrow$ $\mathfrak{a}_+ = i(\mathfrak{t}^{-\sigma} \cap \mathfrak{t}_+) \simeq \mathfrak{p}/K$

Anti-holomorphic involution on (M, Ω)

- $Z := M^\tau$ is a Lagrangian submanifold (that we suppose non-empty).
- Real moment map $\Phi_\mathfrak{p} : Z \to \mathfrak{p}$.
- The set $\Delta_\mathfrak{p}(Z) := \Phi_\mathfrak{p}(Z) \cap \mathfrak{a}_+$ parameterizes the K-orbits in $\Phi_\mathfrak{p}(Z)$.

Theorem (O'S-S, 2000)

$$\Delta_\mathfrak{p}(Z) \simeq \Delta_\mathfrak{u}(M) \cap \mathfrak{t}^{-\sigma}$$

$\Delta_\mathfrak{p}(Z)$ is called the **real moment polytope**.

The example of isotropic representations of symmetric spaces

Let us consider an involution σ on $U \subset \widetilde{U}$.
The involution σ extends to an **antilinear** involution $\sigma_{\mathbb{C}}$ on $U_{\mathbb{C}} \subset \widetilde{U}_{\mathbb{C}}$.

- $G = (U_{\mathbb{C}}^{\sigma_{\mathbb{C}}})^0 \subset \widetilde{G} = (\widetilde{U}_{\mathbb{C}}^{\sigma_{\mathbb{C}}})^0$: real reductive Lie groups
- Maximal compact subgroups $K = (U^{\sigma})^0 \subset \widetilde{K} = (\widetilde{U}^{\sigma})^0$
- Cartan decompositions : $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

Hamiltonian action of $\widetilde{U} \times U$ on $\widetilde{U}_C$ $\rightsquigarrow$ Kirwan polytope $\mathrm{Horn}(\widetilde{U}, U)$.

- $\widetilde{G}$ = Lagrangian submanifold of $\widetilde{U}_{\mathbb{C}}$ is equipped with an action of $\widetilde{K} \times K$
- Restriction of the moment map $\Phi : \widetilde{U}_{\mathbb{C}} \to \tilde{\mathfrak{u}} \times \mathfrak{u}$ defines $\Phi_{\mathfrak{p}} : \widetilde{G} \to \tilde{\mathfrak{p}} \times \mathfrak{p}$
- Real moment polytope:

$$\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K) = \left\{(\tilde{\xi}, \xi) \in \tilde{\mathfrak{a}}_+ \times \mathfrak{a}_+ \mid K \cdot \xi \subset \pi(\widetilde{K} \cdot \tilde{\xi})\right\}$$

Corollary of O'Shea-Sjamaar Theorem

$$\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K) \simeq \mathrm{Horn}(\widetilde{U}, U) \bigcap \tilde{\mathfrak{t}}^{-\sigma} \times \mathfrak{t}^{-\sigma}$$

Isotropic representations of symmetric spaces: examples

Initial question : what are the relations between

1. $\mathrm{e}(X)$, $\mathrm{e}(Y)$ and $\mathrm{e}(X+Y)$ for $X, Y \in Herm(n)$.
2. $\mathrm{s}(X)$, $\mathrm{s}(Y)$ and $\mathrm{s}(X+Y)$ for $X, Y \in M_{p,q}(\mathbb{C})$.
3. $\mathrm{e}(X)$ and $\mathrm{e}(\Re(X))$ where $\Re(X) \in Sym(n)$ is the real part of $X \in Herm(n)$.
4. $\mathrm{e}(X)$ and $\mathrm{s}(X_{12})$ where X_{12} is the off-diagonal bloc of $X \in Herm(n)$.
5. $\mathrm{s}(X)$, $\mathrm{s}(X_{12})$ and $\mathrm{s}(X_{21})$ for $X \in M_{n,n}(\mathbb{C})$.
6. $\mathrm{s}(X)$, $\mathrm{s}(X_{11})$ and $\mathrm{s}(X_{22})$ for $X \in M_{n,n}(\mathbb{C})$.

Answer : compute the real moment polytope $\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$ in the following cases

1. $G = GL_n(\mathbb{C})$ and $\widetilde{G} = G \times G$ $\rightsquigarrow$ $\mathrm{Horn}(n)$
2. $G = U(p,q)$ and $\widetilde{G} = G \times G$ $\rightsquigarrow$ $\mathrm{Singular}(p,q)$
3. $G = GL_n(\mathbb{R})$ and $\widetilde{G} = GL_n(\mathbb{C})$ $\rightsquigarrow$ $\mathcal{E}(n)$
4. $G = U(p,q)$ and $\widetilde{G} = GL_n(\mathbb{C})$
5. $G = U(p,q) \times U(q,p)$ and $\widetilde{G} = U(n,n)$
6. $G = U(p,p) \times U(q,q)$ and $\widetilde{G} = U(n,n)$

Determination of the inequalities of $\mathrm{Horn}(\widetilde{U}, U)$

- Maximal torus $T \subset U$ and $\widetilde{T} \subset \widetilde{U}$, such that $T \subset \widetilde{T}$
- Weyl groups W, $\widetilde{W}$ and longest element $w_o \in W$
- $\mathfrak{R} := \mathfrak{R}(\tilde{\mathfrak{u}}/\mathfrak{u}) \subset \mathfrak{t}^*$ set of roots relatively to the action $T \circlearrowleft \tilde{\mathfrak{u}}/\mathfrak{u} \otimes \mathbb{C}$
- $\gamma \in \mathfrak{t}$ is $\mathfrak{R}$-admissible if γ is rational and $\mathrm{Vect}(\mathfrak{R} \cap \gamma^\perp) = \mathrm{Vect}(\mathfrak{R}) \cap \gamma^\perp$
- Schubert classes $\Theta^\gamma_w \in H^*(U/U^\gamma, \mathbb{Z})$ associated to $w \in W/W^\gamma$
- Schubert classes $\Theta^\gamma_{\tilde{w}} \in H^*(\widetilde{U}/\widetilde{U}^\gamma, \mathbb{Z})$ associated to $\tilde{w} \in \widetilde{W}/\widetilde{W}^\gamma$
- Morphism $\iota^* : H^*(\widetilde{U}/\widetilde{U}^\gamma, \mathbb{Z}) \to H^*(U/U^\gamma, \mathbb{Z})$ associated to $\iota : U/U^\gamma \hookrightarrow \widetilde{U}/\widetilde{U}^\gamma$

Theorem

$(\tilde{\xi}, \xi) \in \mathrm{Horn}(\widetilde{U}, U)$ if and only if the inequality $\boxed{(\tilde{\xi}, \tilde{w}\gamma) \geq (\xi, w_o w \gamma)}$ holds for any $(\gamma, w, \tilde{w}) \in \mathfrak{t} \times W/W^\gamma \times \widetilde{W}/\widetilde{W}^\gamma$ satisfying

- γ is antidominant and $\mathfrak{R}$-admissible,
- Cohomological condition: $\Theta^\gamma_w \cdot \iota^*\left(\Theta^\gamma_{\tilde{w}}\right) = [pt]$ in $H^*(U/U^\gamma, \mathbb{Z})$,
- Numerical condition: $N(\gamma, w, \tilde{w}) = 0$.

Different versions of the theorem due to: Berenstein-Sjamaar (2000), Kapovich-Leeb-Millson (2005), Belkale-Kumar (2006), Ressayre (2010).

Determination of the inequalities of $\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$

- Maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$ and $\tilde{\mathfrak{a}} \subset \tilde{\mathfrak{p}}$, such that $\mathfrak{a} \subset \tilde{\mathfrak{a}}$.
- Restricted Weyl group : $W_{\mathfrak{a}} = N_W(\mathfrak{a})/Z_W(\mathfrak{a})$ and $W_{\tilde{\mathfrak{a}}} = N_{\widetilde{W}}(\tilde{\mathfrak{a}})/Z_{\widetilde{W}}(\tilde{\mathfrak{a}})$.
- Restricted root system $\Sigma \subset \mathfrak{a}^*$: set of roots relatively to the action $\mathfrak{a} \circlearrowleft \tilde{\mathfrak{p}}/\mathfrak{p}$
- $\gamma \in \mathfrak{a}$ is Σ-admissible if γ is rational and $\mathrm{Vect}(\Sigma \cap \gamma^\perp) = \mathrm{Vect}(\Sigma) \cap \gamma^\perp$
- Schubert classes Θ^γ_w parameterized by $(W/W^\gamma)^\sigma \simeq W_{\mathfrak{a}}/W^\gamma_{\mathfrak{a}}$
- Schubert classes $\Theta^\gamma_{\tilde{w}}$ parameterized by $(\widetilde{W}/\widetilde{W}^\gamma)^\sigma \simeq \widetilde{W}_{\tilde{\mathfrak{a}}}/\widetilde{W}^\gamma_{\tilde{\mathfrak{a}}}$

Theorem (PEP, 2021)

$(\tilde{x}, x) \in \mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$ if and only if the inequality $\boxed{(\tilde{x}, \tilde{w}\gamma) \geq (x, w_o w\gamma)}$ holds for any $(\gamma, w, \tilde{w}) \in \mathfrak{a} \times W_{\mathfrak{a}}/W^\gamma_{\mathfrak{a}} \times \widetilde{W}_{\tilde{\mathfrak{a}}}/\widetilde{W}^\gamma_{\tilde{\mathfrak{a}}}$ satisfying

- γ is antidominant and Σ-admissible,
- Cohomological condition: $\Theta^\gamma_w \cdot \iota^*\left(\Theta^\gamma_{\tilde{w}}\right) = [pt]$ in $H^*(U/U^\gamma, \mathbb{Z})$,
- Numerical condition: $N(\gamma, w, \tilde{w}) = 0$.

In 2008, Kapovich-Leeb-Millson obtained a **weaker description** of $\mathrm{Horn}_{\mathfrak{p}}(K \times K, K)$:

- Their "Cohomological condition" holds in $H^*(K/K^\gamma, \mathbb{Z}_2)$.
- They don't have a "Numerical condition".

Determination of the facets of a Kirwan polytope

First case: suppose that $0 \notin \Delta_{\mathfrak{u}}(M)$.

- Let $\gamma =$ orthogonal projection of 0 on $\Delta_{\mathfrak{u}}(M)$.
- Let $C \subset M^{\gamma}$ be the connected component containing $\Phi_{\mathfrak{u}}^{-1}(\gamma)$.
- **Białynicki-Birula's submanifold** : $C^{-} = \{m \in M, \lim_{\infty} e^{-it\gamma} m \in C\}$.

Kirwan-Ness stratification 1

- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $U_{\mathbb{C}} \times_{P_\gamma} C^{-}$.
- $\boxed{(\xi, \gamma) \geq (\Phi_{\mathfrak{u}}(C), \gamma)}$ for all $\xi \in \Delta_{\mathfrak{u}}(M)$.

Second case: suppose that $a \in \mathfrak{t}^*_+$ is a regular element not contained in $\Delta_{\mathfrak{u}}(M)$.

- Let $\gamma_a = a' - a$ where $a' =$ orthogonal projection of a on $\Delta_{\mathfrak{u}}(M)$.
- Let $C_a \subset M^{\gamma_a}$ be the connected component containing $\Phi_{\mathfrak{u}}^{-1}(a')$.
- **Białynicki-Birula's submanifold** : $C_a^{-} = \{m \in M, \lim_{\infty} e^{-it\gamma_a} m \in C_a\}$.

Kirwan-Ness stratification 2

- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $B \times_{B \cap P_{\gamma_a}} C_a^{-}$.
- $\boxed{(\xi, \gamma_a) \geq (\Phi_{\mathfrak{u}}(C_a), \gamma_a)}$ for all $\xi \in \Delta_{\mathfrak{u}}(M)$.

Ressayre's pairs

$\mathfrak{u}$-dimension: If $D \subset M$, we define $\dim_{\mathfrak{u}}(D) = \inf\{\dim(\mathfrak{u}_x), x \in D\}$.

Ressayre's pairs

(C, γ) is a Ressayre's pair if

- γ is rational,
- $C \subset M^\gamma$ and $\dim_{\mathfrak{u}}(C) - \dim_{\mathfrak{u}}(M) \in \{0, 1\}$,
- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $B \times_{B \cap P_\gamma} C^-$.

Rmq: the notion of Ressayre's pair has nothing to do with the symplectic structure.

Theorem: Ressayre, 2010 (algebraic varieties) and PEP, 2020 (Kähler manifolds)

An element $\xi \in \mathfrak{t}^*_+$ belongs to $\Delta_{\mathfrak{u}}(M)$ if and only if $\boxed{(\xi, \gamma) \geq (\Phi_{\mathfrak{u}}(C), \gamma)}$ for any Ressayre's pair (C, γ).

This technique can be adapted to describe **real moment polytopes** by considering Ressayre's pair (C, γ) compatible with the involutions:

- $\sigma(\gamma) = -\gamma$,
- $\tau(C) = C$ and $C \cap Z \neq \emptyset$,
- $\dim_{\mathfrak{p}}(C \cap Z) - \dim_{\mathfrak{p}}(Z) \in \{0, 1\}$.

The End

Thank you for your attention !