Rescaling Ward identities in the random normal matrix model

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Particle systems

A system $\{\zeta_i\}_1^n \in \mathbb{C}$ ("point charges”) in external field nQ.

- **Energy:**

$$H_n = \sum_{j \neq k}^{n} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j).$$

- **Boltzmann–Gibbs law:**

$$d\mathbf{P}_n(\zeta) = \frac{1}{Z_n^\beta} e^{-\beta H_n(\zeta)}\, d^{2n}\zeta, \quad \zeta = (\zeta_j)_1^n. \tag{1}$$

- **Assumptions.** $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is l.s.c., C^ω-smooth, and

$$Q(\zeta) >> \log|\zeta|, \quad (\zeta \to \infty).$$

A minimizer $\{\zeta_j\}_1^n$ of H_n is a *Fekete-configuration*.

Frostman's equilibrium measure

- *Q-energy* of a Borel p.m. μ on $\mathbb{C}$

$$I(\mu) := \iint \log \frac{1}{|\zeta - \eta|} \, d\mu(\zeta) d\mu(\eta) + \int Q \, d\mu.$$

  The *equilibrium measure* σ minimizes $I(\mu)$ where $\mu(\mathbb{C}) = 1$.
- *Droplet*

$$S = S[Q] := \operatorname{supp} \sigma. \tag{2}$$

Theorem

(Frostman)

$$d\sigma(z) = \chi_S(z) \, \Delta Q(z) \, dA(z).$$

(In particular $\Delta Q \geq 0$ on S.)

Nature of droplets

Lemma

Fix $p \in \partial S$. There is a nbh N of p and a "local Schwarz function" $s(\zeta)$ on N obeying

- *s is analytic in $N \setminus S$,*
- *s is continuous on N and*

$$\bar{\zeta} = s(\zeta), \quad \zeta \in (\partial S) \cap N.$$

Theorem

(Sakai, 1991) ∂S is a union of finitely many analytic curves. Possible singularities: cusps pointing out of S and double points.

Complement S^c is an *Unbounded Quadrature Domain* (in wide sense of Shapiro).

Droplets 1

Technical assumptions:

- Q is real-analytic in a nbh of S.
- $\Delta Q > 0$ in a nbh of S.

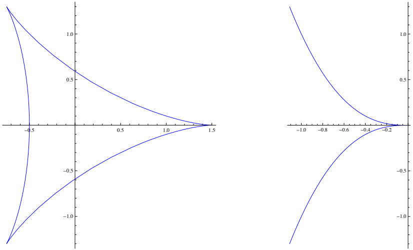


Figure: The Deltoid is not admissible; it has three maximal 3/2 cusps. 5/2 cusp is OK.

Droplets 2

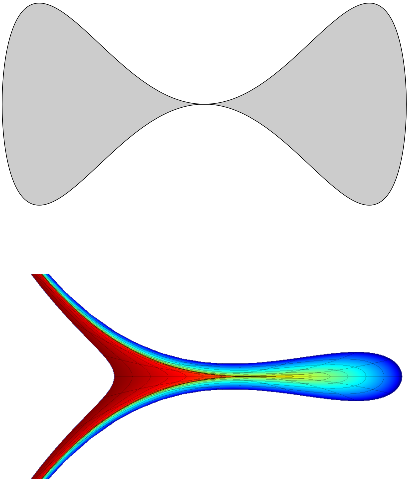


Figure: Double point and 5/2 cusp under Hele-Shaw flow.

Joint intensities

Let $\{\zeta_j\}_1^n$ random sample. k-point function

$$\mathbf{R}_{n,1}(\eta) = \lim_{\epsilon \to 0} \frac{\mathbf{P}_n(D(\eta;\epsilon) \cap \{\zeta_j\}_1^n \neq \emptyset)}{\epsilon^2},$$

$$\mathbf{R}_{n,2}(\eta_1,\eta_2) = \lim_{\epsilon \to 0} \frac{\mathbf{P}_n(D(\eta_l;\epsilon) \cap \{\zeta_j\} \neq \emptyset, \quad l = 1,2)}{\epsilon^4}, \text{ etc.}$$

Asymptotics as $n \to \infty$ should give CFT.

If $\beta = 1$, the process is *determinantal*,

$$\mathbf{R}_{n,k}(\eta_1,\ldots,\eta_k) = \det\left(\mathbf{K}_n(\eta_i,\eta_j)\right)_{i,j=1}^k .$$

Here $\mathbf{K}_n$ is a "correlation kernel" = reprokernel for

$$\mathcal{W}_n := \{q \cdot e^{-nQ/2};\ \text{degree}(q) < n\} \subset L^2.$$

Note: $\mathbf{E}_n(f(\zeta_1,\ldots,\zeta_k)) = \frac{(n-k)!}{n!} \int_{\mathbb{C}^k} f \cdot \mathbf{R}_{n,k}$.

"Classical" convergence result (for all β)

Random measure

$$\mu_n := \frac{1}{n}\sum_1^n \delta_{\zeta_j}.$$

For $f \in W^{1,2}$

$$\sigma_n(f) := \mathbf{E}_n(\mu_n(f)) = \frac{1}{n}\sum_1^n \mathbf{E}_n(f(\zeta_j)) = \frac{1}{n}\int_{\mathbb{C}} f \cdot \mathbf{R}_n.$$

Theorem

(HM) $\frac{1}{n}\mathbf{R}_n\, dA \to \sigma$ *and*

$$\sigma_n(f) \to \int f\, d\sigma, \qquad (n \to \infty).$$

(Here $\mathbf{R}_n = \mathbf{R}_{n,1}$.)

Example: Ginibre ensemble ($\beta = 1$)

Let $Q(\zeta) = |\zeta|^2$. Then $S = \{|\zeta| \le 1\}$ and $\sigma = \chi_S \, dA$.
The process $\{\zeta_i\}_1^n$ can be interpreted as eigenvalues of an $n \times n$-matrix with i.i.d. centered complex Gaussian entries of variance $1/n$.

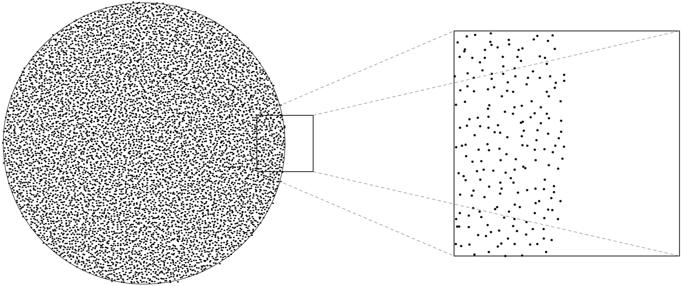


Figure: A sample of the Ginibre process for a large value of n. We will later look at the process near the boundary.

Fluctuation theorem ($\beta = 1$)

Random measures $\text{fluct}_n := n(\mu_n - \sigma) = \sum_1^n \delta_{\zeta_j} - n\sigma$.
Random variables on $(\mathbb{C}^n, \mathbf{P}_n)$

$$\text{fluct}_n(f) = \sum_1^n f(\zeta_j) - n\sigma(f), \quad (f \in C_b^\infty(\mathbb{C})).$$

Theorem

$\text{fluct}_n(f)$ *converges in distribution to the normal* $N(e_f, \sigma_f^2)$*, where*

$$e_f = \frac{1}{8\pi} \int_S f \cdot \Delta(\chi_S + L^S), \quad \sigma_f^2 = \frac{1}{2} \int_{\mathbb{C}} |\nabla f^S|^2, \quad (L = \log \Delta Q).$$

Here f^S *equals* f *in* S *and is harmonic and bounded in* $\mathbb{C} \setminus S$.

Comments

1. There is no $(1/\sqrt{n})$-normalization!
2. Theorem says that random distributions

$$\text{fluct}_n - \Delta(\chi_S + L^S)$$

   converge to GFF on S with free boundary conditions.
3. Test-class should be $f \in W^{1,2}(\mathbb{C})$.
4. The theorem is only proved for *connected* S with everywhere regular boundary.
5. There are "physical" results for arbitrary β; also results by Johansson in dim 1. Our method "should" extend, but we need some estimates.
6. (Expansion of Bergman kernel in the bulk is an easy consequence of the fluctuation theorem.)

Ward's identity

Let $\{\zeta_j\}_1^n$ system. For smooth ψ define r.v.'s

$$A_\psi = \frac{1}{2}\sum_{j\neq k}^{n} \frac{\psi(\zeta_j)-\psi(\zeta_k)}{\zeta_j-\zeta_k},\ B_\psi = n\sum_1^n \partial Q(\zeta_j)\psi(\zeta_j),\ C_\psi = \sum_1^n \partial\psi(\zeta_j).$$

Theorem

For all ψ

$$\mathbf{E}_n(\beta\cdot(A_\psi - B_\psi) + C_\psi) = 0.$$

This is an implicit relation between $\mathbf{R}_{n,1}$ and $\mathbf{R}_{n,2}$.

(Proof: reparametrization invariance of the partition function $Z_n := \int_{\mathbb{C}^n} e^{-\beta H_n}\, dV_n$.)

Rescaling

$\{\zeta_j\}_1^n$ random sample from $\mathbf{P}_n$. Fix $p_n \in S$, $\theta_n \in \mathbb{R}$.

Mesoscopic scale:

$$r_n = 1/\sqrt{n\Delta Q(p_n)}.$$

Rescaled system:

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta_j - p_n).$$

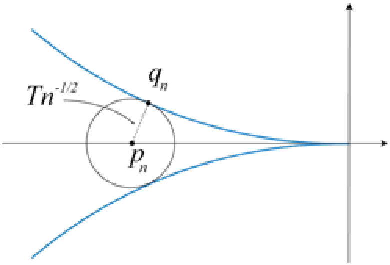


Figure: It is OK to rescale about a moving point.

Rescaled process ($\beta = 1$)

Rescaled process $\Theta_n := \{z_j\}_1^n$ has k-point function

$$R_{n,k}(z_1, \dots, z_k) = r_n^{2k} \mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k).$$

We have:

$$R_{n,k}(z_1, \dots, z_k) = \det(K_n(z_i, z_j))_{i,j=1}^k, \quad K_n(z, w) := r_n^2 \mathbf{K}_n(\zeta, \eta).$$

Known: if $p \in \operatorname{Int} S$, then $R_{n,k}(z_1, \dots, z_k) \to \det(G(z_i, z_j))_{k \times k}$ where

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}.$$

is *Ginibre kernel*.

Compactness and analyticity

A *cocycle* is a function $c(z,w)=g(z)\bar{g}(w)$ where g is continuous and unimodular. Correlation kernels are only determined up to cocycles.

Theorem

There are cocycles c_n such that (on subsequences)

$$c_{n_k}K_{n_k}\to K,\quad \textit{where}\quad K(z,w)=G(z,w)\Psi(z,w).$$

Here $G(z,w)=e^{-|z|^2/2-|w|^2/2+z\bar{w}}$ and $\Psi(z,w)$ is Hermitian entire.

(Proof: Taylor's formula + normal families.)

A limit point $K=G\Psi$ is called a *limiting kernel* at (moving) point p.

Limiting point fields

A limiting kernel K is correlation kernel of a limiting point field $\{\zeta_j\}_1^\infty$ with k-point function $R_k(\eta_1, \ldots, \eta_k) = \det(K(\eta_i, \eta_j))_{i,j=1}^k$.
Limiting 1-pt function

$$R(z) := K(z, z) = \Psi(z, z).$$

R determines Ψ by polarization and $K = G\Psi$ so

R determines all k-point functions.

Forrester–Honner's formula

Theorem

Let $Q = |\zeta|^2$ and rescale about a boundary point in the outer normal direction. Then $R(z) = F(z + \bar{z})$ where F is "free boundary plasma function"

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(z-t)^2/2}\, dt.$$

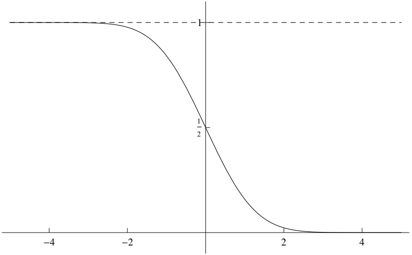


Figure: F is the analytic continuation to $\mathbb{C}$ of the d.f. of the standard normal.

Rescaled Ward identity

Suppose $R(z) = \Psi(z,z) \neq 0$. Put

$$B(z,w) := \frac{|K(z,w)|^2}{R(z)} = e^{-|z-w|^2} \frac{|\Psi(z,w)|^2}{\Psi(z,z)}, \quad C(z) := \int \frac{B(z,w)}{z-w} \, dA(w).$$

Theorem

$R > 0$ everywhere, $C(z)$ is smooth, and we have Ward's equation

$$\bar{\partial} C(z) = R(z) - 1 - \Delta \log R(z).$$

Since $R \mapsto \Psi$ by polarization, this is an equation for R!

Note: Ward's equation holds at any (moving) point s.t. $R \neq 0$. To fix R uniquely, we need side-conditions. These depend on the nature of the point we're zooming on (bulk point, regular boundary pt, singular boundary pt).

Apriori estimates (side conditions)

Rescale about a regular boundary point in outer normal direction. Let $R(z)$ a limiting 1-point function.

1. Exterior estimate:

$$R(z) \leq Ce^{-2x^2}, \quad (x \geq 0).$$

2. $1/8$-formula:

$$\int_{-\infty}^{+\infty} t \cdot (R(t) - \chi_{(-\infty,0)}(t))\, dt = \frac{1}{8}.$$

(Proof: (1) by potential theory; (2) fluctuation theorem.)

Complementarity

A limiting kernel $K = G\Psi$ is a positive matrix in Aronszajn's sense,

$$\sum_{j,k=1}^{N} \alpha_j \bar{\alpha}_k K(z_j, z_k) \geq 0.$$

(Because $\det(K(z_j, z_k))_{N\times N} = R_N(z_1, \ldots, z_N) \geq 0$.)

Theorem

The complementary kernel

$$\tilde{K}(z, w) = G(z, w)(1 - \Psi(z, w))$$

is also a positive matrix. In particular $R(z) = \Psi(z, z) \leq 1$.

Warning: $\tilde{K}$ does not solve Ward, in general.

Translation Invariance (T.I.)

$R(z) = \Psi(z, z)$ is called *t.i.* if $\Psi(z + it, w + it) = \Psi(z, w)$, $t \in \mathbf{R}$.
Equivalently,

$$\Psi(z, w) = \Phi(z + \bar{w})$$

where Φ is entire.

Gaussian representation

Theorem

If $K(z,w) = G(z,w)\Phi(z+\bar{w})$ is a t.i. limiting kernel, then there exists a Borel function f on $\mathbf{R}$ with $0 \le f \le 1$ such that

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z-t) f(t)\, dt,$$

where γ is the Gaussian kernel

$$\gamma(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Examples:

- Bulk: $\Phi \equiv 1 = \gamma * \chi_{\mathbb{R}}$
- The plasma kernel: $F = \gamma * \chi_{(-\infty,0)}$

Proof: Uses Bochner's theorem on positive definite functions (also for the complementary kernel).

T.i. solutions to Ward's equation

Theorem

Let $R(z) = \Phi(z + \bar{z})$ a t.i. limiting 1-point function with $R \not\equiv 0$. Then R solves Ward's equation iff there is an interval $I \subset \mathbb{R}$ such that

$$\Phi = \gamma * \chi_I.$$

Corollary

If $R(z) = \Phi(z + \bar{z})$ is rescaled about a regular boundary point and R is t.i. then

$$\Phi = F = \gamma * \chi_{(-\infty,0)}.$$

Comments:

1. **Conjecture**: An arbitrary limiting kernel is t.i (exception: bulk singularities).
2. In the general (non-t.i.) case, Ward is a twisted convolution equation. The physically relevant ones "should" be the above.

Consequences

For radial $Q(\zeta) = Q(|\zeta|)$ and for "ellipse potential" $Q = |\zeta|^2 - t\mathrm{Re}\,(\zeta^2)$ we know that any limiting 1-point function R is t.i.

Corollary

If Q is one of the types above, rescale about a boundary point. The rescaled systems $\{z_j\}_1^n$ converges to the field with kernel

$$K(z, w) = G(z, w) \cdot F(z + \bar{w}).$$

- This should be true for a general potential at a regular boundary point.
- Lee and Riser obtained this for the ellipse potential $Q(\zeta) = |\zeta|^2 - t\mathrm{Re}\,(\zeta^2)$ using orthogonal polynomials.

Singular boundary points

Now assume that p is a cusp or a double point. If p is a cusp, we assume it has type $(\nu, 2)$ where $\nu > 3$, i.e. it resembles

$$x^\nu = y^2.$$

Theorem

Rescale according to $z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p)$. Then any limiting kernel is trivial: $R = 0$.

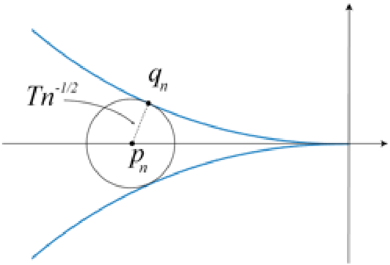
(Proof: exterior estimate (suitable coord system)

$$R(z) = \Psi(z, z) \leq Ce^{-2x^2}.$$

Since $L(z, w) = e^{z\bar{w}}\Psi(z, w)$ is Hermitian-entire and positive definite, $\log L(z, z)$ is subharmonic. This gives $\Psi = 0$, by the maximum principle.)

Cusps; moving points

Assume S has a $(\nu, 2)$ cusp at p and fix $T > 0$.
Let $p_n \in S$ be the point of distance $T/\sqrt{n\Delta Q(p)}$ from the boundary which is closest to p.



Rescale about p_n

$$z_j = e^{-i\theta_n} r_n^{-2}(\zeta_j - p_n), \quad j = 1, \ldots, n$$

where θ_n is chosen so that $e^{-i\theta_n}(p - p_n)$ is positive imaginary.

Existence theorem

Theorem

If T is sufficiently large, then each limiting 1-point function $R(z) = K(z, z)$ is positive, satisfies Ward's equation, and the estimate

$$R(z) \le Ce^{-2(|x|-T)^2}. \tag{3}$$

- Estimate (3) shows that R is associated with a "new" determinantal point field.
- After the rescaling, the droplet looks like the strip

$$\Sigma_T : \quad -T \le \operatorname{Re} z \le T,$$

so it is natural to assume that the field is t.i.

Natural candidates 1

- For $s > 0$ let

$$\Phi_s(z) = \gamma * \chi_{(-s,s)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s}^{s} e^{-(z-t)^2/2}\, dt.$$

  If $s \leq 2T$ then $R_s(z) = \Phi(z + \bar{z})$ satisfies $R_s(x) \leq Ce^{-2(|x|-T)^2}$ and Ward's equation.
- How should we choose s?
- In regular case, we used $1/8$-formula:

$$\int_{\mathbb{R}} t \cdot (R(t) - \chi_{(-\infty,0)}(t))\, dt = \frac{1}{8}.$$

  Something similar should hold at cusps.

Conclusion. We must extend the boundary fluctuation theorem to domains with cusps.

Natural candidates 2

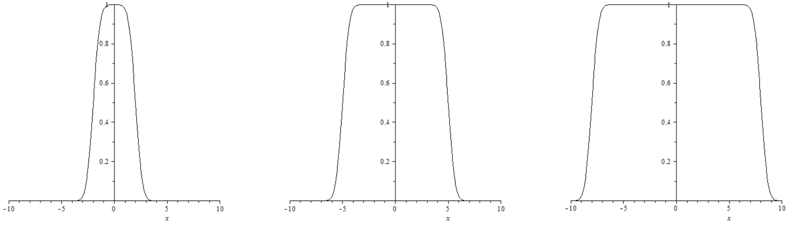


Figure: The graphs of $R_T(x) := \Phi_{T/2}(2x)$ for $T = 2, 5, 8$.

The hard edge

Let Q be a potential. Define $Q^S = Q$ on S and $Q^S = +\infty$ otherwise. Let $\{\zeta_j\}$ be a random sample from $\mathbf{P}_n$ and rescale about a *regular* boundary point p to obtain $\{z_j\}_1^n$.

Theorem

For u.t.i. potentials, the processes $\{z_j\}_1^n$ converge to a unique point field with correlation kernel

$$K(z,w) = G(z,w)H(z+\bar{w})\chi_{\mathbb{L}}(z)\chi_{\mathbb{L}}(w), \quad \mathbb{L} = \{\mathrm{Re}\, z < 0\},$$

where H is the hard edge plasma function,

$$H(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-(z-t)^2/2}}{F(t)}\, dt,$$

($F =$free boundary plasma function).

Hard edge plasma function

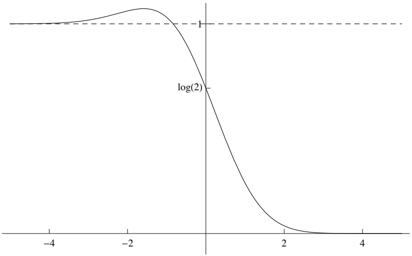


Figure: The graph of H.

The hard-edge theory is parallel to the free boundary; we can obtain existence of new hard edge fields near cusps, and so on.

Natural candidates 3: hard edge near a singular point

For $T > 0$ define

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)}\, dt, \quad (F_T = \gamma * \chi_{(-2T,2T)}).$$

The "1-point function" is then $R_T^h(z) := H_T(z+\bar{z})\chi_{(-T,T)}(\mathrm{Re}\, z)$.

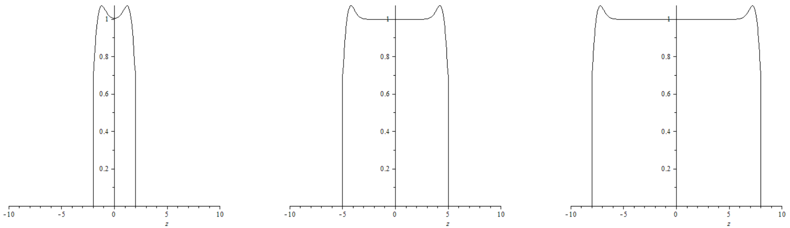


Figure: The graph of R_T^h restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$.

Fekete points

Let p be a regular boundary point, and let $\mathcal{F}_n = \{\zeta_{jn}\}_{j=1}^{n}$ be an n-Fekete configuration, $n = 1, 2, \ldots$. Denote

$$d_n(\zeta_{nj}) = \sqrt{n \Delta Q(\zeta_{nj})} \cdot \min_{k \neq j} |\zeta_{nj} - \zeta_{nk}|,$$

and ("asymptotic separation constant")

$$\Delta(\mathcal{F}) = \liminf_{n \to \infty} \min_{j=1,\ldots,n} \left\{ d_n(\zeta_{nj}) \right\}.$$

Theorem

$\Delta(\mathcal{F}) \geq 1/\sqrt{e}$.

We believe that $\Delta(\mathcal{F}) = \sqrt{2/\sqrt{3}}$.
(This comes close to Abrikosov's conjecture.)

Comparison: the one-dimensional case

If Q is real-analytic on $\mathbb{R}$ and $Q = +\infty$ outside $\mathbb{R}$, then S is a finite union of compact intervals. Let p be a boundary point. Rescale by

$$x_j = cn^{2/3}(\xi_j - p).$$

Theorem

In free boundary case, there is a c such that $\{x_j\}_1^n$ *converges to the Airy process with kernel*

$$K(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}.$$

In hard edge case we get a Bessel process with kernel

$$K(x,y) = \frac{J_0(\sqrt{x})\sqrt{y}\,J_0'(\sqrt{y}) - \sqrt{x}\,J_0'(\sqrt{x})J_0(\sqrt{y})}{2(x-y)}.$$

These are reprokernels for "de Branges spaces”. The two-dimensional plasma kernels are quite different.

β-ensembles

We don't now have a kernel, but we put

$$B(z,w) = \frac{R(z)R(w) - R_2(z,w)}{R(z)}$$

and

$$C(z) = \int_{\mathbb{C}} \frac{B(z,w)}{z-w}\, dA(w).$$

Ward's equation is

$$\bar{\partial} C = R - 1 - \frac{1}{\beta} \Delta \log R.$$

The equation needs to be "closed". When $\beta = 1$ we used the extra structure of existence of a kernel $K = G\Psi$.
What can we do in general?