Quantum Hall effect and Quillen metric

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Lowest Landau level

- Σ compact (connected) Riemann surface,
- L a holomorphic line bundle on Σ.
  $L|_U \simeq U \times \mathbb{C}$, transition functions are holomorphic.
- States on the lowest Landau level : basis of $H^0(\Sigma, L)$, space of holomorphic sections of L on Σ.
  $H^0(\Sigma, L) = \{s \in \mathcal{C}^\infty(\Sigma, L) : \overline{\partial}_L s = 0\}$.
  Locally : $\overline{\partial}_L(f\zeta) = \frac{\partial f}{\partial \overline{z}} \zeta d\overline{z}$, ζ holomorphic frame of $L|_U$.
- collective wave function of free electrons on LLL :
  Integer quantum-Hall state $\det(s_i(z_j))$, here s_i a basis of $H^0(\Sigma, L)$.
  LLL is completed filled : N particles (fermions) at position $z_1, \cdots, z_N$.

Lowest Landau level II

- Example : $\mathscr{O}(1)$ hyperplane line bundle on $\Sigma = \mathbb{CP}^1$. $\mathscr{O}(-1) = \{([z], \xi) \in \mathbb{CP}^1 \times \mathbb{C}^2 : \xi \in [z]\}$.
- $\{f \in \mathbb{C}[z] : \operatorname{grad} f \leqslant k\} \cong \{g \in \mathbb{C}[z_0, z_1] : \operatorname{grad} g = k\}$ $\cong H^0(\mathbb{P}^1, \mathcal{O}(k))$
- $\det(s_i(z_j)) = \prod_{0<i<j\leq k}(z_j - z_i)$,
- If $L|_U \simeq U \times \mathbb{C}$, $|1|^2_{h^L} = e^{-2\phi}$, then Laughlin state : $|\det(s_i(z_j))|^2 = \prod_{0<i<j\leq k}|z_j - z_i|^2 \prod_j e^{-2\phi(z_j)}$

Riemann-Roch Theorem

- Σ compact Riemann surface with genus $\mathbf{g}$.
- $|1|^2_{h^L} = e^{-2\phi}$ on $L|_U$. (Magnetic field) : Curvature of Chern connection ∇^L :

$$R^L = (\nabla^L)^2 = 2\partial\overline{\partial}\phi.$$

  Global $(1,1)$-form on Σ, even ϕ is only defined on U.
- degre of L : $\deg L := \int_\Sigma \frac{i}{2\pi} R^L \in \mathbb{Z}$.
  $K := T^{*(1,0)}\Sigma$ canonical line bundle on Σ.
- Riemann-Roch Theorem :

$$\dim H^0(\Sigma, L) - \dim H^0(\Sigma, L^* \otimes K) = \deg L + 1 - \mathbf{g}.$$

  If $\deg L > \deg K = 2\mathbf{g} - 2$, then $H^0(\Sigma, L^* \otimes K) = 0$.

Generating functional

- (Σ, g) compact Riemann surface with genus $\mathbf{g}$.
  (L, h) Hertimiam holomorphic line bundle over Σ.
- s_i basis of $H^0(\Sigma, L)$, Hermitian norm of $\frac{1}{N!}\det(s_i(z_j))$

$$|\Psi(z_1, \ldots, z_N)|^2 := \frac{1}{N!}|\det s_j(z_l)|^2 \prod_{j=1}^{N} h(z_j, \bar{z}_j).$$

  (measure on configuration of N points.)
- partition function is

$$Z = \frac{1}{(2\pi)^N N!}\int_{\Sigma^N} |\det s_i(z_j)|^2 \prod_{j=1}^{N} h(z_j, \bar{z}_j)\sqrt{g} d^2 z_j,$$

  and $\log Z$ is called the generating functional.

Quillen metric

- determinantal formula :

$$Z = \det \frac{1}{2\pi} \int_\Sigma \bar{s}_j(\bar{z}) s_l(z) h \sqrt{g} d^2 z = \det \langle s_j, s_l \rangle.$$

  $\log Z - \log Z_0$ independ. choice of basis in $H^0(\Sigma, L)$.
- Kodaira Laplacian :
  $\Delta_{\mathrm{L}} = 2\bar{\partial}_L^* \bar{\partial}_L : \mathcal{C}^\infty(\Sigma, L) \to \mathcal{C}^\infty(\Sigma, L)$.
- regularized spectral determinant :

$$\det{}' \Delta_{\mathrm{L}} = \exp(-\zeta'(0)), \quad \zeta(s) = \sum_{0 \neq \lambda \in spec(\Delta_{\mathrm{L}})} \lambda^{-s}$$

- For $\deg L > 2(\mathbf{g} - 1)$, Quillen metric on $\det H^0(\Sigma, L)$:

$$\| s_1 \wedge \cdots \wedge s_N \|^2 = \frac{\det \langle s_j, s_l \rangle}{\det' \Delta_{\mathrm{L}}} = \frac{Z}{\det' \Delta_{\mathrm{L}}},$$

Dolbeault complex

- X compact complex manifold, $n = \dim X$.
- E a holomorphic vector bundle on X.
- $\overline{\partial}_E : \Omega^{0,q}(X, E) := \mathcal{C}^\infty(X, \Lambda^q(T^{*(0,1)}X) \otimes E) \to \Omega^{0,q+1}(X, E)$ the Dolbeault operator :

$$\overline{\partial}_E(\sum_j \alpha_j \xi_j) = \sum_j (\overline{\partial}\alpha_j)\xi_j.$$

ξ_j local holomorphic frame of E, and $\alpha_j \in \Omega^{0,q}(X)$.

$$(\overline{\partial}_E)^2 = 0.$$

- Dolbeault cohomology of X with values in E :

$H^q(X, E) := H^{(0,q)}(X, E) := \ker(\overline{\partial}_E|_{\Omega^{0,q}}) / Im(\overline{\partial}_E|_{\Omega^{0,q-1}})$.

Determinant line $\lambda(E) = \otimes_j (\det H^j(X, E))^{(-1)^j}$

Quillen metric

- $D = \overline{\partial}_E + \overline{\partial}_E^*$. ($\sqrt{2}D$= Dirac operator)
  Hodge Theory : $H^\bullet(X,E) \simeq \operatorname{Ker} D$.
- $\zeta(s) = -\frac{\partial}{\partial s}\sum_q (-1)^q q \operatorname{Tr}[((D^2|_{\Omega^{0,q}})')^{-s}]$.
  $T = \exp(\frac{\partial \zeta}{\partial s}(0))$ Ray-Singer analytic torsion (1973).
- Quillen metric $\| \quad \|_{\lambda(E)} := \| \quad \|^{L^2}_{\lambda(E)} \exp(\frac{\partial \zeta}{\partial s}(0))$
- Bismut-Gillet-Soulé anomaly formula (1988) :
  g_0^X, g_1^X Kähler metrics, h_0^E, h_1^E Hermitian metrics on E

$$\log\left(\frac{\| \quad \|_{\lambda(E),0}}{\| \quad \|_{\lambda(E),1}}\right)^2 = \int_X \widetilde{\operatorname{Td}}(g_0^X, g_1^X)\operatorname{ch}(E, h_0^E) + \operatorname{Td}(TX, g_1^X)\widetilde{\operatorname{ch}}(h_0^E, h_1^E).$$

Determinant line bundle of cohomology

- W, S compact complex manifolds.
  $\pi : W \to S$ holomorphic submersion with compact fiber X.
- E a holomorphic vector bundle on W.
- $\lambda(E)_b = \otimes_j (\det H^j(X_b, E|_{X_b}))^{(-1)^j}$, $b \in S$.
  Grothendieck-Knusden-Mumford : $\lambda(E)$ holomorphic line bundle on S, (algebraic construction)

Curvature formula of Bismut-Gillet-Soulé

- Bismut-Gillet-Soulé (1988) : Assume that $\pi : W \to S$ is locally Kähler, i.e., for any local chart $U \subset S$, there is a Kähler form on $\pi^{-1}(U)$. Then
- $\| \quad \|_{\lambda(E)} := \| \quad \|^{L^2}_{\lambda(E)} \exp(\frac{\partial \zeta}{\partial s}(0))$ is smooth metric on $\lambda(E) = \otimes_j (\det H^j(X, E|_X))^{(-1)^j}$ over S.
- Curvature Ω of Chern connection on $(\lambda(E), \| \quad \|_{\lambda(E)})$

$$\Omega = -2\pi i \Big\{ \int_{W|S} \big[\mathrm{ch}(E, h^E)\, \mathrm{Td}(TW|S, h^{TW|S}) \big] \Big\}^{(1,1)}.$$

Chern class

- h^E Hermitian metric on E, ∇^E Chern (holomorphic and Hermitian) connection on (E, h^E), its curvature
$$R^E = (\nabla^E)^2 \in \Omega^{(1,1)}(W, \mathrm{End}(E)).$$
- $$\mathrm{Td}(E, h^E) = \det\left(\frac{R^E/2\pi i}{e^{R^E/2\pi i} - 1}\right),$$
$$\mathrm{ch}(E, h^E) = \mathrm{Tr}\left[e^{-R^E/2\pi i}\right].$$
They are closed forms in $\oplus_q \Omega^{(q,q)}(W)$.
- $\mathrm{Td}(E) := [\mathrm{Td}(E, h^E)], \mathrm{ch}(E) := [\mathrm{ch}(E, h^E)] \in \oplus_q H^q_{dR}(W)$ do not depend on h^E. $\mathrm{Td}(E)$ Todd class of E, $\mathrm{ch}(E)$ Chern class of E.

Bott-Chern class

- h_0^E, h_1^E Hermitian metrics on E,
- $\exists$ unique classes $\widetilde{\mathrm{Td}}(h_0^E, h_1^E), \widetilde{\mathrm{ch}}(h_0^E, h_1^E)$ $\in \oplus_q \Omega^{(q,q)}(W)/\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$ s. t.

$$\frac{i\partial\bar{\partial}}{2\pi}\widetilde{\mathrm{Td}}(h_0^E, h_1^E) = \mathrm{Td}(E, h_0^E) - \mathrm{Td}(E, h_1^E),$$

$$\frac{i\partial\bar{\partial}}{2\pi}\widetilde{\mathrm{ch}}(h_0^E, h_1^E) = \mathrm{ch}(E, h_0^E) - \mathrm{ch}(E, h_1^E).$$

Let h_t^E a path of metrics, then

$$\widetilde{\mathrm{ch}}(h_0^E, h_1^E) := \int_0^1 \mathrm{Tr}\left[(h_t^E)^{-1}\frac{\partial h_t^E}{\partial t}\exp\left(\frac{iR_t^E}{2\pi}\right)\right]dt.$$

Quillen

- Quillen's result (1985) : E smooth complex vector bundle over a compact Riemann surface Σ.
  $\mathcal{A}$ holomorphic structure on E is affine space w.r.t. $\Omega^{0,1}(\Sigma, \mathrm{End}(E))$. Curvature formula for $W = \Sigma \times \mathcal{A} \to S = \mathcal{A}$
- Physics : Belavin-Knizhnik (1986)
  - Avron-Seiler-Zograf 1994 and 1995 ! - first noticed relation of adiabatic transport in QHE and Quillen theory
  - Levay 1997

Torus : Ray-Singer (1973)

- $\Sigma = \mathbb{C}/\Gamma$ torus, $\Gamma = lattice\{1, \tau\}$, $\tau \in \mathbb{H}$ upper half-plane.
  $E = \mathbb{C}$ trivial smooth line bundle on Σ.
- holomorphic structure on E parametrized by

$$Jac(\Sigma) = \{\chi : \pi_1(\Sigma) = \Gamma \to S^1 : \\ \chi(m\tau + n) = e^{2\pi i(mu+n\tau)}, 0 \leq u, v < 1\}.$$

- If $u \neq 0$ or $v \neq 0$, then $H^0(\Sigma, E_\chi) = 0$,

$$T = \left| e^{\pi i v^2 \tau} \frac{\theta_1(u - \tau v, \tau)}{\eta(\tau)} \right|.$$

  If $u = v = 0$, then $H^0(\Sigma, E_\chi) = \mathbb{C}$,

$$T = (\mathrm{Im}\tau) \left| \eta(\tau) \right|^2.$$

Universal line bundle

- Σ compact oriented surface of genus $\mathbf{g}$.
  $Y = \mathcal{M}_{\mathbf{g}} \times Jac(\Sigma)$.
  Σ_b the Riemann surface at $b \in \mathcal{M}_{\mathbf{g}}$.
  X = union of all $\Sigma_b \times Jac(\Sigma_b)$ over $\mathcal{M}_{\mathbf{g}}$ (universal curve),
  $\sigma : X \to Y$ natural holomorphic projection of complex manifolds.
- Observation : $\sigma : X \to Y$ is a Kähler fibration !
  We can apply Bismut-Gillet Soulé curvature formula.
- L universal line bundle of deg $L|_{\Sigma} = 1$ over X,
  $K = T^{*(1,0)}\Sigma$ universal canonical line bundle over X.
  For $k > 0$, $s \in \frac{1}{2}\mathbb{Z}$, $E = L^k \otimes K^s$ hol. line bundle on X.

BGS curvature formula

- Assume $k + 2(\mathbf{g} - 1)(s - 1) > 0$. Then $\mathcal{L} = \det H^0(\Sigma, L^k \otimes K^s)$ hol. line bundle over Y.
- g^Σ, h^L any metrics on $T^{(1,0)}\Sigma, L$.
- BGS curvature formula : $F = R^{L^k}$,

$$\Omega^{\mathcal{L}} = \frac{i}{4\pi} \int_{X|Y} \Big[\mathrm{F} \wedge \mathrm{F} + (1 - 2s)\, \mathrm{F} \wedge \mathrm{R}^{TX|Y} + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \mathrm{R}^{TX|Y} \wedge \mathrm{R}^{TX|Y} \Big].$$

Chern-Simons functional : Main result

- $\mathrm{F} = dA_{(X)}$ and $\mathrm{R}^{TX|Y} = d\omega_{(X)}$ for 1-forms $A_{(X)}$ and $\omega_{(X)}$ on X.
- $\Omega^{\mathcal{L}} = d_Y \mathcal{A}^{\mathcal{L}}$.

  Then we choose an adiabatic process, i.e., a smooth (open or closed) contour $\mathcal{C}$ in Y.
- Klevtsov-Ma-Marinescu-Wiegmann

$$\int_{\mathcal{C}} \mathcal{A}^{\mathcal{L}} = \frac{1}{4\pi} \int_{\sigma^{-1}(\mathcal{C})} A \wedge dA + \frac{1-2s}{2}(A \wedge d\omega + dA \wedge \omega) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega \wedge d\omega \,.$$

This is abelian Chern-Simons action with Wen-Zee terms.