Laplacian growth on a branched Riemann surface

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Abstract

Abstract

Laplacian growth refers to domain evolution driven by harmonic gradients, for example the gradient of the Green's function with a fixed pole. It makes sense on Riemannian manifolds of arbitrary dimension. In the talk I will discuss a particular case when the manifold is a branched covering surface of the complex plane. There turn out to be unexpected couplings to topics in complex analysis, such as contractive zero divisors in Bergman space. The talk is based on joint work with Yu-Lin Lin.

Overview

Moments

Let $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ be a normalized conformal map, say

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{n+1} \quad (a_0 > 0,\ \limsup |a_n|^{1/n} < 1).$$

The **harmonic moments** of $\Omega = f(\mathbb{D})$ are

$$M_k = \frac{1}{\pi}\int_{\Omega} z^k dA(z) = \frac{1}{2\pi i}\int_{\partial\Omega} z^k \bar{z} dz =$$

$$= \left[f^*(\zeta) := \overline{f(1/\bar{\zeta})}\right] = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)^k f^*(\zeta) f'(\zeta) d\zeta = \qquad (1)$$

$$= \textbf{Richardson's formula} = \sum (j_0 + 1) a_{j_0} \cdots a_{j_k} \bar{a}_{j_0+\ldots+j_k+k}.$$

Summation over all $(j_0, \ldots, j_k) \geq (0, \ldots, 0)$.

Remark

By (1), the moments M_k make sense for arbitrary analytic functions f (not necessarily univalent) on $\overline{\mathbb{D}}$ (or even $\partial\mathbb{D}$), and for arbitrary $k \in \mathbb{Z}$.

Moments, cont. I

Locally there are one-to-one correspondencies

$$(a_0, a_1, a_2, \dots,) \leftrightarrow \Omega \leftrightarrow (M_0, M_1, M_2, \dots),$$

so we can write

$$f(\zeta) = f(\zeta; a_0, a_1, a_2, \dots) = f(\zeta; \Omega) = f(\zeta; M_0, M_1, M_2, \dots).$$

Clearly

$$\frac{\partial f}{\partial a_k} = \zeta^{k+1},$$

but what is

$$\frac{\partial f}{\partial M_k} \quad ??$$

This, and many similar questions, have been answered in a beautiful way by **Mark Mineev-Weinstein, Paul Wiegmann, Anton Zabrodin** and others in series of papers on integrability of the Dirichlet problem.

Moments, cont. II

To exhibit analytic dependence (for degree 3 polynomials):

$$f(\zeta;\Omega) = a_0\zeta + a_1\zeta^2 + a_2\zeta^3,$$

$$(\bar{a}_2, \bar{a}_1, a_0, a_1, a_2) \leftrightarrow \Omega \leftrightarrow (\bar{M}_2, \bar{M}_1, M_0, M_1, M_2).$$

$$\begin{cases} M_0 = a_0^2 + 2a_1\bar{a}_1 + 3a_2\bar{a}_2, \\ M_1 = a_0^2\bar{a}_1 + 3a_0a_1\bar{a}_2, \\ M_2 = a_0^3\bar{a}_2. \end{cases}$$

Even in the case of degree 2 polynomials with real coefficients,

$$\begin{cases} M_0 = a_0^2 + 2a_1^2, \\ M_1 = a_0^2a_1, \end{cases}$$

the system is not explicitly invertible (without solving a third degree algebraic equation). If $2M_0^3 > 27|M_1|^2$ there are two solutions (a_0, a_1) of the above system, one of which makes f univalent.

Jacobian for moments

In view of the above, it was a remarkable achievement by **O. Kuznetsova** and **V. Tkachev** to find, in the general polynomial case, an explicit formula for the Jacobi determinant of the system in terms of the **meromorphic resultant** $\mathrm{Res}\,(f', f'^*)$ between f' and f'^*:

Theorem (Kuznetsova, Tkachev, 2004/2005)

$$\frac{\partial(\bar{M}_n, \ldots, M_0, \ldots, M_n)}{\partial(\bar{a}_n, \ldots, a_0, \ldots, a_n)} = 2a_0^{2n+1}\mathrm{Res}\,(f', f'^*)$$

A version of this a formula was conjectured by **C. Ullemar** in 1981.

The resultant $\mathrm{Res}\,(f', f'^*)$ can be defined as the multiplicative action of f'^* on the divisor of f. It vanishes if and only if f' and f'^* have a common zero, i.e., if and only if f' has a zero on $\partial\mathbb{D}$ or a pair of zeros which are reflections of each other with respect to $\partial\mathbb{D}$.

String equation

Mineev-Weinstein *et al* introduce the **Poisson bracket**

$$\{f, g\} = \zeta \frac{\partial f}{\partial \zeta} \frac{\partial g}{\partial M_0} - \zeta \frac{\partial g}{\partial \zeta} \frac{\partial f}{\partial M_0}$$

between two functions, f and g, holomorphic in a neighborhood of $\partial\mathbb{D}$ and depending on $(M_0, M_1, M_2, \dots)$.

Theorem (Mineev-Weinstein, Wiegmann, Zabrodin, 2000-...)

With f univalent in a neighborhood of $\overline{\mathbb{D}}$, the **string equation**

$$\{f, f^*\} = 1$$

holds. Moreover, for suitable **Hamiltonians** $\mathcal{H}_k$*,*

$$\frac{\partial f}{\partial M_k} = \{f, \mathcal{H}_k\}.$$

Recall $f^*(\zeta) = \overline{f(1/\bar{\zeta})}$.

Proof of string equation

Proof.

For $z \in \partial\Omega$, define

$$S(z) = \Big[\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{w}\,dw}{w-z}\Big]_{\text{jump across } \partial\Omega} = S_{\text{ext}}(z) - S_{\text{int}}(z)$$

$$= \sum_{k=0}^{\infty}\frac{M_k}{z^{k+1}} - (\text{function holomorphic in } \Omega) = \sum_{k=-\infty}^{\infty}\frac{M_k}{z^{k+1}}.$$

Each of the two terms extends holomorphically across $\partial\Omega$, hence their difference $S(z)$, the **Schwarz function** for $\partial\Omega$, is holomorphic in a neighborhood of $\partial\Omega$ and satisfies (by a well-known jump formula)

$$S(z) = \bar{z} \quad \text{on } \partial\Omega; \quad \text{consequently}$$

$$S \circ f = f^* \quad \text{identically near } \partial\mathbb{D}.$$

□

Proof of string equation, cont.

Proof (continued).

We have $S(z) = S(z; M_0, M_1, \ldots)$, and by the above

$$\frac{\partial S}{\partial M_0} = \frac{1}{z} + \text{(function holomorphic in } \Omega\text{)}.$$

This gives

$$\{f, f^*\} = \zeta \frac{\partial f}{\partial \zeta} \cdot \left(\frac{\partial S}{\partial M_0} \circ f + \frac{\partial S}{\partial z} \circ \frac{\partial f}{\partial M_0}\right) - \zeta \frac{\partial S}{\partial z} \frac{\partial f}{\partial \zeta} \cdot \frac{\partial f}{\partial M_0}$$

$$= \zeta \frac{\partial f}{\partial \zeta} \cdot \left(\frac{1}{f(\zeta)} + \text{holomorphic in } \mathbb{D}\right) = \text{holomorphic in } \mathbb{D}.$$

For symmetry reasons $\{f, f^*\}$ will be holomorphic also in the exterior (including $\{\infty\}$), hence must be constant, which turns out to be $= 1$. □

Laplacian growth

Note that $\partial f/\partial M_0$ refers to changes of f with $M_1, M_2, \ldots$ kept constant. Thus, with M_0 interpreted as time, say $M_0 = 2t$, the string equation describes **Laplacian growth** (LG). This is the domain evolution by which $\partial\Omega$ moves with **speed** = **harmonic measure** on $\partial\Omega$ = normal derivative of Green's function with pole at origin.

A convenient characterization of LG, which immediately connects to the moment preservation, is

$$\frac{d}{dt}\int_{\Omega(t)} h\, dA = h(0) \quad \forall h \in \mathrm{Harm}(\overline{\Omega(t)}).$$

In physics LG first appeared as the evolution of a *Hele-Shaw* blob of viscous fluid between two plates (experiments 1898 by **H.S. Hele-Shaw**), and the counterpart of the string equation was set up around 1945 by **P.Ya. Polubarinova** and **L.A. Galin**.

Laplacian growth, classical description

For any bounded subdomain $\Omega \subset \mathbb{C}$, let

$$G_\Omega(x, a) = -\frac{1}{2\pi} \log |x - a| + \text{harmonic}$$

be the *Green's function* for Ω, vanishing on $\partial\Omega$. The *harmonic measure* with respect to a is

$$d\nu = -\frac{\partial G_{\Omega(t)}(\cdot, a)}{\partial n}\, ds \quad \text{on } \partial\Omega$$

Definition

Laplacian growth (=**motion by harmonic measure)** can be defined as a smooth evolution $\Omega(t)$ such that

$$\frac{d}{dt} \int_{\Omega(t)} \varphi\, dA = -\int_{\partial\Omega(t)} \varphi \frac{\partial G_{\Omega(t)}(\cdot, a)}{\partial n} ds$$

for every test function φ in $\mathbb{C}$.

Non-univalent case

Questions:

- Does the string equation make sense for non-univalent functions? And if so, does it hold?
- Global question: if $(M_0, M_1, M_2, \dots)$ are the moments for some $f(\zeta)$, can M_0 be increased indefinitely? That is, do the moments $(M_0 + 2t, M_1, M_2, \dots)$ correspond to some $f(\zeta, t)$ for all $t > 0$?

  This is equivalent to asking for a global, simply connected but possibly non-univalent, LG evolution.

Partial answers:

- By Kuznetsova-Tkachev theorem: $\frac{\partial f}{\partial M_k}$ has a natural meaning for polynomials f of any fixed degree, whenever $\mathrm{Res}\,(f', f'^*) \neq 0$.
- In general, $\partial f/\partial M_k$ is ambiguous because f will depend on more parameters than $(M_0, M_1, M_2, \dots)$.

Non-univalent functions. Difficulties

Yu-Lin Lin and B.G. (arXiv:1411.1909) study the string equation in cases when f is not locally univalent. Difficulties:

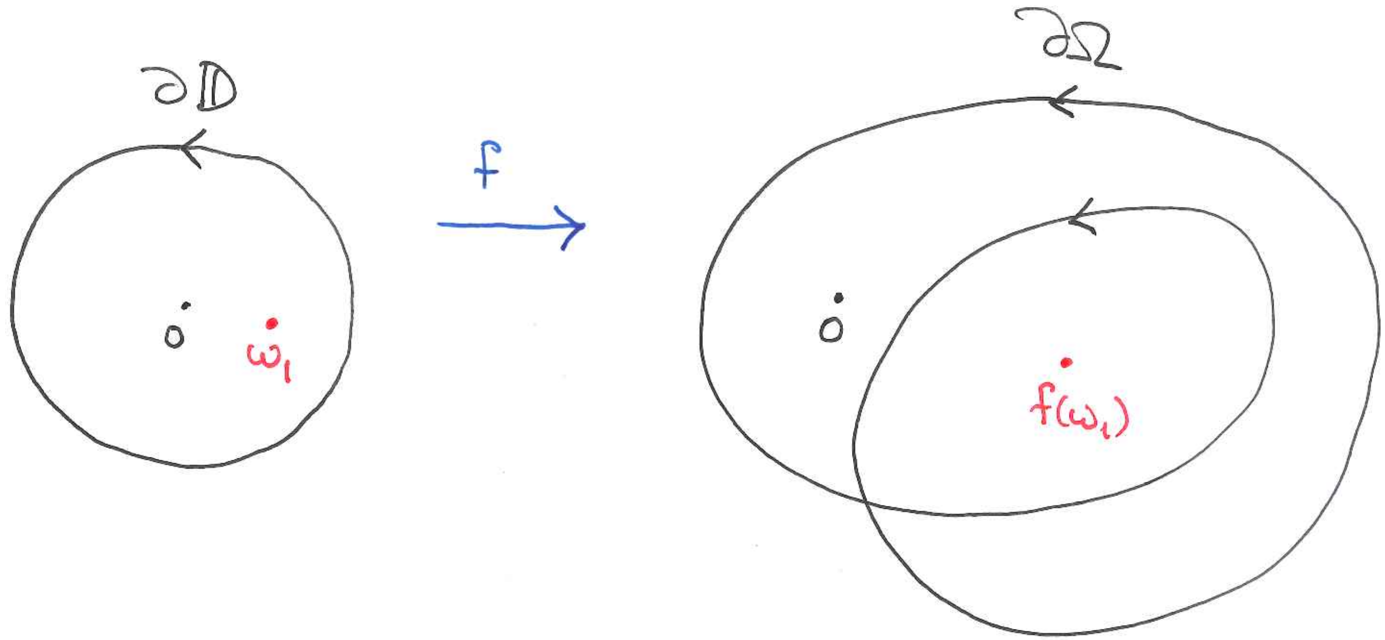
- $\partial f/\partial M_0$ does not make obvious sense, because f is not determined by the moments $M_0, M_1, M_2, \dots$ alone. One has to find more parameters. It turns out that

$$f(\zeta) = f(\zeta; M_0, M_1, M_2, \dots; b_1, b_2, \dots),$$

  is a good choice, where $b_k = f(\omega_k)$ and $\omega_1, \omega_2, \dots$ are the zeros of f' inside $\mathbb{D}$. In fact, f can be viewed as a conformal map onto a **Riemann surface** above $\mathbb{C}$, which then has **branch points** over $b_1, b_2, \dots$. Keeping these fixed means that f all the time maps into one and the same Riemann surface.

  Equivalent to keeping the b_k fixed is:

- $f(\zeta; M_0, \dots)$ is a **subordination chain** with respect to M_0.



f depends on location of branch point $f(\omega_1) =: b_1$
$(\omega_1 \in \mathbb{D},\ f'(\omega_1) = 0)$. So

$$f = f(\xi;\, M_0, M_1, \dots;\, b_1)$$

LG directional derivative

The string equation can be solved for $\partial f/\partial M_0$:

$$\{f, f^*\} = 1 \quad \Longleftrightarrow \quad \frac{\partial f}{\partial M_0} = \frac{\zeta f'(\zeta)}{4\pi i} \int_{\partial\mathbb{D}} \frac{1}{|zf'(z)|^2} \frac{z+\zeta}{z-\zeta} \frac{dz}{z},$$

for $t \in \mathbb{D}$. The right member,

$$\nabla(0)f(\zeta) := \frac{\zeta f'(\zeta)}{4\pi i} \int_{\partial\mathbb{D}} \frac{1}{|zf'(z)|^2} \frac{z+\zeta}{z-\zeta} \frac{dz}{z} \quad (\zeta \in \mathbb{D}),$$

is the **LG directional derivative**, and it makes sense also if f' has zeros in $\mathbb{D}$ (even though zeros on $\partial\mathbb{D}$ cause some troubles).

Identifying M_0 with **time** t we arrive at the **LG evolution equation**

$$\dot{f} = \nabla(0)f,$$

which we set out to solve for $0 \leq t < \infty$, given f at $t = 0$.

Weak solutions

Laplacian growth makes sense on Riemannian manifolds, and there is a **good notion of weak solution** that is global (allows $t \to \infty$).

- But this is **not always simply connected**, and hence is not always on the form $f(\mathbb{D})$. If we insist on having a solution on the form $\Omega(t) = f(\mathbb{D}, t)$ we must allow $\Omega(t)$ to spread on a Riemann surface above $\mathbb{C}$.
- However, the Riemann surface we would need will **not be given in advance**, it has to be created along with the solution. Whenever a zero of f' approaches $\partial\mathbb{D}$, one has to add a branch point to make sure that the solution can spread on a covering surface.
- Still, we claim that the above problems can be handled:

Main theorem (?)

Theorem (almost. . .)

Starting with any function

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{n+1} \quad (a_0 > 0,\ \limsup |a_n|^{1/n} < 1)$$

there is a global evolution, satisfying $\dot{f} = \nabla(0) f$ *in a weak sense. More precisely, there exists a Riemann surface* $\mathcal{M}$ *and a covering map* $p : \mathcal{M} \to \mathbb{C}$ *such that for each* t, $f(\cdot, t) \to \mathbb{C}$ *lifts to*

$$\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M}$$

and then becomes univalent. The image domains $\tilde{\Omega}(t) = \tilde{f}(\mathbb{D}, t)$ *make up a global weak LG evolution on* $\mathcal{M}$*.*

The evolution is not unique, but presumably there is a unique minimal choice, introducing no more branch points than necessary.

Example of evolution on Riemann surface

Example

$\mathcal{M}$ = Riemann surface of $\sqrt{z-1}$ = the two-sheeted surface

$$\mathcal{M} = (\mathbb{C} \setminus \{1\}) \cup \{1\} \cup (\mathbb{C} \setminus \{1\})$$

over $\mathbb{C}$. Local coordinate (actually global) on $\mathcal{M}$: $\tilde{z} = \sqrt{z-1}$.
Covering map: $p : \mathcal{M} \to \mathbb{C}$, $\tilde{z} \mapsto z = \tilde{z}^2 + 1$.

Laplacian growth $\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M}$ started at $\tilde{z} = +i$ becomes

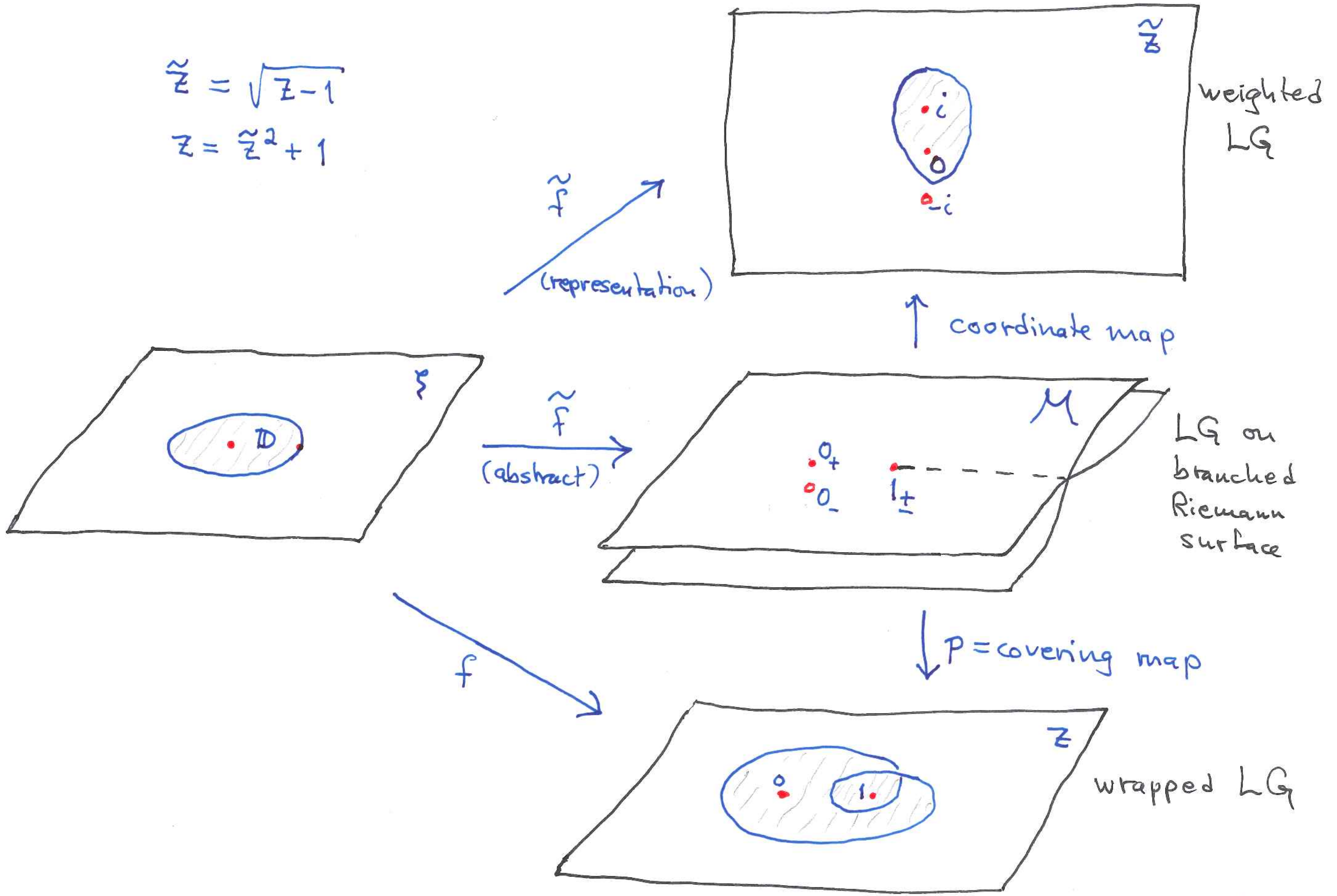
$$\tilde{z} = \tilde{f}(\zeta, t) = \begin{cases} \sqrt{t\zeta - 1}, & (0 < t < 1) \\ \sqrt{\frac{t(t\zeta-1)^2}{\zeta - t}} & (1 < t < \infty) \end{cases}$$

and when pushed down to $\mathbb{C}$

$$z = f(\zeta, t) = \begin{cases} t\zeta, & (0 < t < 1) \\ \frac{\zeta(t^3\zeta - 2t^2 + 1)}{\zeta - t} & (1 < t < \infty) \end{cases}$$

$$\tilde{z} = \sqrt{z-1}$$

$$z = \tilde{z}^2 + 1$$



Example, cont. I

Example

The derivative

$$f'(\zeta, t) = \begin{cases} t & (0 < t < 1) \\ t \cdot \frac{(t\zeta - 1)(t\zeta - 2t^2 + 1)}{(\zeta - t)^2} & (1 < t < \infty) \end{cases}$$

adopts the factor $G(\zeta) = \frac{(t\zeta - 1)(t\zeta - 2t^2 + 1)}{(\zeta - t)^2}$ at critical time $t = 1$. This has

- Zeros: $\omega_1 = 1/t$ (in $\mathbb{D}$), $\omega_2 = 2t - 1/t$ (outside).
- Poles: $\zeta_1 = \zeta_2 = t$.

With suitable scaling G is a **contractive zero divisor** (in the sense of H. Hedenmalm) for Bergman space. This means for example that

$$h(0) = \int_{\mathbb{D}} h(z)|G(z)|^2 dA(z) \quad \forall h \in \mathrm{Harm}(\overline{\mathbb{D}}).$$

Example, cont. II

Example

For a more general G, of the form

$$G(\zeta) = \frac{(\zeta-\omega_1)(\zeta-\omega_2)}{(\zeta-\zeta_1)^2},$$

one has an identity

$$\frac{1}{\pi}\int_{\mathbb{D}} h(z)|G(z)|^2 dA(z) = a_0 h(0) + a_1 h(1/\bar{\zeta}_1) + c\int_0^{1/\bar{\zeta}_1} hG\, d\zeta.$$

If here $1/\bar{\zeta}_1 = \omega_1$ (or $= \omega_2$) then $a_1 = 0$, and if $\zeta_1 = \frac{1}{2}(\omega_1+\omega_2)$, then $c = 0$. This is exactly what we had on the previous slide, and it is what happens in general in the LG-evolution when zeros of f' penetrate into $\mathbb{D}$: a pair of zeros and a double pole, subject to the above relations, are created.

Several evolutions of a cardioid I

Example

We start LG with

$$f(\zeta, 0) = \zeta - \frac{1}{2}\zeta^2,$$

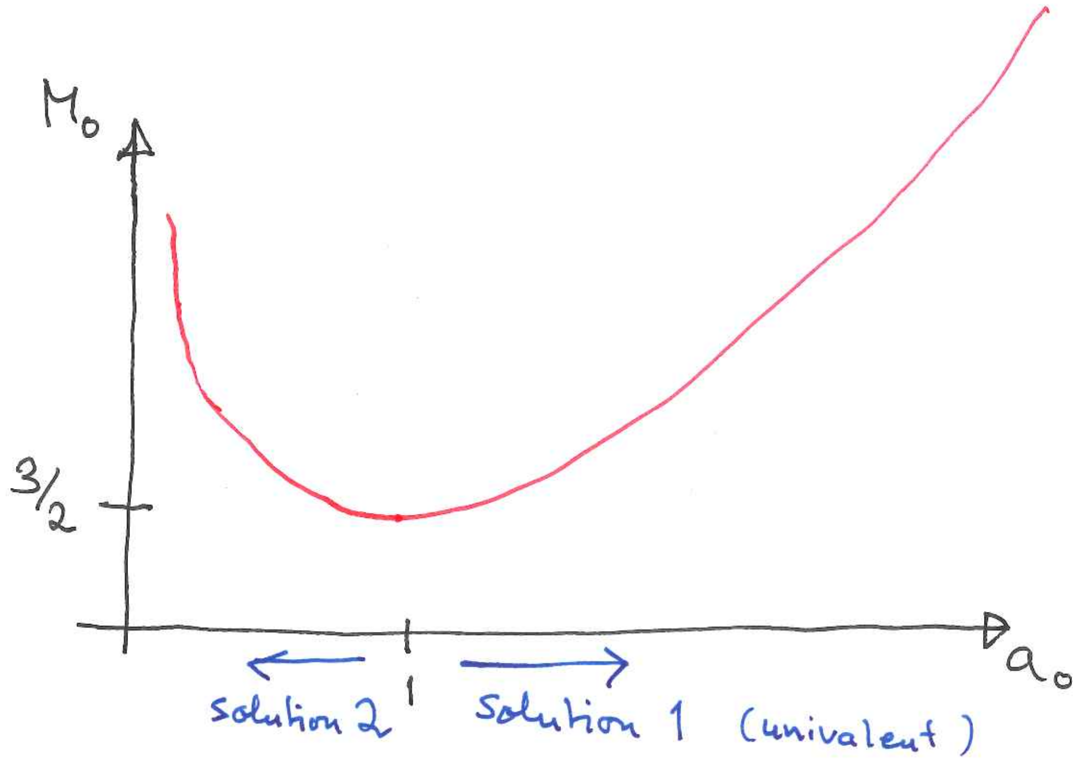
for which $M_0 = 3/2$, $M_1 = -1/2$, $M_2 = M_3 = \cdots = 0$. For convenience we allow a free (monotone) relation between time t and M_0. Normalizing so that the leading coefficient in f is e^t we have a perfectly good global LG-solution

$$f(\zeta, t) = e^t\zeta - \frac{1}{2}e^{-2t}\zeta^2, \quad 0 < t < \infty,$$

for which $M_1, M_2, \dots$ remain fixed and

$$M_0 = a_0^2 + 2|a_1|^2 = e^{2t} + \frac{1}{2}e^{-4t}.$$

Cardioid : $f(\xi) = a_0 \xi + a_1 \xi^2 = a_0 \xi - \frac{1}{2a_0^2} \cdot \xi^2$



$$M_0 = a_0^2 + \frac{1}{2} \frac{1}{a_0^4}$$

Both solutions represent injection ($M_0 \nearrow$)

Several evolutions of a cardioid II

Example

M_0 is a convex function of t for all $-\infty < t < \infty$ and it attains its minimum value at $t = 0$. Therefore M_0 increases also when t decreases from $t = 0$, and we get a new LG evolution by changing sign of t:

$$f(\zeta, t) = e^{-t}\zeta - \frac{1}{2}e^{2t}\zeta^2, \quad 0 < t < \infty.$$

This is however not univalent, f' has a zero $\omega_1(t) = e^{-3t}$ in $\mathbb{D}$, but f still satisfies the string equation. Also, $b_1(t) = f(\omega_1(t), t)$ does not stay fixed, so $f(\cdot, t)$ are not conformal maps into a fixed Riemann surface.

Several evolutions of a cardioid III

Example

Since $f'(\zeta, 0) = 1 - \zeta$ has a zero $\omega_1 = 1$ on $\partial\mathbb{D}$ one might want to lift solutions to a Riemann surface with a branch point over $f(\omega_1, 0) = \frac{1}{2}$, in order to make sure that one does not run into troubles. We then let

$$f'(\zeta, 0) = (1 - \zeta)\,\frac{(\zeta - 1)(\zeta - 1)}{(\zeta - 1)^2}$$

continue as

$$f'(\zeta, t) = b(t)\,\frac{(\zeta - \omega_1(t))(\zeta - \omega_2(t))(\zeta - \omega_3(t))}{(\zeta - \zeta_1(t))^2},$$

with the zeros and poles related according to certain principles:

Several evolutions of a cardioid IV

Example

- The reflected point of $\zeta_1(t)$ is to be a zero of f':

$$g(1/\zeta_1(t), t) = 0.$$

- $f(\cdot, t)$ shall map the above point $1/\zeta_1(t)$ to a point which does not move:

$$f(1/\zeta_1(t), t) = \text{constant} = f(1, 0) = \frac{1}{2}.$$

- The moment M_1 is conserved in time:

$$M_1(t) = \mathrm{Res}_{\zeta=0}(ff^* f' d\zeta) = M_1(0) = -\frac{1}{2}.$$

- The dependence of $M_0(t)$ on t has to be specified.

Several evolutions of a cardioid V

Example

The above may be worked out to a solution

$$f(\zeta, t) = \frac{b_1\zeta + b_2\zeta^2 + b_3\zeta^3}{\zeta_1 - \zeta}$$

where

$$\begin{cases} \zeta_1(t) = \sqrt{\frac{1}{2}(1 + 2e^t - e^{-2t})}, \\ b_1(t) = e^t, \\ b_2(t) = -\frac{1}{4\sqrt{2}}(1 + 2e^t + 3e^{-2t})\sqrt{1 + 2e^t - e^{-2t}}, \\ b_3(t) = \frac{1}{4}(2e^{-t} + e^{-2t} - e^{-4t}). \end{cases}$$

The relation between M_0 and t is here

$$M_0(t) = \frac{1}{8}(4e^{2t} + 2e^t + e^{-2t} + 6e^{-3t} + 2e^{-4t} - 3e^{-6t}).$$

Several evolutions of a cardioid VI

Remark

An interesting aspect is that the above fully explicit solution $f(\zeta, t)$ is not only smooth at $t = 0$, it even has a real analytic continuation across $t = 0$. This extended solution, defined on $-\varepsilon < t < \infty$ (say), has the drawback that it has a pole inside $\mathbb{D}$ when $t < 0$). But in some sense it still represents suction out of the cardioid as t decreases to negative values.

The Huntingford example

Example (A real polynomial of degree three)

$$\begin{aligned} f(\zeta,t) &= a_0(t)\zeta + a_1(t)\zeta^2 + a_2(t)\zeta^2 \\ &= e^t\zeta + \frac{M_1\zeta^2}{e^{2t}+3e^{-2t}M_2} + e^{-3t}M_2\zeta^3, \end{aligned}$$

with t suitably related to M_0 (see below) and with M_1, M_2 fixed. Solution exists for all $-\infty < t < \infty$ if $M_2 \geq 0$. Relations:

$$\begin{cases} M_0 = a_0^2 + 2a_1^2 + 3a_2^2, \\ M_1 = a_0^2 a_1 + 3a_0 a_1 a_2, \\ M_2 = a_0^3 a_2. \end{cases}$$

The choice

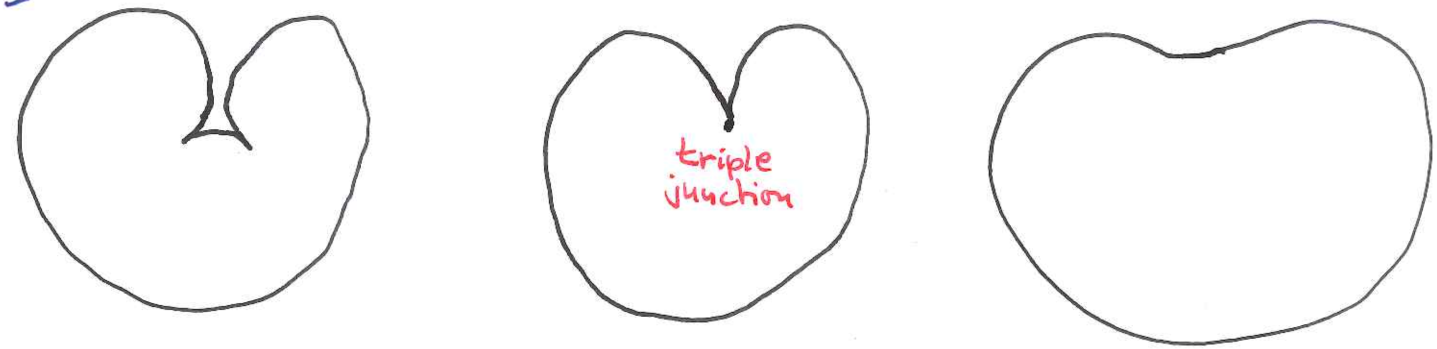
$$\begin{cases} M_1 = 32/25, \\ M_2 = 1/5 \end{cases}$$

is particularly interesting.

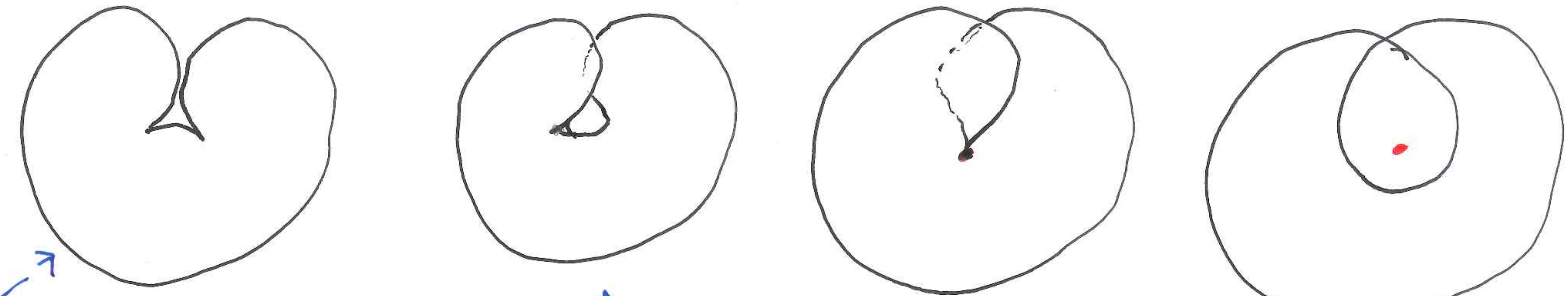
Hunting ford:

$$f(\xi) = a_0\xi + a_1\xi^2 + a_2\xi^3 = a_0\xi + \frac{M_1\xi^2}{a_0^2 + 3a_0^{-2}M_2} + a_0^{-3}M_2\xi^3 .$$

Critical case: $M_1 = 32/25,\ M_2 = 1/5.$ (M_1, M_2 fixed)



time → ($a_0 \nearrow$)



time →

Overcritical case:

↑ loss of univalence

↑ loss of local univalence

↑ continuation on branched Riemann surf.

The complex Hamiltonians $\mathcal{H}_k$

On using

$$M_k = \frac{1}{2\pi i}\int_{\partial\Omega} z^k \bar{z} dz,$$

the harmonic moments M_k make sense for all $k \in \mathbb{Z}$. The complete expansion of $S(z)$ and its primitive $W(z)$ (with respect to z) reads

$$S(z; M_0, M_1, \dots) = \sum_{k\in\mathbb{Z}} \frac{M_k}{z^{k+1}}.$$

$$W(z; M_0, M_1, \dots) = M_0 \log z - \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{M_k}{kz^k} - \frac{1}{\pi}\int_\Omega \log|z|\, dA.$$

Now the k:th **Hamiltonian**, for $k = 1, 2, \dots$, is

$$\mathcal{H}_k(\zeta; M_0, M_1, \dots) = -\frac{\partial W(z; M_0, M_1, \dots)}{\partial M_k},$$

where $z = f(\zeta; M_0, M_1, \dots)$ is kept fixed under the differentiation.

The complex Hamiltonians $\mathcal{H}_k$, cont.

For $k = 0$ we simply set

$$\mathcal{H}_0(\zeta; M_0, M_2, \dots) = -\log \zeta.$$

Now it is a straight-forward task to verify the evolution equations for the conformal map:

$$\frac{\partial f}{\partial M_k} = \{f, \mathcal{H}_k\} \quad k = 0, 1, 2, \dots.$$

For the negative moments we mention the beautiful relations

Theorem (Mineev-Weinstein, Wiegmann, Zabrodin)

$$\frac{1}{k}\frac{\partial M_{-k}}{\partial M_j} = \frac{1}{j}\frac{\partial M_{-j}}{\partial M_k} \quad (k, j \geq 1).$$

Note that $M_{-k} = M_{-k}(\Omega) = M_{-k}(M_0, M_1, \dots)$.

The complex Hamiltonians $\mathcal{H}_k$, cont.

The theorem can be proved by exhibiting a **prepotential** $\mathcal{E}(\Omega) = \mathcal{E}(M_0, M_1, \dots)$ such that

$$\frac{\partial \mathcal{E}(\Omega)}{\partial M_k} = \frac{1}{k} M_{-k}.$$

In fact, one can take

$$\mathcal{E}(\Omega) = -\frac{1}{\pi^2} \int_{B\setminus\Omega} \int_{B\setminus\Omega} \log |z - \zeta| \, dA(z) dA(\zeta)$$

(B a large disk for example), the **renormalized energy** for the exterior domain, in some contexts referred to as (the logarithm of) a **"tau"-function**, which is a kind of partition function in mathematical models in statistical mechanics.

Traditional Hamiltonian point of view

Introducing time t as $M_0 = 2t$ and setting $x + iy = f(e^{i\theta}, t)$ on the curve, the string equation reads, considering x, y as functions of t, θ,

$$\frac{\partial(x,y)}{\partial(t,\theta)} = 1, \quad \text{equivalently} \quad dt \wedge d\theta = dx \wedge dy.$$

In **extended phase space** with coordinates (x, y, t) one may introduce $H = \theta$ as the **Hamiltonian** function. The **action** 1-form is

$$\omega = y\,dx - H\,dt,$$

and the string equation says that $d\omega = 0$ along a Laplacian growth trajectory. This means that $i(\xi)d\omega = 0$ for the vector field $\xi = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{t}\frac{\partial}{\partial t}$, and spelling this out gives the traditional Hamilton equations, expressing stationarity of action, as

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{t} = 1.$$

Hamiltonian point of view, cont.

Recall now that $H = \theta$ is the polar angle on $\partial\mathbb{D}$. It extends to a (multi-valued) harmonic function in $\mathbb{D}$, and the conjugate harmonic function is $G = -\log r$, the **Green's function** of $\mathbb{D}$. In terms of G the equations become

$$\dot{x} = -\frac{\partial G}{\partial x}, \quad \dot{y} = -\frac{\partial G}{\partial y}, \quad \dot{t} = 1.$$

Since the Green's function is conformally invariant one can also interpret the above as equations on $\partial\Omega$.

The last equation just says that $t =$ time, and first two describe an evolution of $\partial\Omega$ with speed $-\nabla G$, as expected.

The Hele-Shaw directional derivative

Instead of considering the partial derivative $\partial/\partial M_0$, one may consider the corresponding **directional derivative**, $\nabla(0)$. More generally, let $\nabla(a)$ denote the directional derivative corresponding to Laplacian growth with a source at $a \in \mathbb{C}$. This can be regarded as a tangent vector in the infinite-dimensional space $\mathcal{M}$ of bounded domains $\Omega \subset \mathbb{R}^n$ with analytic boundary, and it acts on smooth functionals

$$\mathcal{F} : \mathcal{M} \to \mathbb{R}$$

as follows. Given $\Omega \in \mathcal{M}$, let $\{\Omega(t) : 0 \leq t < \varepsilon\}$ be the Hele-Shaw evolution with a source at a and initial domain $\Omega(0) = \Omega$. Then

$$(\nabla(a)\mathcal{F})(\Omega) := \frac{d}{dt}\bigg|_{t=0} \mathcal{F}(\Omega(t)).$$

The Hele-Shaw directional derivative, example

Example

Let h be a fixed harmonic function defined in a neighbourhood of the closure of Ω and define $\mathcal{F}$ by

$$\mathcal{F}(\Omega) = \int_\Omega h\,dA.$$

Then

$$(\nabla(a)\mathcal{F})(\Omega) = \frac{d}{dt}\bigg|_{t=0} \int_{\Omega(t)} h\,dA = -\int_{\partial\Omega} h(z)\frac{\partial G_\Omega(z,a)}{\partial n}ds = h(a).$$

Hadamard

Example

Choose two points $b, c \in \Omega$ and take $\mathcal{F}(\Omega)$ to be

$$\mathcal{F}(\Omega) = G_\Omega(b, c).$$

Using the *Hadamard variational formula*

$$\delta G(b, c) = \int_{\partial\Omega} \delta n \frac{\partial G(\cdot, b)}{\partial n} \frac{\partial G(\cdot, c)}{\partial n} ds$$

with $\delta n = -\frac{\partial G(\cdot, a)}{\partial n} \delta t$, i.e., in the case of Hele-Shaw injection at a, one gets a beautiful formula, completely symmetric in a, b and c:

$$\nabla(a) G(b, c) = -\int_{\partial\Omega} \frac{\partial G(\cdot, a)}{\partial n} \frac{\partial G(\cdot, b)}{\partial n} \frac{\partial G(\cdot, c)}{\partial n} ds.$$

Integrability of the Dirichlet problem

The above symmetry can be related to an integrability statement: Hele-Shaw injection at a point *a* followed by injection at a point *b* gives the same result as if the injections had been performed in the opposite order. In infinitesimal form this statement becomes

$$\nabla(a)\nabla(b) = \nabla(b)\nabla(a).$$

Therefore, the symmetry in (16) would be explained by exhibiting a functional $\mathcal{E}$ such that

$$G_\Omega(b, c) = \nabla(b)\nabla(c)\mathcal{E}(\Omega) + \text{constant}.$$

Such a functional indeed exists and can be realized as a renormalized energy of the complementary domain. In the case $n = 2$ this is (up to an additive constant, and with $\Omega \subset B$, B fixed)

$$\mathcal{E}(\Omega) = \frac{1}{4\pi}\int_{B\setminus\Omega}\int_{B\setminus\Omega} \log|z-\zeta|\, dA(z)dA(\zeta).$$

Prepotential (energy functional)

Theorem

The result of single and repeated actions of ∇ on $\mathcal{E}(\Omega)$ is given by

- $$\nabla(a)\mathcal{E}(\Omega) = -\frac{1}{2\pi}\int_{B\setminus\Omega} \log|z-a|\, dA(z) = U^{B\setminus\Omega}(a).$$
- $$\nabla(a)\nabla(b)\mathcal{E}(\Omega) = H(a,b) = G(a,b) + \frac{1}{2\pi}\log|a-b|.$$
- $$\nabla(a)\nabla(b)\nabla(c)\mathcal{E}(\Omega) = \nabla(a)G(b,c).$$

Here $G(z,\zeta) = \frac{1}{2\pi}\log|z-\zeta| + H(z,\zeta)$, i.e., $H(z,\zeta)$ is the regular part of the Green function. Note that the term $\frac{1}{2\pi}\log|a-b|$ above is just a "constant" with respect to variations of Ω.

Proof

In the definition (36) of $\mathcal{E}(\Omega)$ the domain Ω occurs symmetrically at two places. Thus, the result of $\nabla(a)$ acting on Ω in $\mathcal{E}(\Omega)$ will be twice the result of its action on only one of the occurencies. This gives

$$\nabla(a)\mathcal{E}(\Omega) = \frac{1}{2\pi}\frac{d}{dt}\bigg|_{t=0} \int_{\mathbb{D}_R\setminus\Omega}\int_{\mathbb{D}_R\setminus\Omega(t)} \log|z-\zeta|\, dA(\zeta)dA(z)$$

$$= \frac{1}{2\pi}\int_{\mathbb{D}_R\setminus\Omega}\int_{\partial\Omega} \log|z-\zeta| \frac{\partial G(\zeta,a)}{\partial n} ds_\zeta dA(z)$$

$$= -\frac{1}{2\pi}\int_{\mathbb{D}_R\setminus\Omega} \log|z-a|\, dA(z) = U^{\mathbb{D}_R\setminus\Omega}(a).$$

Here we used that $\log|z-\zeta|$ is harmonic in Ω in the variable ζ when $z \in \mathbb{D}_R \setminus \Omega$. This proves (17).

Proof, cont

Applying next $\nabla(b)$ to the above and using that $G(z,a)=0$ on $\partial\Omega$ we obtain

$$\nabla(b)\nabla(a)\mathcal{E}(\Omega) = -\frac{1}{2\pi}\int_{\partial\Omega}\log|z-a|\frac{\partial G(z,b)}{\partial n}ds_z$$

$$= \int_{\partial\Omega}(G(z,a)-H(z,a))\frac{\partial G(z,b)}{\partial n}ds_z$$

$$= H(b,a) = G(b,a) + \frac{1}{2\pi}\log|b-a|.$$

This gives (17) (with reversed roles of a and b), and (17) finally is immediate by another application of ∇.

Thanks

Thank you for your attention!