AdS/CFT and Geometric Aspects of Chern-Simons Theories

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Geometric Aspects of QHE
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QHE in the canonical Holography

QHE and Holography

Motivation

$$\text{Quantum} \qquad \longleftrightarrow \qquad \text{Geometry}$$

Black Holes

Black holes in AdS-space

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)\mathrm{d}t^2 + \sum_i \mathrm{d}x_i^2 + \frac{\mathrm{d}z^2}{f(z)}\right), \qquad f(z) = 1 - \frac{z^d}{z_h^d}$$

In AdS_4

$$T = \frac{3}{4\pi z_h}$$

$$S = \frac{L^2 \Delta x \Delta y}{4G z_h^2} \qquad \text{– Bekenstein-Hawking entropy}$$

- Black holes are thermodynamical systems
- TD quantities are typically defined for an infinitely remote observer

Black holes

Charge density

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \sum_i dx_i^2 + \frac{dz^2}{f(z)}\right)$$

Charge density/Chemical potential $\longleftrightarrow$ bulk gauge field

$$A_0 = \mu - \langle\rho\rangle z^{d-2}, \qquad \mu = \langle\rho\rangle z_h^{d-2}, \qquad f(z) = 1 - \left(1 + Q^2\right)\frac{z^d}{z_h^d} + Q^2\frac{z^{2d-2}}{z_h^{2d-2}}$$

In AdS_4

$$T = \frac{3 - Q^2}{4\pi z_h}$$

Magnetic Field

Magnetic field [Hartnoll,Kovtun'07]

Dyonic *AdS* black hole

$$\frac{\mathrm{d}s^2}{L^2} = \frac{\alpha^2}{z^2}\left(-f(z)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{\mathrm{d}z^2}{z^2 f(z)}$$

$$A_t = \mu - \frac{\rho}{\chi\alpha} z, \qquad A_x = -By$$

Regularity at the horizon $A_t = 0$ relates $\mu = \mu(\rho)$

$$T = \alpha\,\frac{3 - Q^2}{4\pi}, \qquad Q^2 = \frac{\rho^2 + \chi^2 B^2}{\chi^2\alpha^4} \qquad S = \frac{L^2}{4G}\,\pi\alpha^2\,\Delta x\Delta y$$

- Extremal ($T = 0$) black hole: $(\rho^2 + B^2) z_h^4 = 3$ has $S \neq 0$
- Quantity $\chi = \frac{L^2}{4G}$ can be related to the central charge c of the dual CFT

Transport

Green's functions [Hartnoll,Kovtun'07]

Find the response of the system to a small perturbation of electric field and temperature gradient. The holographic prescription for calculation of correlators gives the following for the retarded Green's functions:

- for 2 currents $\langle[\mathcal{J}_i(t), \mathcal{J}_j(0)]\rangle_R$

$$G^R_{ij}(\omega) = -i\omega\epsilon_{ij}\,\frac{\rho}{B}$$

  By Kubo formula

$$\sigma_{ij} = -\lim_{\omega\to 0}\frac{\operatorname{Im} G^R_{ij}(\omega)}{\omega} = \epsilon_{ij}\sigma_H\,, \qquad \sigma_H = \frac{\rho}{B}$$

  σ_{ij} is antisymmetric, but not quantized. ρ and B so far independent

- for $\langle[\mathcal{J}_i(t), \mathcal{T}_{tj}(0)]\rangle_R$ and $\langle[\mathcal{T}_{ti}(t), \mathcal{T}_{tj}(0)]\rangle_R$

$$G^R_{i\pi_j}(\omega) = -i\omega\epsilon_{ij}\,\frac{3\varepsilon}{2B}\,, \qquad G^R_{\pi_i\pi_j}(\omega) = \frac{\chi s^2T^2\, i\omega\delta_{ij}}{\rho^2+\chi^2B^2} - \frac{9\rho\varepsilon^2\, i\omega\epsilon_{ij}}{4B\left(\rho^2+\chi^2B^2\right)}$$

Transport

Thermal conductivities [Hartnoll et al'07][DM,Orazi,Sodano'12]

Low temperature expansions of the conductivities yield

$$\alpha_{xx} = \alpha_{yy} = 0\,, \qquad \alpha_{xy} = -\alpha_{yx} = \frac{s}{B} = \frac{\pi}{\sqrt{3}}\sqrt{1+\sigma_H^2} + O(T)$$

$$\kappa_{xx} = \kappa_{yy} = \frac{\chi s^2 T}{\rho^2 + \chi^2 B^2} \to \chi\frac{\pi^2}{3}T + O(T^2)$$

$$\kappa_{xy} = -\kappa_{yx} = \frac{\rho s^2 T}{B(\rho^2 + \chi^2 B^2)} \to \frac{\pi^2}{3}\sigma_H T + O(T^2)$$

Wiedemann-Franz law

$$\frac{\kappa_H}{\sigma_H} = \mathcal{L}T\,, \qquad \mathcal{L} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2$$

Edges

If the system has an edge [Takayanagi'11]

$$S = \frac{1}{2\kappa}\int_N \mathrm{d}^{d+1}x\,\sqrt{-g}\,(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial N}\mathrm{d}^d x\,\sqrt{-h}K + S_{\partial N}[\text{matter}]$$

h_{ab}-induced metric on ∂N, K-extrinsic curvature, $K_{ab} = h^\mu_a h^\nu_b \nabla_\mu n_\nu$

$$\delta S = \frac{1}{2\kappa}\int_{\partial N}\mathrm{d}^d x\,\sqrt{-h}\,(K_{ab} - K h_{ab} + \Sigma h_{ab} - T_{ab})\,\delta h^{ab}$$

Neumann boundary conditions [Compere,Marolf'08]

$$K_{ab} - (K-\Sigma)h_{ab} = T_{ab}$$

Edges

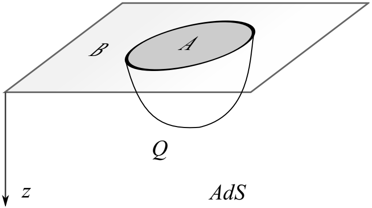
Same for the gauge fields [Fujita,Kaminski,Karch'12][DM,Orazi,Sodano'12]

$$c_1 \int_N \mathrm{d}^4x\, \sqrt{-g} F_{\mu\nu}^2 + c_2 \int_N F \wedge F + k \int_Q A \wedge F - k \int_P \mathrm{d}^2x\, A_x A_t$$

Neumann boundary conditions imply

$$c_1 F + (c_2 + k) * F|_Q = 0$$

- Density and magnetic field are locked together $\rho \sim B$



Top-down AdS/CFT

Models of QHE

- Keski-Vakkuri, Kraus'08; Davis, Kraus, Shah'08 Construction of an effective Chern-Simons theory. D-brane theory of plateau transitions
- Fujita, Ryu, Takayanagi'09 D-brane engineering of an effective Chern-Simons theory in low dimensions. Model massless edge modes and stripes of states with different ν. Proposal hierarchical FQHEs, using IIA string on C^2/Z_n
- Bergman, Jokela, Lifschytz, Lippert'10 D-brane engineering. Model a gapped system with massless edge modes. Quantization of conductivity as a result of quantization of a flux through a compact manifold. Irrational filling fractions

Low Dimensional AdS/CFT

AdS$_3$ and Chern-Simons

3d Gravity

$$S = \frac{1}{8\pi G}\int \mathrm{d}^3x\,\sqrt{-g}(R-\Lambda) \qquad \Lambda = -\frac{2}{\ell^2}$$

Vacuum solution – anti de Sitter space (*e.g.* in global coordinates)

$$\frac{\mathrm{d}s^2}{\ell^2} = \mathrm{d}\rho^2 - \sinh^2\rho\,\mathrm{d}t^2 + \cosh^2\rho\,\mathrm{d}\phi^2$$

The (2+1)d gravity is said to be topological: any two metrics are related by a diffeomorphism, but some of those are non-trivial (large gauge transforms) and lead to new physical solutions:

$$t \to 2\sqrt{M}t\,, \qquad \phi \to 2\sqrt{M}\phi\,, \qquad \rho \to \rho - \frac{1}{2}\log M$$

$$\frac{\mathrm{d}s^2}{\ell^2} = \mathrm{d}\rho^2 - \left(e^\rho - Me^{-\rho}\right)^2\mathrm{d}t^2 + \left(e^\rho + Me^{-\rho}\right)^2\mathrm{d}\phi^2$$

AdS_3 and Chern-Simons

CFT connection

Restrict to diffeomorphisms preserving the boundary condition

$$\frac{\mathrm{d}s^2}{\ell^2} = \mathrm{d}\rho^2 + \frac{1}{4}\,\mathrm{e}^{2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\phi^2\right) + O(\rho^0)$$

Large gauge transformations change the subleading asymptotic modifying physical charges (mass H and angular momentum J). The charge components generate 2 copies of the Virasoro algebra

$$\{Q_m, Q_n\} = (n-m)Q_{n+m} + \frac{c}{12}\,n(n^2-1)\delta_{n+m,0} \qquad c = \frac{3\ell}{2G}$$

[Brown,Henneaux'86]

AdS_3 and Chern-Simons

3d Gravity as Chern-Simons [Witten'88]

$$A = \omega + \frac{1}{\ell}\, e \qquad \bar{A} = \omega - \frac{1}{\ell}\, e$$

$$S = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}]$$

where $A, \bar{A}$ are $SL(2,R)$-valued flat connections

for $SL(N,R) \times SL(N,R)$ one also obtains higher spin fields $s \leq N$

$$g_{\mu\nu} = \mathrm{Tr}\ \left(e_\mu e_\nu\right) \qquad \phi_{\mu\nu\rho} = \mathrm{Tr}\ \left(e_{(\mu} e_\nu e_{\rho)}\right)$$

Flat connections are mapped to solutions of Einstein eqs. Gauge transforms become diffeos

AdS_3 and Chern-Simons

Black holes from flat connections

Gauge transformation ($w = t + i\phi$) $\qquad L_0, L_{\pm 1} \in sl(2)$

$$A = b^{-1} a b + b^{-1}\,\mathrm{d}b \qquad b = \exp(-L_0 \rho) \qquad a = a_w dw + a_{\bar{w}} d\bar{w}$$

If one chooses

$$a_w = L_1 + M L_{-1}\,, \qquad \bar{a}_{\bar{w}} = L_{-1} + M L_1$$

one gets

$$\frac{\mathrm{d}s^2}{\ell^2} = \mathrm{d}\rho^2 - \left(e^{\rho} - M e^{-\rho}\right)^2 \mathrm{d}t^2 + \left(e^{\rho} + M e^{-\rho}\right)^2 \mathrm{d}\phi^2$$

AdS_3 and Chern-Simons

CFT connection revisited [Gaberdiel,Gopakumar'10]

- $SL(2,R) \times SL(2,R)$ Chern-Simons corresponds to a Virasoro CFT

  [Verlinde'89][Witten'91]

- $SL(N,R) \times SL(N,R)$ Chern-Simons (higher spin $s \leq N$ theory) coupled to a set of matter fields corresponds to a W_N CFT
- for a generic non-integer N – higher spin (Vasiliev) theory

AdS/CFT can test this conjecture in the $c \to \infty$ limit

AdS_3/CFT_2

Matching with CFT spectrum [Castro *et al*'11][Perlmutter,Prochazka,Raeymaekers'12]

- In the 't Hooft limit, $N \to \infty$ ($c \to \infty$) and $\lambda = N/(k+N)$ fixed the spectrum of the CFT side is not well understood
- Alternative is the *semiclassical* limit, $c \to \infty$, but $N = -\lambda$ fixed (non-unitary)

In the non-unitary regime one can match the spectra. There is a descrete set of gauge connections of $SL(N, C)$ with trivial holonomies around the contractible cycle.

- spectrum of M matches the $(0, \Lambda_-)$ irreps of minimal model CFT's (heavy states)
- fluctuations around those connections produce the spectrum of (Λ_+, Λ_-)

AdS_3/CFT_2

Entaglement entropy [Ryu,Takayanagi'06]

Holographic formula for computing entanglement entropy

$$S_{\mathrm{EE}}(A) = \frac{\mathrm{Area}(\gamma(A))}{4G}\,, \qquad \gamma(A)\ -\ \text{minimal area surface}$$

In AdS_3 it reproduces the known CFT_2 result [Calabrese,Cardy'04]

$$S_{\mathrm{EE}} = \frac{c}{6}\log\frac{\sqrt{\epsilon^2 + x^2/4} + x/2}{\sqrt{\epsilon^2 + x^2/4} - x/2} \to \frac{c}{3}\log\frac{x}{\epsilon}$$

- The relation opens up a rich source of speculations on the meaning of quantum geometry

AdS_3/CFT_2

Entanglement entropy from Chern-Simons [Ammon,Castro,Iqbal'13]

Natural observables in Chern-Simons theory are (vevs of) Wilson loops

$$W_R(C) = \mathrm{Tr}_R\, \mathrm{P} \exp \oint_C A$$

– gauge invariants, topological invariants.

Less obvious – Wilson lines: looking at the data defining W_R one can guess

$$W_R(x_i, x_f) \sim \exp\left(-\sqrt{2c_2(R)}L(x_i, x_f)\right)$$

Wilson line computes the proper geodesic distance for a particle of mass $m^2 = 2c_2$

$\mathrm{AdS}_3/\mathrm{CFT}_2$

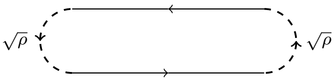
Example

$$W(C) = \mathrm{Tr}\,\mathbf{P}\exp\left(-\int_{\bar{C}} A\right)\mathbf{P}\exp\left(-\int_{C} \bar{A}\right)$$

Wilson line between points $(u, -x/2, 0)$ and $(u, x/2, 0)$

$$A_x = \begin{pmatrix} 0 & 1/u \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{P}\exp\int_{-x/2}^{x/2} A_x dx = \exp A_x \cdot x = \begin{pmatrix} 1 & x/u \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{P}\exp\int_{-x/2}^{x/2} A_x dx\,\mathbf{P}\exp\int_{x/2}^{-x/2} \bar{A}_x dx = \begin{pmatrix} 1 + x^2/u^2 & x/u \\ x/u & 1 \end{pmatrix}$$



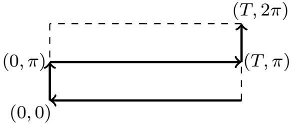


Image credit Nair'15

$\mathrm{AdS}_3/\mathrm{CFT}_2$

General behavior [Hegde,Kraus,Perlmutter'15]

$SL(N)$, any representation $w = t + i\phi$

$$W_R(C) \underset{\epsilon \to 0}{\longrightarrow} \langle\, \mathrm{hw}_R \,| W |\, \mathrm{hw}_R \,\rangle$$

$$= \mathrm{e}^{-4h_R} \langle\, \mathrm{hw}_R \,| \mathrm{e}^{-a_w w - a_{\bar{w}} \bar{w}} | -\mathrm{hw}_R \,\rangle \langle\, -\mathrm{hw}_R \,| \mathrm{e}^{\bar{a}_w w + \bar{a}_{\bar{w}} \bar{w}} | \mathrm{hw}_R \,\rangle$$

- Entanglement entropy case corresponds to $\mathrm{hw}_R = \rho$
- For general R the Wilson line computes a semiclassical ($c \to \infty$) conformal block

AdS_3/CFT_2

(AdS/)CFT interpretation

Wilson lines compute the coupling of a probe particle of mass $m = \sqrt{2c_2(R)}$ to the classical background provided by the connection A, $\bar{A}$. From the AdS/CFT point of view this is

$$\langle\, O(\infty)O(0)O(w)O(1)\,\rangle = \langle\, O_H \,|\, O_L(0)O_L(w) \,|\, O_H \,\rangle$$

For O_L corresponding to the ρ-primary one gets the von Neumann entropy (*cf.* talk by V.P. Nair)

AdS$_3$/CFT$_2$

From matrix elements to tau-functions [DM,Mironov,Morozov]

Calculation of Wilson lines reduces to determination of matrix elements

$$\langle -\mathrm{hw}_R \,|\mathrm{e}^{a_w w + a_{\bar{w}}\bar{w}}|\mathrm{hw}_R \,\rangle\,, \qquad a_w = L_{-1} + \sum_{s=2}^{N} Q_s L^{(s)}_{s-1}$$

It turns out that physically interesting matrix elements are described by special τ-functions

$$\tau^{(k)}(s,\bar{s}|G) = \langle\, \mathrm{hw}_k | e^H \, G \, e^{\bar{H}} |\, \mathrm{hw}_k \rangle\,, \qquad e^H = \exp \sum_{i=1}^{s} s_s R_k (L^s_{-(s-1)})$$

Toda recursion relation

$$\tau^{(k)}\partial_1\bar{\partial}_1\tau^{(k)} - \partial_1\tau^{(k)}\bar{\partial}_1\tau^{(k)} = \tau^{(k+1)}\tau^{(k)}$$

AdS_3/CFT_2

Skew tau-function [DM,Mironov,Morozov]

$$\tau_-^{(k)}(s,G) = \langle\, \mathrm{hw}_k \,|\, e^H\, G \,|\, -\mathrm{hw}_k \,\rangle = \left(\frac{\partial}{\partial \bar{s}_1}\right)^{k(N-k)} \tau^{(k)}(s,\bar{s},G)$$

Recursion relation

$$\tau_-^{(k)} \frac{\partial^2 \tau_-^{(k)}}{\partial t^2} - \left(\frac{\partial \tau_-^{(k)}}{\partial t}\right)^2 = \tau_-^{(k+1)}\, \tau_-^{(k-1)}$$

Other τ-functions? Integrable structures? (work in progress)

Conclusions

- Holography provides an interesting connection between quantum physics and geometry
- In low dimensions it connects to something quite well understood (*e.g.* connection between CS and CFT), relevant QHE
- Low-dimensional examples expand the AdS/CS/CFT connection. In particular gravity and thermofield dynamics (see talk of V.P Nair)
- Virasoro CFT's make accidental appearances in the QHE theory (see other talks, *e.g.* by Cappelli, Klevtsov). AdS_3/CFT_2 calls for a further look into this story