A classification 2+1D topological orders with/without symm. for boson/fermion systems

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We used to think symmetry breaking describe all phases of matter
→ phases of matter classified by group theory

Now we know there are new phases of matter beyond symmetry breaking
→ **topological orders**
classified by braided fusion categories with modular extension in 2+1D

Examples of topo. orders – long range entanglement (LRE)

Abelian topological order: $\to$ fractional statistics $N \to \infty$

- IQH and Laughlin **many-body** state Laughlin PRL **50** 1395 (1983)

$$\Psi^F_{\nu=1} = \prod_{1\le i<j\le N}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi^{F,B}_{\nu=1/m} = \prod(z_i - z_j)^m \mathrm{e}^{-\frac{m}{4}\sum |z_i|^2} = (\Psi^F_{\nu=1})^m$$

where $z_i = x_i + \mathrm{i}\, y_i$.

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Non-abelian topological order: $\to$ non-abelian statistics

- $SU(N)_2$ state via slave-particle Wen PRL **66** 802 (Feb. 1991)

$$\Psi^B_{SU(2)_2} = (\Psi^F_{\nu=2})^2,\ \nu = 1; \qquad \Psi^F_{SU(3)_2} = (\Psi^F_{\nu=2})^3,\ \nu = \frac{2}{3};$$

$\to$ $SU(N)_2$ Chern-Simons effective theory $\to$ non-abelian statistics

- Pfaffien state via CFT Moore-Read NPB **360** 362 (Aug. 1991)

$$\Psi^B_{Pfa} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}, \qquad \nu = 1$$

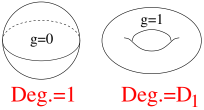
- *The Pfaffien and $SU(2)_2$ have the same non-abelian statistics*
- *The $SU(3)_2$ state has the Fibonacci non-abelian statistics*

Topological invariants that define LRE and topo. orders

Topological order describes the order in gapped quantum liquds.
We conjectured that topological order can be completely defined via only two topological properties (at least in 2D):

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

- (1) Ψ = space of **locally indistinguishable (LI)** many-body states for one of z_i.
  - Given $\Psi_1(z_i)$, $\exists$ other LI $\Psi_2(z_i), \cdots$.
  - Topo. degeneracy $D_g \equiv \dim\Psi$, depends on topology of space



Wen PRB 40, 7387 (89), Wen-Niu PRB 41, 9377 (90)

- The notion of LI states is defined respect to the notion of **local operators**: symmetric function $O_\xi(z_1, z_2, \cdots)$ which is non-zero only when $|\xi - z_i| < l$

$$\int \prod_i \mathrm{d}^2 z_i\ \Psi_1^* O_\xi \Psi_1 = \int \prod_i \mathrm{d}^2 z_i\ \Psi_2^* O_\xi \Psi_2, \quad \forall O_\xi$$

- Also known as **topological degeneracy**

The degeneracy is robust against any local purtabations

Topological invariants that define LRE and topo. orders

- (2) **Vector bundle on the moduli space**
  i. Consider a torus Σ_1 w/ metrics g_{ij}. **ii**. Different metrics g_{ij} form the moduli space $\mathcal{M} = \{g_{ij}\}$. **iii**. The LI states depend on spacial metrics: $\Psi_\alpha(g_{ij}) \to$ a vector bundle over $\mathcal{M}$ with fiber $\Psi_\alpha(g_{ij})$.

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- Loops $\pi_1(\mathcal{M}) = SL(2,\mathbb{Z})$: 90° rotation $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$
  Dehn twist: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$

  S, T generate a rep. of modular group: $S^2 = (ST)^3 = C,\ C^2 = 1$

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Conjecture: **The vector bundles from all genus-g Σ_g (*ie* the data (S, T, c), ...) completely characterize the topo. orders**

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Classify 2+1D topo. orders (*ie* patterns of entanglement)

via the topological invariants (S, T, c)

- A 2+1D topological order $\to$ a (S, T, c)
- An arbitary (S, T, c) $\not\to$ a 2+1D topological order
- (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ 2+1D topo. orders

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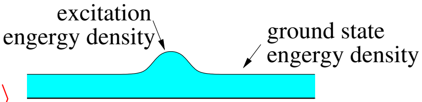
- How to find the conditions,
  beyond $S^2 = (ST)^3, S^4 = 1$?
  Study topological excitations above the ground states.
  ie consider vector bundle from the degenerate ground states on Σ_g *with punctures (quasiparticles)*.

\- In particular, the vector bundles from the degenerate ground states
  on $\Sigma_0 = S^2$ with punctures (quasiparticles)
  $\to$ unitary modular tensor category theory (UMTC)

Local and topological quasiparticle excitations

In a system: $H = \sum_{\boldsymbol{x}} H_{\boldsymbol{x}}$



- a particle-like excitation:
  energy density $= \langle \Psi_{exc} | H_{\boldsymbol{x}} | \Psi_{exc} \rangle$
- **Local quasiparticle excitation**: $|\Psi_{exc}\rangle = \hat{O}_\xi |\Psi_{grnd}\rangle$
  can be created by local operators O_ξ
- **Topological quasiparticle excitation**: $|\Psi_{exc}\rangle \neq \hat{O}_\xi |\Psi_{grnd}\rangle$ cannot be created by local operators O_ξ
- **Topological types**: equivalent classes defined by local op. O_ξ
  if $|\Psi'_{exc}\rangle = \hat{O}_\xi |\Psi_{exc}\rangle$, then $|\Psi'_{exc}\rangle$ $|\Psi_{exc}\rangle$ belong to the same type.
- With symmetry, we require O_ξ to be symmetric local operators.

- **Example:** Symmetry $SO(3)$
  Trivial type: spin-0 excitations.
  Non-trivial type: spin-1, spin-2, ... excitations.
  Non-trivial type: excitations with spin-$1/2$, fractional statistics.

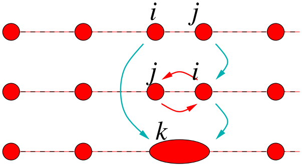
Theory of topological excitations = fusion category

Number of topological types, N, is an important topological invariant that (partially) characterizes the topogical order.

- Topological excitations can fuse (form bound states) $\to$ { Topological excitations } $\leftrightarrow$ A tensor (fusion) category $\mathcal{C}$.
- **Data to describe fusion of topological types**:
  Bound state of two topogical types i, j may not be a topogical type
  May correspond to serveral types k's with accidental degeneracy:
  $i \otimes j = k_1 \oplus k_1 \oplus k_2 \oplus k_3 \oplus \cdots = \oplus_k N^{ij}_k k$
  Ex. with $SO(3)$ symm.: spin-1 $\otimes$ spin-1 $=$ spin-0 $\oplus$ spin-1 $\oplus$ spin-2
- Associativity condition:
  $(i \otimes j) \times k = i \otimes (j \otimes k) \to \sum_m N^{ij}_m N^{mk}_l = \sum_m N^{im}_l N^{jk}_m$
  N^{ij}_k **are the data to describe fusion of the tensor category.**
  N^{ij}_k = topological inv. $\to$ fusion ring of a tensor category.
- **Quantum dim**. d_i: degree of freedom for type-i: $d_i d_j = \sum_k N^{ij}_k d_k$

Theory of topological excitations = braided fusion category

- Particles can also braid → unitary braided fusion category
- Braiding requires that
  $N^{ij}_k = N^{ji}_k$.



Theory of topological excitations = braided fusion category

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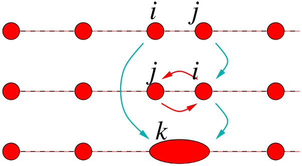
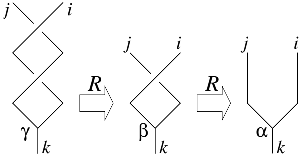


- Braiding → **mutual statistics** $e^{i\theta_{ij}^{(k)}}$ and non-trivial **spin** s_i

  2π rotation of (i, j) = 2π rotation of k

  2π rotation of (i, j) = 2π rotation of i and j and exchange i, j twice

$$e^{i2\pi s_i} e^{i2\pi s_j} e^{i\theta_{ij}^{(k)}} = e^{i2\pi s_k}$$



A unitary braided fusion category (UBFC) is a set of topological types with fusion and braiding, which is described by data (N_k^{ij}, s_i)

Relation between (S, T, c) and (N^{ij}_k, s_i, c)

Conjecture: **A bosonic topological order [*ie* a non-degenerate UBFC $\equiv$ an unitary modular tensor category (UMTC)] is fully characterized by data (S, T, c) or by data (N^{ij}_k, s_i, c) .**

- From (S, T, c) to (N^{ij}_k, s_i, c): Verlinde formula

$$N^{ij}_k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}, \qquad \mathrm{e}^{\mathrm{i}\, 2\pi s_i} \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{c}{24}} = T_{ii}.$$

- From (N^{ij}_k, s_i, c) to (S, T, c):

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i}\,(s_i + s_j - s_k)} d_k, \qquad T_{ii}. = \mathrm{e}^{\mathrm{i}\, 2\pi s_i} \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{c}{24}}$$

Conditions on (N^{ij}_k, s_i, c) $\leftrightarrow$ Conditions on (S, T, c)
$\rightarrow$ **A theory of unitary modular tensor category (UMTC)**

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Conditions on (N^{ij}_k, s_i, c) $\leftrightarrow$ Conditions on (S, T, c)
$\rightarrow$ A theory of unitary modular tensor category (UMTC)
simplified theory of UMTC Rowell-Stong-Wang arXiv:0712.1377

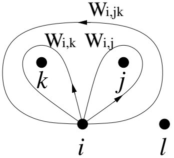
A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

- **Fusion ring**: N^{ij}_k are non-negative integers that satisfy

$$N^{ij}_k = N^{ji}_k, \quad N^{1i}_j = \delta_{ij}, \quad \sum_{k=1}^{N} N^{ik}_1 N^{kj}_1 = \delta_{ij},$$

$$\sum_{m=1}^{N} N^{ij}_m N^{mk}_l = \sum_{m=1}^{N} N^{im}_l N^{jk}_m \text{ or } \boldsymbol{N}_k \boldsymbol{N}_i = \boldsymbol{N}_i \boldsymbol{N}_k$$

where $i, j, \cdots = 1, 2, \cdots, N$, and the matrix $\boldsymbol{N}_i$ is given by $(\boldsymbol{N}_i)_{kj} = N^{ij}_k$. N^{ij}_1 defines a charge conjugation $i \to \bar{i}$: $N^{ij}_1 = \delta_{\bar{i}j}$. We refer N as the rank.



- N^{ij}_k and s_i satisfy Anderson-Moore 1988; Vafa 1988

$$\det(W_{i,j}) \det(W_{i,k}) = \det(W_{i,jk}) \quad \to \quad \sum_r V^r_{ijkl} s_r = 0 \text{ mod } 1$$

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar{r}} + N^{il}_r N^{jk}_{\bar{r}} + N^{ik}_r N^{jl}_{\bar{r}} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar{m}}$$

Elements of (diagonal) $W_{i,j} \to s_i$, Dim. of $W_{i,j} \to N^{ij}_k$.

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

From $(N^{ij}_k, s_i, c) \to (S, T)$

- Let d_i be the largest eigenvalue of the matrix $\boldsymbol{N}_i$. Let

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi\,\mathrm{i}\,(s_i+s_j-s_k)} d_k, \quad D^2 = \sum_i d_i^2.$$

Then, S satisfies

$$S_{11} > 0, \quad \sum_k S_{kl} N^{ij}_k = \frac{S_{li} S_{lj}}{S_{1l}}, \quad S = S^\dagger C, \quad C_{ij} \equiv N^{ij}_1.$$

- Let $T_{ij}. = \mathrm{e}^{\mathrm{i}\,2\pi s_i} \mathrm{e}^{-\mathrm{i}\,2\pi \frac{c}{24}} \delta_{ij}$ then ($SL(2,\mathbb{Z})$ modular representation)

$$S^2 = (ST)^3 = C.$$

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \mathrm{e}^{4\pi\,\mathrm{i}\,(s_j-s_k)}$. Then $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$.

Rowell-Stong-Wang arXiv:0712.1377

2+1D bosonic topo. orders (up to E_8-states) via (N^{ij}_k, s_i, c)

$\zeta_n^m = \frac{\sin(\pi(m+1)/(n+2))}{\sin(\pi/(n+2))}$ Rowell-Stong-Wang arXiv:0712.1377; Wen arXiv:1506.05768

N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.
1_1^B	1	0					
2_1^B	$1, 1$	$0, \frac{1}{4}$	$\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$	$\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$		$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$	
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	(221) double-layer	3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$		$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$		$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	$\Psi^2_{\nu=2}$ $SU(2)_2$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$		$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	Z_2 gauge	4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	$\prod(z_i - z_j)^4$	4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$	
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	(220) double-layer	4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$	
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$		4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$	
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	double semion	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$	
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$		$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$	
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	$\Psi^2_{\nu=3}$ $SU(2)_3$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$	Fibonacci^2
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$		$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$	
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$		$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$	
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	(223) DL	5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$		$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$		$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$		$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	
$5_{18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$		$5_{-18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	

Remote detectability: why those (N^{ij}_k, s_i, c) are realizable

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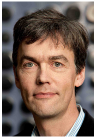

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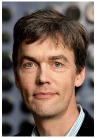

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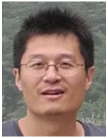

Levin arXiv:1301.7355, Kong-Wen arXiv:1405.5858

- **Remote detectable = Realizable (anomaly-free)**:
  Every non-trivial topo. excitation i can be remotely detected by at least one other topo. excitation j via the non-zero mutual braiding $\theta^{(k)}_{ij} \neq 0 \to S_{ij} = \frac{1}{D}\sum_k N^{ij}_k \mathrm{e}^{-\mathrm{i}\theta^{(k)}_{ij}} d_k$ is unitary (one of conditions) $\to$ the topological order is realizable in the same dimension.
- The centralizer of BFC $\mathcal{C}$ = the set of particles with trivial mutual statistics respecting to all others: $Z_2(\mathcal{C}) \equiv \{i \mid \theta^{(k)}_{ij} = 0,\ \forall j, k\}$.
  Remote detectable $\leftrightarrow$ $Z_2(\mathcal{C}) = \{1\}$ $\leftrightarrow$ Realizable (anomaly-free)

Bosonic/fermionic topo. orders with/without symmetry

- "Topological" excitations with symmetry: Two particles are equivalent iff they are connected by **symmetric** local operators.
  Equivalent classes = topological types with symmetry

- **Example**: for $G = SO(3)$:
- Trivial "topogical" types: spin-0. (centralizer=SFC)
- Non-trivial "topogical" types: spin-1, spin-2, $\cdots \sim$ irreducible reps.
  (Cannot be created by local symmetric operators, but can be created by local asymmetric operators.)
- Really non-trivial "topogical" types. (Other types)
  (Cannot created by local symmetric operators, nor by local asymmetric operators.)

- *How to classify topological orders with symmetry?*
  How to classify fermionic topo. orders with/without symmetry?
  Consider braided fusion category whose centralizer is non-trivial.
  centralizer = symmetric fusion catgeory (SFC) = symmetry

SFC = Exc. in bosonic/fermionic product states with symmetry = a categorical description of symmetry

Symmetric fusion catgeories (SFC):

- For *bosonic product states*, 1) Particle are bosonic with **trivial mutual statistics (not remotely detectable)**;
  2) Particles are labeled by irrep. R_i.
  Topological types = irreducible representation $R_i \in Rep(G)$
  The fusion and the trivial braiding of R_i define a spectial UBFC, called **symmetric fusion category (SFC)** and denoted as $\text{Rep}(G)$

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- For *fermionic product states*, 1) Some particles are bosonic, and others are fermionic, and all have trivial mutual statistics
  2) Particles are labeled by irrep. R_i. The full symm. group G^f contain fermion-number-parity $\hat{f} = (-)^{\hat{N}_f} \in G^f$.

- Topological types = irreducible representation R_i (ex. spin-s)
  The particle R_i has a Fermi statistics if $\hat{f} \neq 1$ in R_i (ex. spin-1)
  The particle R_i has a Bose statistics if $\hat{f} = 1$ in R_i (ex. spin-$\frac{1}{2}$)
- The fusion and bosonic/fermionic braiding of $R_i \to \text{SFC} = \text{sRep}(G^f)$

Classifcation of bosonic/fermionic topo. orders with symm.

Classify 2+1D topological orders using braided fusion category (particles with fusion and braiding):

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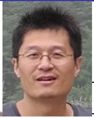
- **Bosonic topo. orders with symm. G: $\mathbf{Rep}(G)$ are the only particles that has trivial mutual statistics with all particles.**
- **Fermionic topo. orders with symm.: $\mathbf{sRep}(G^f)$ are the only particles that have trivial mutual statistics with all particles.**

2+1D fermionic topo. orders (up to $p + \mathrm{i}p$) via (N^{ij}_k, s_i, c)

Classified by **UBFC's with centralizer** $\{1, f\}$.

Lan-Kong-Wen arXiv:1507.04673

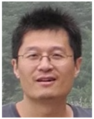



$N^F_c(^{\lvert\Theta_2\rvert}_{\angle\Theta_2/2\pi})$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	com
$2^F_0(^{\zeta_2^1}_{0})$	2	$1, 1$	$0, \frac{1}{2}$	trivial $\mathcal{F}_0$
$4^F_0(^{0}_{0})$	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2^B_1(^0_0)$, $K = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$
$4^F_{1/5}(^{\zeta_2^1\zeta_3^1}_{3/20})$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{-14/5}(^{\zeta_3^1}_{3/20})$
$4^F_{-1/5}(^{\zeta_2^1\zeta_3^1}_{-3/20})$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{14/5}(^{\zeta_3^1}_{-3/20})$
$4^F_{1/4}(^{\zeta_6^3}_{1/2})$	13.656	$1, 1, \zeta_6^2, \zeta_6^2 = 1 + \sqrt{2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_{(A_1,6)}$
$6^F_0(^{\zeta_2^1}_{1/4})$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_{-2}(^{1}_{1/4})$, $K = (3)$, $\Psi_{1/3}(z_i)$
$6^F_0(^{\zeta_2^1}_{-1/4})$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_{2}(^{1}_{-1/4})$, $K = (-3)$, $\Psi^*_{1/3}(z_i)$
$6^F_0(^{\zeta_6^3}_{1/16})$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1 = \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{1/2}(^{\zeta_6^1}_{1/16})$, $\mathcal{F}_{U(1)_2/\mathbb{Z}_2}$
$6^F_0(^{\zeta_6^3}_{-1/16})$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-1/2}(^{\zeta_6^1}_{-1/16})$
$6^F_0(^{1.0823}_{3/16})$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{3/2}(^{0.7653}_{3/16})$
$6^F_0(^{1.0823}_{-3/16})$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-3/2}(^{0.7653}_{-3/16})$
$6^F_{1/7}(^{\zeta_2^1\zeta_5^2}_{-5/14})$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{8/7}(^{\zeta_5^2}_{-5/14})$
$6^F_{-1/7}(^{\zeta_2^1\zeta_5^2}_{5/14})$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{-8/7}(^{\zeta_5^2}_{5/14})$
$6^F_0(^{2\zeta_{10}^1}_{-1/12})$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1,-10)}$
$6^F_0(^{2\zeta_{10}^1}_{1/12})$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1,10)}$

2+1D bosonic topo. orders with Z_2 symmetry

Classified by **UBFC's with centralizer Rep(Z_2)**.

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
$2_0^{\zeta_2^1}$	2	1, 1	0, 0	$\mathcal{E} = \text{Rep}(Z_2)$
$3_2^{\zeta_2^1}$	6	1, 1, 2	$0, 0, \frac{1}{3}$	$K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
$3_{-2}^{\zeta_2^1}$	6	1, 1, 2	$0, 0, \frac{2}{3}$	$K = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$
$4_1^{\zeta_2^1}$	4	1, 1, 1, 1	$0, 0, \frac{1}{4}, \frac{1}{4}$	$\Psi_{\nu=1/2} \boxtimes \text{Rep}(Z_2)$
$4_1^{\zeta_2^1}$	4	1, 1, 1, 1	$0, 0, \frac{1}{4}, \frac{1}{4}$	$\Psi_{\nu=1/2} \boxtimes^t \text{Rep}(Z_2)$
$4_{-1}^{\zeta_2^1}$	4	1, 1, 1, 1	$0, 0, \frac{3}{4}, \frac{3}{4}$	$\Psi_{\nu=-1/2} \boxtimes \text{Rep}(Z_2)$
$4_{-1}^{\zeta_2^1}$	4	1, 1, 1, 1	$0, 0, \frac{3}{4}, \frac{3}{4}$	$\Psi_{\nu=-1/2} \boxtimes^t \text{Rep}(Z_2)$
$4_{14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{2}{5}, \frac{2}{5}$	$2^B_{14/5} \boxtimes \text{Rep}(Z_2)$
$4_{-14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{3}{5}, \frac{3}{5}$	$2^B_{-14/5} \boxtimes \text{Rep}(Z_2)$
$4_0^{\zeta_2^1}$	10	1, 1, 2, 2	$0, 0, \frac{1}{5}, \frac{4}{5}$	$K = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
$4_4^{\zeta_2^1}$	10	1, 1, 2, 2	$0, 0, \frac{2}{5}, \frac{3}{5}$	$K = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$



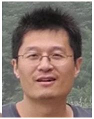
Lan-Kong-Wen to appear

Fermionic topo. orders with mod-4 fermion number conser.

Classified by **UBFC's with centralizer Rep**(Z_4^f).

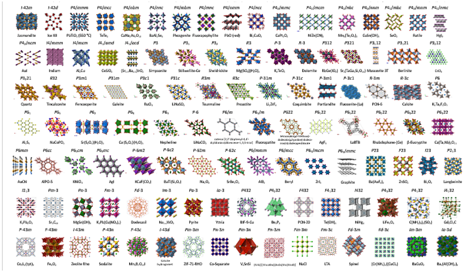
$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment
4_0^0	4	1, 1, 1, 1	$0, 0, \frac{1}{2}, \frac{1}{2}$	$\mathcal{E} = \text{sRep}(Z_4^f)$
6_0^0	12	1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{2}{3}$	$K = (3)$
6_0^0	12	1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}$	$K = (-3)$
8_*^0	8	1, 1, 1, 1, 1, 1, 1, 1	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$\Psi_{\nu=1/2} \boxtimes \text{sRep}(Z_4^f)$
8_*^0	8	1, 1, 1, 1, 1, 1, 1, 1	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	
8_*^0	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{3}{5}, \frac{3}{5}$	
8_*^0	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{9}{10}, \frac{9}{10}$	
8_*^0	20	1, 1, 1, 1, 2, 2, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{10}, \frac{2}{5}, \frac{3}{5}, \frac{9}{10}$	
8_*^0	20	1, 1, 1, 1, 2, 2, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{3}{10}, \frac{7}{10}, \frac{4}{5}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{5}{8}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{5}{8}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{7}{8}$	
10_*^0	16	1, 1, 1, 1, 1, 1, 1, 1, 2, 2	$0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{7}{8}$	

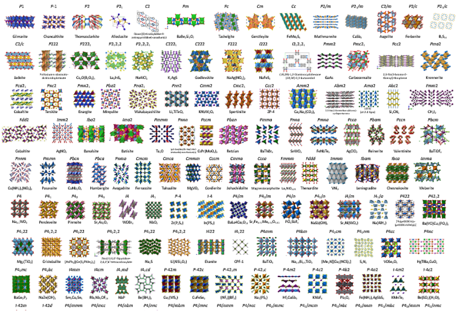
Lan-Kong-Wen to appear



Zoo of quantum phases of matter

- **230 crystals** from group theory





- **Infinity many topological orders in 2+1D** from category theory