Approximate representations of symplectomorphisms via quantization

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What is this lecture about?

The group of Hamiltonian diffeomorphisms $Ham(M, \omega)$, central object in symplectic topology.

Under certain assumptions it admits **an approximate projective unitary representation** with respect to p-Schatten norm (in the sense of De Chiffre, Glebsky, Lubotzky, Thom)

Yields **obstructions to Hamiltonian actions** of finitely generated groups which do not admit such approximate representations (Lubotzky-Oppenheim)

Main tool: **Berezin-Toeplitz quantization** with optimal remainders

Mathematical model of classical mechanics

(M^{2n}, ω)–symplectic manifold
ω– **symplectic form**. Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.
M-phase space of mechanical system.

Energy determines evolution: $h : M \times [0,1] \to \mathbb{R}$ – Hamiltonian function (energy). Hamiltonian system:

$$\begin{cases} \dot{q} = \frac{\partial h}{\partial p} \\ \dot{p} = -\frac{\partial h}{\partial q} \end{cases}$$

Family of **Hamiltonian diffeomorphisms**

$$\phi_t : M \to M, \quad (p(0), q(0)) \mapsto (p(t), q(t))$$

Key feature: $\phi_t^* \omega = \omega$.
$Ham(M, \omega)$-group of Hamiltonian diffeomorphisms

$Ham(M,\omega)$ as a group

$Ham(M,\omega) = \text{Symp}_0(M,\omega)$ (identity component of all symplectic diffeomorphisms) if $H^1(M,\mathbb{R}) = 0$.

Geometry (group of symmetries) meets Dynamics (motions of classical mechanics).

Infinite-dimensional Lie group with Lie algebra $C_0^\infty(M)$-functions with zero mean. Lie bracket - Poisson bracket.

Algebra of ***Ham*****:**

- simple group (Banyaga, 1978);
- has non-trivial quasi-morphisms Entov-P., Gambaudo-Ghys, Shelukhin;
- Constraints on torsion (P., Atallah-Shelukhin), and on finitely generated subgroups including non-uniform lattices (P., Franks-Handel, Brown-Fisher-Hurtado), symplectic facets of Zimmer program;
- RAAGs (i.e., all relations are commutators between specified generators) do act (M. Kapovich)

Asymptotic projective representations

$A : H \to H$, $\|A\|_p = \left(\mathrm{tr}\left(\left(\sqrt{A^*A}\right)^p\right)\right)^{1/p}$ - p-th Schatten norm.
δ_p - corresponding distance on $\mathbb{PU}(H)$.

Group Γ - *p-norm projectively approximated* De Chiffre, Glebsky, Lubotzky, Thom, 2018,
if exist maps $\rho_k : \Gamma \to \mathbb{PU}(H_k)$, $k \to \infty$:

- **(asymptotic projective rep.)**
  $\lim \delta_p(\rho_k(x)\rho_k(y), \rho_k(xy)) = 0, \quad \forall x, y \in \Gamma,$
- **(faithful)** $\liminf \delta_p(\rho_k(x), \mathbb{1}) > 0, \quad \forall x \in \Gamma, x \neq 1.$

[DGLT] considered unitary asymptotic representations; we need projective - tiny adjustment
Theorem:(Lubotzky-Oppenheim, 2019) $\forall p \in (1, +\infty)$ $\exists$ finitely presented **non** p-norm projectively approximated groups.
Notation: $\mathcal{PLO}_p$- class of such groups.

Example: $\Gamma_0 = \mathbb{U}(2n) \cap Sp(2n, \mathbb{Z}[\sqrt{-1}, 1/\ell])$ a cocompact lattice in $Sp(2n, \mathbb{Q}_\ell)$.

Class of symplectic manifolds

(M, ω)-closed Kähler manifold, $\dim M = 2n$,
$[\omega]/(2\pi)$-integral (quantizable),
Assumption (♠): First Chern class $c_1(TM)$ is even on $\mathrm{Ker}([\omega])$.

Example: $\mathbb{C}P^n$, closed complex curves.
Here $H^2(M)$ is 1-dimensional so $c_1(TM) = 0$ on $\mathrm{Ker}([\omega])$.

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Theorem (Charles -P., 2020)

If $p > n$, $Ham(M, \omega)$ is p-norm projectively approximated.

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Corollary

Non p-norm projectively approximated groups, $p > n$, do not admit a faithful Hamiltonian action on (M^{2n}, ω).

Idea of proof: quantization!

Math. model of fin. volume quantum mechanics

H - finite dimensional Hilbert space over $\mathbb{C}$

$\mathcal{L}(H)$ - Hermitian operators on H

$\mathcal{S}$- density operators $\rho \in \mathcal{L}(H)$, $\rho \geq 0$, $Trace(\rho) = 1$.

$\hbar$-Planck constant.

Quantum mechanics contains classical in the limit $\hbar \to 0$.

Table: Quantum-Classical Correspondence

	CLASSICAL	QUANTUM
	Symplectic mfd (M, ω)	$\mathbb{C}$-Hilbert space H
OBSERVABLES	$f \in C^\infty(M)$	$A \in \mathcal{L}(H)$
STATES	Probability measures on M	Density ops $\rho \in \mathcal{S}$
BRACKET	Poisson bracket $\{f, g\}$	Commutator $\frac{i}{\hbar}[A, B]$

Fine quantization

- sequence $\hbar_k > 0$ with $\lim_{k\to\infty} k\hbar_k = 1$;
- finite-dimensional complex Hilbert spaces H_k, $k \in \mathbb{N}$
- $\mathbb{R}$-linear maps $Q_k : C^\infty(M) \to \mathcal{L}(H_k)$, $Q_k(1) = \mathbb{1}$:

(P1) (**norm correspondence**) $\|Q_k(f)\|_{op} = \|f\| + \mathcal{O}(1/k)$;

(P2) (**bracket correspondence**) $[Q_k(f), Q_k(g)] = \frac{\hbar_k}{i} Q_k(\{f, g\}) + \mathcal{O}(1/k^3)$;

(P3) (**dimension**) $\dim H_k = \left(\frac{k}{2\pi}\right)^n \mathrm{Vol}(M, \omega) + \mathcal{O}(k^{n-1})$,

Remainder (in op. norm) in (P2) is $\mathcal{O}(1/k^3)$, as opposed to "usual" $\mathcal{O}(1/k^2)$.

Existence

(M, ω)-closed, $\dim M = 2n$,

$[\omega]/(2\pi)$-integral (quantizable).

Assumption (♠): $c_1(TM)$ is even on $\mathrm{Ker}([\omega])$.

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Theorem

M admits a fine quantization.

Quantization of symplectomorphisms

f_t-Hamiltonian generating Hamiltonian flow ϕ_t, $\phi_1 = \phi$
$\phi \in Ham(M, \omega)$- group of Hamiltonian diffeomorphisms
$\widetilde{Ham}(M, \omega)$ - universal cover

Schroedinger equation: $U_k(t) : H_k \to H_k$ - quantum evolution
$\dot{U}_k(t) = -\frac{i}{\hbar_k} Q_k(f_t) U_k(t), \;\; U_k(0) = \mathbb{1}$.
$U_k = U_k(1)$ - quantization of $\widetilde{\phi} \in \widetilde{Ham}$ generated by f_t

Family of maps: $\mu_k : \widetilde{Ham}(M, \omega) \to \mathbb{U}(H_k), \;\; \widetilde{\phi} \mapsto U_k$.

Depends on choice of a Hamiltonian path joining $\mathbb{1}$ with ϕ
in class of paths homotopic with fixed endpoints to be settled

Thm. (Charles-P.) Assume μ_k comes from fine quantization.

(i) The unitaries $\mu_k(\widetilde{\phi})$ and $\mu'_k(\widetilde{\phi})$ defined via two different choices of paths homotopic with fixed endpoints representing $\phi \in \widetilde{Ham}(M,\omega)$, satisfy $\|\mu_k(\widetilde{\phi}) - \mu'_k(\widetilde{\phi})\|_{op} = \mathcal{O}(1/k)$.

(ii) $\forall \widetilde{\phi}, \widetilde{\psi} \in \widetilde{Ham}(M,\omega)$ $\|\mu_k(\widetilde{\phi})\mu_k(\widetilde{\psi}) - \mu_k(\widetilde{\phi}\widetilde{\psi})\|_{op} = \mathcal{O}(1/k)$.

(iii) $\|\mu_k(\widetilde{\phi}) - \mathbb{1}\|_{op} \geq 1/2 + \mathcal{O}(1/k)$ $\forall \phi \neq \mathbb{1}$.

Idea of proof: (i),(ii) - Egorov theorem: quantization intertwines Hamiltonian and Schroedinger evolutions up to small error.

(iii) - displacement: $\phi \neq \mathbb{1} \Rightarrow \phi B \cap B = \emptyset$ for some open $B \subset M$ $\Rightarrow U_k$ "orthogonalizes" a state supported in B.

***p*-norm approximation:** $\|A\|_{op} \leq \|A\|_p \leq d_k^{1/p}\|A\|_{op}$
$d_k := \dim H_k \sim k^n$, $\dim M = 2n$.
If $p > n$, $(k^n)^{1/p} \cdot k^{-1} \to 0$.

Effect of loops

So far found asymptotic representation of $\widetilde{Ham}$.

ϕ_t - loop in Ham, $\phi_0 = \phi_1 = \mathbb{1}$, f_t-Hamiltonian.

Floer theory: all orbits $\gamma = \{\phi_t x\}$ contractible, D-spanning disc

Action $\mathcal{A}(\gamma, D) = \int_D \omega - \int_0^1 f_t(\phi_t x)dt$, Maslov $m(\gamma, D)$.

Mixed Action-Maslov: (cf. Weinstein 1989, P. 1997)

$r(\gamma) := \lambda A(\gamma, D) - \frac{\pi}{2} m(\gamma, D) \pmod{2\pi}$ does not depend on D.

Defines homomorphism $\pi_1(Ham) \to \mathbb{R}/(2\pi\mathbb{Z})$. ($\lambda$ defined below).

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Theorem

Assume M-Kähler, $\widetilde{\phi} = [\gamma] \in \pi_1(Ham)$, $U_k = \mu_k(\widetilde{\phi})$- quantization. Then $U_k = e^{ikA(\gamma)+ir(\gamma)}\mathbb{1} + \mathcal{O}(k^{-1})$.

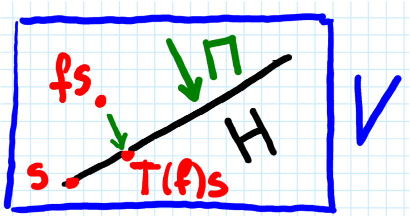
Proof: Use Charles-Le Floch, 2020 sharp quantum propagation.

Corollary: μ_k descends to projective asymptotic rep. of Ham.

Berezin-Toeplitz quantization-1

(M, ω, J)- closed Kähler manifold, quantizable:
$[\omega]/(2\pi) \in H^2(M, \mathbb{Z})$
L- a holomorphic Hermitian line bundle over M
Curvature of Chern connection $= i\omega$.

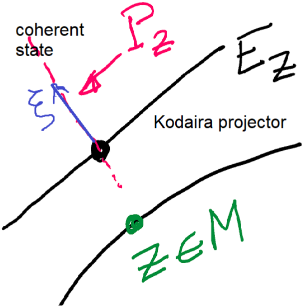
$H_\hbar := H^0(M, L^{\otimes k}) \subset V_\hbar := L_2(M, L^{\otimes k})$, $\hbar = 1/k$.
$\Pi_\hbar : V_\hbar \to H_\hbar$ – the orthogonal projection.
The Toeplitz operator: $T_\hbar(f)(s) := \Pi_\hbar(fs)$, $f \in C^\infty(M)$, $s \in H_\hbar$.

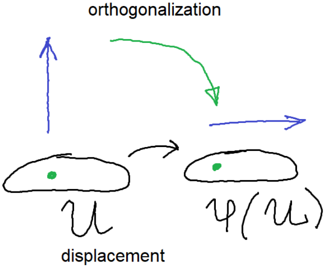


Boutet de Monvel - Guillemin, 1981; Bordemann, Meinrenken and Schlichenmaier, 1994

Berezin-Toeplitz quantization -2

Hyperplane $E_z \subset H$, $E_z := \{s \in H_\hbar \ : \ s(z) = 0\}$.
Kodaira embedding $M \to \mathbb{P}(H_\hbar^*)$, $z \mapsto E_z$
$P_{z,\hbar}$– coherent state projector $H_\hbar$ to $E_z^\perp$





Exists Rawnsley function $R_\hbar (2\pi\hbar)^{-m}(1 + \mathcal{O}(\hbar)) \in C^\infty(M)$:

$T_\hbar(f) = \int_M f(z) R_\hbar(z) P_{x,\hbar} d\mathrm{Vol}(z)$

Used in **displacement** $\Rightarrow$ **orthogonalization**

Existence of fine quantization

(M, ω)-closed, $\dim M = 2n$,
$[\omega]/(2\pi)$-integral (quantizable).
Assumption (♠): $c_1(TM)$ is even on $\mathrm{Ker}([\omega])$.

.

Theorem

M admits a fine quantization.

Proof: (based on Charles, 2007) T_k-Berezin-Toeplitz for $L^k \otimes E$
L, E - line bundles, $c_1(L) = [\omega]/(2\pi)$

By (♠) $\exists \lambda \in \mathbb{Q}$: $e := c_1(TM)/2 - \lambda c_1(L) \in H^2(M, \mathbb{Z})$

Choose E: $c_1(E) = e$.

$h_k := (k + \lambda)^{-1}$

$Q_k(f) = T_k(f - (1/4k)\Delta f)$

Geometry of *Ham*: Carries a bi-invariant Finsler metric (Hofer's metric) associated to uniform norm on the Lie algebra, a central object in modern symplectic topology. Interesting coarse geometry. Completely elusive in the context of quantization

Rescale the p-Schatten distance $\rho_p := k^{-n/p}\delta_p$, so that $PU(H_k)$ becomes bounded. For instance, $\rho_\infty = \delta_\infty$, the operator norm.

Question: Do we still have approximation?

Evidence in favor of YES based on an analysis of displacement

Utraproduct $G := \prod_{k \to \mathcal{U}} (PU(H_k), \rho_p)$ is equipped with natural metric $\rho(\mathbf{1}, \{g_i\}) := \lim \rho_p(\mathbf{1}, g_i)$, the ultrafilter-limit. (Hypothetical) approximation yields monomorphism $Ham \to G$.

Question: What is the pullback metric on *Ham*?
For $p = \infty$ (i.e., operator norm) it is discrete.
For $p < \infty$ the answer might be related to Halmos metric $\text{dist}(\mathbf{1}, \phi) := \sup_{A \subset M} \text{Vol}(A \triangle \phi(A))$.

The End

THANK YOU!