A No-Go Theorem about Tensor Network States and Chern Bands

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based on joint work with N. Read (Yale), see arXiv:1307.7726
and discussions with Barry Bradlyn (Princeton)

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Outlook

- (i) what is a Tensor Network State, (ii) what is a Chern band
- motivation and statement of the theorem
- sketch of the proof

Crash course on Tensor Network States (or TPS, or PEPS)

- Hilbert space: $\mathcal{H} = \bigotimes_x h_x = h^{\otimes L^2}, \quad x = \vec{x} \in \vec{u}\mathbb{Z}_L + \vec{v}\mathbb{Z}_L$ and h is finite-dim.
- vector space V, $(\dim V < \infty)$, for each lattice edge and dual vector space V^*, and canonical pairing

$$pair : V \otimes V^* \to \mathbb{C}$$

- pick one tensor

$$\begin{aligned} T_x \quad \in \quad & h \otimes V \otimes V^* \otimes V \otimes V^* \\ & = h_x \otimes V_{x+u/2} \otimes V^*_{x-u/2} \otimes V_{x+v/2} \otimes V^*_{x-v/2} \end{aligned}$$

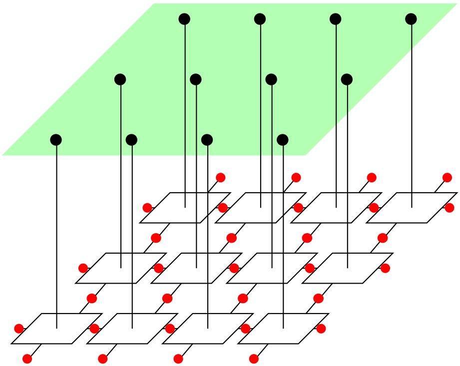
- one gets a translation-invariant $(T_x = T)$ state:

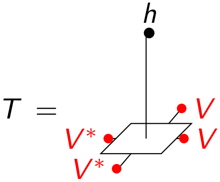
$$|\psi\rangle = pair\left[\bigotimes_x T_x\right] \quad \in \mathcal{H}\,.$$

Crash course on TNSs (or TPS, or PEPS)

what it means:

physical degrees of freedom





auxiliary degrees of freedom

Chern band

- keep things simple: example with only two orbitals per site

$$h = span\left(|0\rangle, c_1^\dagger |0\rangle, c_2^\dagger |0\rangle, c_1^\dagger c_2^\dagger |0\rangle\right)$$

- single-particle space: $span\left(c_1^\dagger |0\rangle, c_2^\dagger |0\rangle\right) \simeq \mathbb{C}^2$.
- there is one such space $\mathbb{C}^2$ for each $k \in BZ$ (Brillouin zone)

$$BZ \times \mathbb{C}^2$$

  which is a trivial complex bundle.
- a 'band' is a rank-1 subbundle of this trivial bundle
- locally generated by (locally non-vanishing) section

$$k \quad \mapsto \quad u_k c_{1k}^\dagger |0\rangle + v_k c_{2k}^\dagger |0\rangle \ \propto \ \exp(\tfrac{v_k}{u_k} c_{2k}^\dagger c_{1k}) c_{1k}^\dagger |0\rangle$$

- if the subbundle is non-trivial (as a complex bundle), it is called a Chern band

Chern band

- keep things simple: only two orbitals per site

$$h = span\left(c_1 |1\rangle, |1\rangle, c_2^\dagger c_1 |1\rangle, c_2^\dagger |1\rangle\right)$$

- single-particle space: $span\left(|1\rangle, c_2^\dagger c_1 |1\rangle\right) \simeq \mathbb{C}^2$.
- there is one such space $\mathbb{C}^2$ for each $k \in BZ$ (Brillouin zone)

$$BZ \times \mathbb{C}^2$$

  which is a trivial bundle.
- a 'band' is a subbundle of this trivial bundle
- locally generated by a section

$$k \quad \mapsto \quad \exp(g_k c_{2k}^\dagger c_{1k}) |1\rangle$$

- if the rank-1 subbundle is non-trivial (as a complex bundle), it is called a Chern band

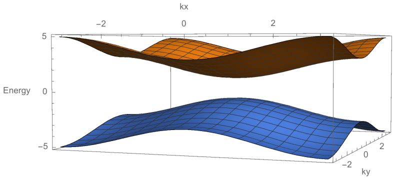
Question

is it possible to find a TNS $|\psi\rangle$ which is a filled Chern band?

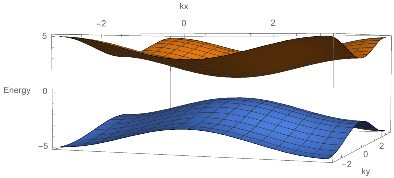
Why do we care?

- tremendous success of Matrix Product States in 1d: provide enough variational freedom to capture all gapped ground states, explain the success of DMRG, TEBD, etc., allow for complete classification of (symmetry protected) topological phases, etc.
- it is hoped that such results extend to $d > 1$ with TNSs
- however, it is unclear whether TNSs can represent all kinds of gapped ground states. Some theories with anomalous edges seem to not admit TNS representatives. A Chern band in d dimension is one simple example of those.

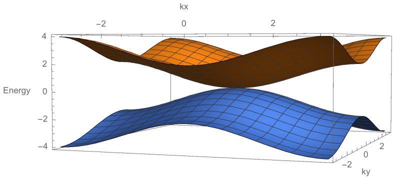
What can or cannot be done with TNS



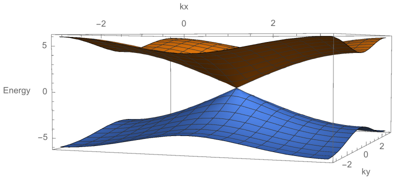
Chern $= 0$



Chern $\neq 0$



Chern $\neq 0$



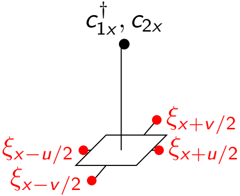
C not defined (not a bundle)

How do we do it?

- goal: construct a TNS which, when written in k-space, is

$$|\psi\rangle \propto \exp\left(\sum_k g_k c_{2k}^\dagger c_{1k}\right)|1\rangle$$

- fermions $\Rightarrow$ use $\mathbb{Z}_2$-graded vector space $V = V_{even} + V_{odd}$. Equivalently, one can use Grassmann variables:



- the 'tensor' $T \in h \otimes V^* \otimes V \otimes V^* \otimes V$ must be 'even', and gaussian. For example:

$$T_x = e^{\xi_{x+u/2}\xi_{x-u/2}+(1-i)\xi_{x+v/2}\xi_{x-u/2}}\, e^{\sqrt{3}\xi_{x-v/2}c_{2x}^\dagger+(\xi_{x-v/2}-\xi_{x+u/2})c_{1x}}$$

- multiplying all the T_x's and integrating out the Grassmann variables,

$$
\begin{aligned}
|\psi\rangle &= \int [d\xi] \prod_x T_x = \int [d\xi] \exp\left(\sum_x \xi_{x+u/2}\xi_{x-u/2} + \cdots + \xi_{x-u/2} c_{2x}^\dagger + \ldots \right) \\
&= \int [d\xi] \exp\left(\sum_k e^{ik_x} \xi_k \xi_{-k} + \cdots + \xi_k c_{2k}^\dagger + \ldots \right) \\
&= \exp\left(\sum_k g_k c_{2k}^\dagger c_{1k} \right) |1\rangle \quad \checkmark
\end{aligned}
$$

- consequence of $|\psi\rangle$ being a TNS:

$$
\boxed{g_k = v_k / u_k, \qquad u_k, v_k \in \mathbb{C}[e^{ik_x}, e^{ik_y}]}
$$

(ratio of two polynomials in e^{ik_x}, e^{ik_y}). The converse is also true: if g_k is ratio of polynomials, then $|\psi\rangle = e^{\sum g c_2^\dagger c_1} |1\rangle$ is a TNS.

- an example that is a Chern band? $g_k = \frac{\sin k_x - i \sin k_y}{\sin^2 k_x + \sin^2 k_y + (2 - \cos k_x - \cos k_y)^2}$ does the job (gives Chern number one, but is gapless).

the No-Go Theorem (arbitrary dimension d and number of bands)

in physics language:

Thm: if a translation-invariant free-fermion state $|\psi\rangle$ is

1. a TNS
2. the ground state of a local, gapped Hamiltonian

then it is in a topologically trivial phase.

in mathematical language:

Thm: if a vector bundle over the d-dim torus is

1. a polynomial* bundle
2. an analytic* bundle

then it is topologically trivial as a complex vector bundle.

* non-standard notions to be defined on next slides

def. 1: analytic bundles

- def: a section of $T^d \times \mathbb{C}^n$ with components that are (real-) analytic in $k_1, k_2, \ldots, k_d$ in a neighborhood of k_0 is said to be analytic in the neighborhood.

- def: a rank-m sub-bundle of $T^d \times \mathbb{C}^n$ is said to be analytic in a neighborhood of k_0 if it possesses m linearly-independent analytic sections in the neighborhood

- physically: a (free-fermion, transl. inv.) state is the ground state of a local, gapped Hamiltonian iff the corresponding bundle is analytic everywhere in the Brillouin Zone.

def. 2: polynomial bundles

- a state $|\psi\rangle$ for m filled bands, written in the form

$$|\psi\rangle \propto \exp\left(\int \frac{d^d k}{(2\pi)^d} \sum_{\alpha\overline{\alpha}} g_{k\alpha\overline{\alpha}} c^\dagger_{k\overline{\alpha}} c_{k\alpha}\right) |11\cdots, 00\cdots 0\rangle$$

  is a TNS iff g_k is an $m \times (n-m)$ matrix with entries that are ratios of polynomials in $R = \mathbb{C}[e^{ik_1}, \ldots, e^{ik_d}]$ (polynomial ring in d var.)

- def: a sub-bundle of $T^d \times \mathbb{C}^n$ that can be obtained from a g_k with entries that are ratios of polynomials is called a polynomial bundle

- def: a section of the trivial bundle $T^d \times \mathbb{C}^n$ with polynomial comp.

$$(k_1, \ldots, k_n) \in T^d \;\mapsto\; \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} \in \mathbb{C}^n\,, \qquad P_j \in R$$

  is called a polynomial section (also a basis-independent prop.)

- example: $d = 2,\; n = 2,\; m = 1$

$$g_k = \frac{\sin k_1 - i \sin k_2}{\sin^2 k_1 + \sin^2 k_2 + (2 - \cos k_1 - \cos k_2)^2}$$

  which defines a rank-1 polynomial bundle with Chern number 1. It has polynomial sections, for instance

$$k = (k_1, k_2) \;\mapsto\; \begin{pmatrix} e^{i2k_1} e^{i2k_2} (\sin^2 k_1 + \sin^2 k_2 + (2 - \cos k_1 - \cos k_2)^2) \\ e^{i2k_1} e^{i2k_2} (\sin k_1 - i \sin k_2) \end{pmatrix} \in R^2$$

  Notice that this section vanishes at $(k_1, k_2) = (0, 0)$. In fact, one can show that, in that example, all polynomial sections of the rank-1 bundle vanish at this point.

Remark: there is some dissymmetry in the definitions: 'polynamiality' is an algebraic/global property, 'analyticity' is a local property. Also, 'analytic bundles' may be locally spanned by analytic sections, while 'polynomial bundles' cannot, in general, be locally spanned by polynomial sections. However:

Proposition

if a polynomial sub-bundle (of rank m) is analytic at k_0, then there exist m polynomial sections that are linearly independent in a neighborhood of k_0

In the above example, $g_k = \frac{\sin k_1 - i \sin k_1}{\sin^2 k_1 + \sin^2 k_2 + (2 - \cos k_1 - \cos k_2)^2}$, the bundle is indeed not analytic at $k = (0, 0)$, meaning that it cannot be the ground state of a gapped, local Hamiltonian.

proof of No-Go Thm for a single band

We are ready to prove the No-Go thm in the case $m = 1$. We assume that the bundle E is polynomial and analytic everywhere.

- not difficult to exhibit one polynomial section w such that any other polynomial section must be a multiple of w
- fix some point k_0. By the Proposition, there exist a non-vanishing polynomial section in a neighborhood of k_0. This section is a multiple of w_k, so w_k cannot vanish at k_0.
- we have exhibited a non-vanishing section, so the bundle is trivial. $\square$

general proof (with additional assumption)

Assuming there is a set of m generators $w_1, \ldots, w_m$ such that any polynomial section is of the form

$$p_1 w_1 + \cdots + p_m w_m \ , \qquad p_j \in R,$$

one can repeat the previous proof (replacing 'non-vanishing section' by 'linearly-independent sections').

However, the set of polynomial sections is a module over R, and it might be a non-free module.

Example: $d = 3, n = 3, m = 2$. The sub-bundle E is spanned by two polyn. sections

$$w_1 = \begin{pmatrix} 0 \\ -e^{ik_3} \\ e^{ik_2} \end{pmatrix} \qquad w_2 = \begin{pmatrix} e^{ik_3} \\ 0 \\ -e^{ik_1} \end{pmatrix}$$

Is it true that any polynomial section of E can be written as a combination of w_1 and w_2 with coefficients in $R = \mathbb{C}[e^{ik_1}, e^{ik_3}, e^{ik_3}]$?

No!

$$w_3 = \begin{pmatrix} -e^{ik_2} \\ e^{ik_1} \\ 0 \end{pmatrix}$$

is not a combination of w_1 and w_2 with coefficients in R. However, any polynomial section is a combination of w_1, w_2 and w_3. One says that w_1, w_2, w_3 are generators for the module M, which is a sub-module of R^3. Notice that the three generators are not linearly independent over R:

$$e^{ik_1} w_1 + e^{ik_2} w_2 + e^{ik_3} w_3 = 0$$

equivalently, one has the exact sequence of modules over R:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} e^{ik_1} \\ e^{ik_2} \\ e^{ik_3} \end{pmatrix}} R^3 \xrightarrow{(w_1, w_2, w_3)} R^3 \longrightarrow M \longrightarrow 0$$

The element $\begin{pmatrix} e^{ik_1} \\ e^{ik_2} \\ e^{ik_3} \end{pmatrix} \in R^3$ is called a syzygy.

What's a syzygy? From the web:

Syzygy (astronomy)

From Wikipedia, the free encyclopedia

In astronomy, a **syzygy** /ˈsɪzɨdʒi/ (from the Ancient Greek *suzugos* (σύζυγος) meaning, "yoked together"[2]) is a straight-line configuration of three celestial bodies in a gravitational system.[3]

Syzygy (mathematics)

From Wikipedia, the free encyclopedia

For other uses, see Syzygy (disambiguation).

In mathematics, a **syzygy** (from Greek συζυγία 'pair') is a relation between the generators of a module *M*.

The syzygy theorem

The Hilbert syzygy theorem states that every module M over the polynomial ring R fits in such an exact sequence

$$0 \longrightarrow R^{n_p} \longrightarrow \ldots \longrightarrow R^{n_2} \longrightarrow R^{n_1} \longrightarrow M \longrightarrow 0$$

and that the sequence is finite.

Back to the proof of the No-Go thm

Relying on the analyticity assumption and on the Proposition, it is possible to turn the exact sequence of modules

$$0 \longrightarrow R^{n_p} \longrightarrow \ldots \longrightarrow R^{n_2} \longrightarrow R^{n_1} \longrightarrow M \longrightarrow 0$$

into an exact sequence of vector bundles,

$$0 \longrightarrow BZ \times \mathbb{C}^{n_p} \xrightarrow{\phi_p} \ldots \longrightarrow BZ \times \mathbb{C}^{n_2} \xrightarrow{\phi_2} BZ \times \mathbb{C}^{n_1} \xrightarrow{\phi_1} E \longrightarrow 0\,.$$

It follows that $Im(\phi_j)$ is a trivial bundle for all j. In particular, the filled band bundle $E = Im(\phi_1)$ is trivial. $\square$

For more details, see the paper: arXiv:1307.7726

Conclusion

- is it possible to have a TNS for a Chern band? No.
- No-Go Thm valid in any dimension d and for arbitrary number of bands
- mathematics involved in the proof: vector bundles, commutative algebra (because TNS $\leftrightarrow$ polynomial) and local geometric properties (because local and gapped Hamiltonian $\leftrightarrow$ analytic bundle)

Thanks!