Geometry and Large N limits in Laughlin states

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[SK, JHEP'13], [Ferrari-SK, JHEP'14], [SK-Wiegmann, PRL'15],
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Laughlin state

On the plateaus QHE is described by collective Laughlin state

$$\Psi(z_1,\dots,z_N)=\prod_{i<j}^{N}(z_i-z_j)^\beta e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \beta\in\mathbb{Z}_+$$

[Laughlin'83]

$\beta=1$: Integer QHE, non-interacting electrons.
$\beta=3,5,7,..$: Fractional QHE, interacting (via Coulomb forces) system.
Hall conductance $\sigma_H=1/\beta$.
Other candidate states were proposed for other plateaus (e.g. Pfaffian state $\mathrm{Pf}\frac{1}{z_i-z_j}\Delta(z)^2$).
Mathematically, the Laughlin state defines a sequence of probability measures μ_N on the configuration space $\mathbb{C}^N/S_N$ of N point-particles

$$\mu_N=|\Psi(z_1,\dots,z_N)|^2\prod_{j=1}^{N}d^2z_j$$

Main problem: partition function

The partition function (L^2 norm of Laughlin state)

$$Z = \int_{\mathbb{C}^N} \prod_{i<j}^{N} |z_i - z_j|^{2\beta} e^{-\frac{B}{2}\sum_i |z_i|^2} \prod_{j=1}^{N} d^2 z_j$$

Central object in Log-gases (Coulomb gas, random matrix β-ensemble).

Main goal: Define Laughlin state(s) $\Psi_r(z_1, .., z_N)$, $r = 1, .., \beta^{\mathrm{g}}$ ([Wen-Niu'90]) on a genus-g Riemann surface Σ with arbitrary geometry: metric g, complex structure J, inhomogeneous magnetic field B, flat connections moduli $A = \frac{\varphi}{\tau-\bar{\tau}} d\bar{z} - \frac{\bar{\varphi}}{\tau-\bar{\tau}} dz$, $\int_a A = \varphi_1$, $\int_b A = \varphi_2$. (solenoid fluxes, $Pic(\Sigma)$) φ etc. Determine the partition function $Z = \sum_r \langle \Psi_r, \Psi_r \rangle^2_{L^2}$ as a function of these geometric parameters

$$Z = Z[g, J, B, \varphi, ...]$$

in the limit of large number of particles.

Why: geometric adiabatic transport

Main idea behind this is *geometric adiabatic transport* [Thouless et.al.; Avron, Seiler, Simon, Zograf, ...]. Laughlin states on a Riemann surface (Σ, g, J) form a vector bundle over the parameter space Y (e.g. moduli space of flat connections $Y = Jac(\Sigma)$ or complex structure moduli $Y = \mathcal{M}_{\mathrm{g}}$). Let d_Y be an exterior derivative along the parameter space. Then adiabatic (Berry) connection and curvature are

$$\mathcal{A} = \langle \Psi, d_Y \Psi \rangle_{L^2}, \quad \mathcal{R} = d_Y \mathcal{A} = -d_Y \bar{d}_Y \log Z.$$

- Hall conductance σ_H is a Chern number of this vector bundle over $Y = Jac(\Sigma)$ (flat connections moduli) [Thouless et.al.'82'85, Tao-Wu'85, Avron-Seiler'85, Avron-Seiler-Zograf'94]
- Anomalous Hall viscosity η_H, associated with the adiabatic transport on the moduli space of a torus $\mathcal{M}_1$. IQHE: [Avron-Seiler-Zograf'95, Levay'95], FQHE: [Tokatly-Vignale'07, Read'09]
- Transport on higher genus. IQHE: [Levay'97], FQHE [SK-Wiegmann'15].

Results: Partition function for IQHE

$$\log Z = \frac{1}{2\pi}\int_\Sigma \left[A_z A_{\bar z} + \frac{1-2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar z}\right] d^2z$$
$$+ \mathcal{F}[B,R].$$

$$\mathcal{F}[B,R] = -\frac{1}{2\pi}\int_\Sigma \left[\frac{1}{2}B\log B + \frac{2-3s}{12}R\log B + \frac{1}{24}(\log B)\Delta_g(\log B)\right]\sqrt{g}d^2z$$
$$+ \mathcal{O}(1/B).$$

[SK'13; SK-Ma-Marinescu-Wiegmann'15]

This holds for any genus, and is related to Bergman kernel expansion. Terminology: $A_z, A_{\bar z}$ are components of the gauge-connection 1-form for the magnetic field $F = dA$. Also $\omega_z, \omega_{\bar z}$ are components of spin-connection $\omega_z = i\partial\log g_{z\bar z}$, where $\mathrm{Ric}(g) = d\omega$, and $s \in \mathbb{Z}/2$ is gravitational spin. R is scalar curvature.

Results: relation to Chern-Simons theory

Note that generating functional $\log Z$ and 3d Chern-Simons (Wen-Zee) actions look similar:

$$\log Z = \frac{1}{2\pi}\int_\Sigma \left[A_z A_{\bar z} + \frac{1-2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar z}\right] d^2 z$$

$$S_{CS} = \frac{1}{4\pi}\int_{\Sigma\times R}\left[AdA + (1-2s)Ad\omega + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega d\omega\right]$$

These two actions are obviously very similar, but one is 2d and another one is in 3d. What is the precise relation?

Results: relation to Chern-Simons theory

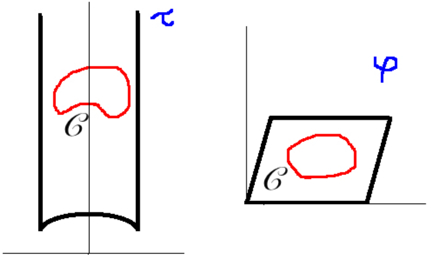
Consider geometric adiabatic transport of IQHE wave function along a contour $\mathcal{C}$ in the moduli space $Y = Pic(\Sigma) \times \mathcal{M}_g$

Define adiabatic connection:

$$\mathcal{A}_Y = \langle \Psi, d_Y \Psi \rangle_{L^2},$$

and adiabatic phase:

$$e^{i \int_\mathcal{C} \mathcal{A}_Y}.$$



Theorem ([SK-Ma-Marinescu-Wiegmann'15]):

$$\int_\mathcal{C} \mathcal{A}_Y = \frac{1}{4\pi} \int_{\Sigma_Y \times \mathcal{C}} \left[AdA + (1-2s)Ad\omega + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega d\omega \right]$$

(proof relies on Quillen-Bismut theory: stay for next talk by Xiaonan Ma for the proof)

Results: Partition function for Laughlin states

$$\log Z = \frac{1}{2\pi\beta}\int_\Sigma \left[A_z A_{\bar z} + \frac{\beta - 2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(\beta-2s)^2}{4} - \frac{\beta}{12}\right)\omega_z\omega_{\bar z}\right] d^2z + \mathcal{F}[B,R].$$

$$\mathcal{F}[B,R] = -\frac{1}{2\pi}\int_\Sigma \left[\frac{2-\beta}{2\beta} B\log B\right]\sqrt{g}d^2z + \dots$$

[Can-Laskin-Wiegmann'14; Ferrari-SK'14, SK-Wiegmann'15]

Lowest Landau level (LLL) and IQHE state

Consider compact connected Riemann surface (Σ, g, J) and positive holomorphic line bundle (L^k, h^k). The latter corresponds to the magnetic field. The curvature $(1,1)$ form of the hermitian metric $h^k(z,\bar{z})$ is given by $F = -i\partial\bar{\partial}\log h^k$. This is the magnetic field strength of total flux k though the surface $\frac{1}{2\pi}\int_\Sigma F = k$. Magnetic field: $B = g^{z\bar{z}}F_{z\bar{z}}$. On the plane and for constant magnetic field $B = k$, this corresponds to $h^k = e^{-\frac{B}{2}|z|^2}$. LLL wave functions

$$\bar{\partial}_{L^k}\psi = 0$$

are holomorphic sections of L^k,

$$\psi_i = s_i(z), \quad i = 1, \ldots, N_k = \dim H^0(\Sigma, L^k) = k + 1 - \mathrm{g}$$

IQHE state: take N_k points on Σ: $z_1, z_2, \ldots, z_{N_k}$. The (holomorphic part F of the) IQHE state is Slater determinant:

$$F(z_1, \ldots, z_{N_k}) = \det[s_i(z_j)]_{i,j=1}^{N_k}$$

Definition of Laughlin state (FQHE)

Consider now line bundle $(L^{\beta k}, h^{\beta k})$. But number of points is still $N_k = k + 1 - \mathrm{g}$, i.e. only fraction of LLL states is occupied (thus *fractional* QHE). The (holomorphic part F of the) Laughlin state satisfies

- $F(z_1, ..., z_{N_k})$ is completely anti-symmetric
- Fix all z_j except one, say z_m. Then $F(\cdot, .., \cdot, z_m, \cdot, .., \cdot)$ is a holomorphic section of $L^{\beta k}$.
- Vanishing condition near diagonal $z_i \sim z_j$ in local complex coordinate system on Σ,

$$F(z_1, ..., z_{N_k}) \sim \prod_{i<j} (z_i - z_j)^{\beta}$$

.

Examples

1. Round sphere S^2, constant magnetic field: $h_0^k = \frac{1}{(1+|z|^2)^k}$ Then $s_j = z^{j-1}$, $j = 1, ..k+1$. FQHE state: $F(z_1, ..., z_{k+1}) = \prod_{i<j}(z_i - z_j)^\beta$

$$|\Psi(z_1, ..., z_{k+1})|^2 = \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{j=1}^{k+1} h^{\beta k}(z_j)$$

[Haldane'83]

2. Flat torus, constant magnetic field: $s_j = \theta_{\frac{j}{k},0}(kz + \varphi, k\tau), j = 1, \ldots, k$, $h_0^k = e^{-2\pi i \frac{(z-\bar{z})^2}{\tau - \bar{\tau}}}$. FQHE states:

$$F_r(z_1, .., z_k) = \theta_{\frac{r}{\beta},0}(\beta z_c + \varphi, \beta\tau) \prod_{i<j} \left(\frac{\theta_1(z_i - z_j, \tau)}{\eta(\tau)} \right)^\beta$$

[Haldane-Rezayi'85]

3. Higher genus $\Sigma_{\mathrm{g}>1}$: β^{g} states.

Arbitary metric and inhomogeneous magnetic field

The advantage of the language of holomorphic line bundles is that it gives us a clear idea how to put the Laughlin state on Σ with arbitrary metric g and inhomogeneous magnetic field B. Consider some fixed (constant scalar curvature) metric g_0, and constant magnetic field B_0 (and corresponding hermitian metric $h_0^k(z,\bar{z})$). Arbitrary metrics are parameterized by:

- Kähler potential $\phi(z,\bar{z})$: $g_{z\bar{z}} = g_{0\bar{z}z} + \partial_z \bar{\partial}_{\bar{z}} \phi$,
- "magnetic" potential $\psi(z,\bar{z})$: $F = F_0 + \partial\bar{\partial}\psi$, $B = g^{z\bar{z}} F_{z\bar{z}}$

Partition function

For the integer QHE ($\beta = 1$), the partition function on arbitrary Σ is

$$Z = \int_{\Sigma^{N_k}} |\det s_i(z_j)|^2 \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) e^{-k\psi(z_j,\bar{z}_j)} \sqrt{g}^{1-s}(z_j) d^2 z_j$$

For the fractional QHE

$$Z = \sum_{r=1}^{n_\beta} \int_{\Sigma^{N_k}} |F_r(z_1, .., z_{N_k})|^2 \prod_{j=1}^{N_k} h_0^{\beta k}(z_j, \bar{z}_j) e^{-k\beta\psi(z_j,\bar{z}_j)} \sqrt{g}^{1-s}(z_j) d^2 z_j.$$

Derivation of $\log Z$ in IQHE

For $\beta = 1$ the partition function satisfies determinantal formula:

$$Z = \int_{\Sigma^{N_k}} |\det s_i(z_j)|^2 \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) e^{-k\psi(z_j, \bar{z}_j)} \sqrt{g}(z_j) d^2 z_j$$
$$= \det\langle s_i, s_j \rangle_{L^2}$$

Denoting $G_{jl} = \langle s_j, s_l \rangle$, we get

$$\delta \log Z = \delta \, \mathrm{Tr} \, \log\langle s_j, s_l \rangle =$$
$$= -\frac{1}{2\pi} \sum_{j,l} G^{-1}_{lj} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g \delta\phi) + k\delta\psi \right) \bar{s}_j s_l h^k \sqrt{g}^{1-s} d^2 z$$
$$= -\frac{1}{2\pi} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g B_k(z, \bar{z})) \, \delta\phi + k B_k(z, \bar{z}) \, \delta\psi \right) \sqrt{g}^{1-s} d^2 z,$$

where $B_k(z, \bar{z})$ is the Bergman kernel on diagonal.

Bergman kernel

B_k is the Bergman kernel on the diagonal. For orthonormal basis of sections $\{s_j\}$:

$$B_k(z,\bar z) = \sum_{i=1}^{N_k} ||s_i||^2_{h^k} =$$
$$= B + \frac{1-2s}{4} R + \frac{1}{4}\Delta_g \log B + \frac{1}{12}\Delta_g (B^{-1} R) + \mathcal{O}(1/k^2)\,.$$

[Zelditch'98, Catlin'99]

In QM, Bergman kernel is the density of states ψ_i on "completely filled" LLL

$$B_k(z,\bar z) = \sum_{i=1}^{N_k} |\psi_i(z)|^2 = \lim_{T\to\infty} \int_{x(0)=z}^{x(T)=z} e^{-\int_0^T (\dot x^2 + A\dot x) dt} \mathcal{D}x(t)$$

[Douglas, SK'09]

$\log Z$ in Integer QHE

$$\log Z = \frac{1}{2\pi}\int_\Sigma \left[A_z A_{\bar z} + \frac{1-2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \right.$$
$$\left. + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar z}\right] d^2 z + \mathcal{F}[B,R].$$

Liouville action is a hallmark of gravitational (conformal) anomaly. CFT partition function transforms within the conformal class $g = e^{2\sigma} g_0$

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = -\frac{c}{12\pi} S_L(\sigma) = -\frac{c}{24\pi}\int_\Sigma \omega_z\omega_{\bar z} d^2 z$$

where c is central charge. What we derived is Laughlin state is the mixed electromagnetic-gravitational anomaly

Since the theory is not conformal (there is a scale, magnetic area: $l^2 \sim V/k$) we now have infinite asymptotic expansion.

$\log Z$ for Laughlin states: derivation

The proof is based on the free field representation of Laughlin states

$$\sum_r^{n_\beta} |\Psi_r|^2 = \int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_{N_k})} e^{-\frac{1}{2\pi}S(g,X)} \mathcal{D}_g X \qquad \text{[Moore-Read'91]}$$

where sum goes over all degenerate Laughlin states on Riemann surface and the free field action is

$$S = \int_M \left(\partial X \bar{\partial} X + i \frac{\beta - 2s}{\sqrt{\beta}} X R \sqrt{g} + \frac{i}{\sqrt{\beta}} A \wedge dX \right)$$

for compactified boson: $X \sim X + 2\pi\sqrt{\beta}$.
Novelty: "background charge" $Q = \frac{\beta - 2s}{\sqrt{\beta}}$, gauge connection coupling.

$\log Z$ for Laughlin states: derivation

Step 1. The "anomalous part" of the expansion comes from transformation properties under the deformation of the metric and the magnetic field $g_0 \to g = g_0 + \partial_z \partial_{\bar{z}} \phi, A_0 \to A = A_0 + \partial \psi$,

$$\int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_{N_k})} e^{-\frac{1}{2\pi}S(g,X)} \mathcal{D}_g X$$
$$= e^{S_{\text{ano}}} \int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_{N_k})} e^{-\frac{1}{2\pi}S(g_0,X)} \mathcal{D}_{g_0} X$$

Step 2. The remainder term $\mathcal{F}[R, B]$ of the expansion of $\log Z$ comes from the interacting path integral

$$\frac{1}{\Gamma(s)} \int_0^\infty d\mu \, \mu^{s-1} \int e^{-\frac{1}{2\pi}S(g,X) - \mu \int_M e^{i\sqrt{\beta}X(z)} \sqrt{g} d^2 z} \mathcal{D}_g X \,,$$

at $s = -N_k$.

[Ferrari-SK(JHEP2014)]

$\log Z$ for Laughlin states

$$\log Z = \frac{1}{2\pi\beta}\int_\Sigma \left[A_z A_{\bar z} + \frac{\beta - 2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(\beta-2s)^2}{4} - \frac{\beta}{12}\right)\omega_z\omega_{\bar z}\right] d^2z$$
$$- \frac{1}{2\pi}\int_\Sigma \left[\frac{2-\beta}{2\beta} B\log B\right]\sqrt{g}d^2z + ...$$

Laughlin states form a non-trivial vector bundle of rank β^{g} over $Y = Jac(\Sigma)\times\mathcal{M}_{\mathrm{g}}$ (In particular, Ψ_r have non-trivial monodromies under S, T transformations on torus.)
Yet, For Laughlin states, adiabatic connection and curvature on $Y = Jac(\Sigma)\times\mathcal{M}_{\mathrm{g}}$ the are controlled by the partition function, due to the property $\Psi_r(y,\bar y) = \frac{F_r(y)}{\sqrt{Z}(y,\bar y)}$.

$$\mathcal{A}_{rs} = \langle F_r, d_y F_s\rangle_{L^2} = (\partial_y \log Z)\delta_{rs}$$
$$\mathcal{R}_{rs} = (i d_y d_{\bar y}\log Z)\delta_{rs}$$

New transport coefficient on higher genus

Consider complex structure deformations $g_{z\bar{z}}|dz|^2 \to g_{z\bar{z}}|dz + \mu d\bar{z}|^2$, where Beltrami differential is $\mu = g_{z\bar{z}}^{-1} \sum_{\kappa=1}^{3g-3} \eta_\kappa \delta y_\kappa$ and η_κ is a basis of holomorphic quadratic differentials.
Berry curvature, associated with these deformations is

$$\mathcal{R} = i d_y d_{\bar{y}} \log Z = \left(\frac{1}{4} k\beta + \frac{c_H}{12}\chi(M)\right) \Omega_{WP},$$

where $\Omega_{WP} = i \int_M d_y \mu \wedge d_{\bar{y}} \bar{\mu}\; g_{z\bar{z}} d^2 z$ is the Weil-Petersson form on the moduli space. Here

$$c_H = 1 - 3\frac{(\beta - 2s)^2}{\beta}$$

is a new transport coefficient, transpiring on higher genus surfaces, since on torus $\chi(M) = 0$.

[SK-Wiegmann'15]

Thank you