Problems for Hausel's Lecture, Monday

1. Let ρ be a representation of a quiver over $\mathbb{K}$. Prove that up to isomorphism there is a unique way to write it as a direct sum of indecomposables. (Krull-Schmidt - see last sheet for an elementary proof.)

2. Recall the notation $X^\alpha = X_1^{k_1} \dots X_n^{k_n}$ for $\alpha = \sum k_i \alpha_i \in \mathbb{N}^{\mathcal{V}}$. Prove that a formal power series of the form $1 + \sum_{\alpha \in \mathbb{N}^{\mathcal{V}} \setminus \{0\}} a_\alpha X^\alpha$ with $a_\alpha \in \mathbb{Z}$ can be uniquely written as an infinity product of the form $\prod_{\alpha \in \mathbb{N}^{\mathcal{V}} \setminus \{0\}} (1 - X^\alpha)^{b_\alpha}$, where $b_\alpha \in \mathbb{Z}$.

3. By induction on the length of $w \in W$ (that is the minimal number of fundamental reflections needed to generate w) prove that for every $w \in W$ $\rho - w(\rho)$ is the sum of distinct positive real roots. Deduce that $\rho - w(\rho) \in \mathbb{N}^I$. (One can prove the stronger statement that $\rho - w(\rho)$ is the sum of positive real roots α for which $-w^{-1}\alpha \in \Delta_+$.)

4. Check the Weyl denominator formula for the A_2 quiver. What are the roots for an A_n quiver? What is the Weyl denominator formula?

5. By applying the Weyl group to the fundamental roots find the 24 roots in the D_4 root system.

6. Consider the dimension vector $\alpha \in (2, 1, \dots, 1)$ on the quiver V_k where 2 is on the central vertex and 1 everywhere else. Show that it is a root when $k \geq 3$. What can we say about the Weyl orbit of $W\alpha \subset \mathbb{N}^I$ for $k = 3, 4, 5$? (Start to apply the fundamental reflections to α and see what happens.)

7. What are the real and imaginary roots for the $\hat{A}_0$ quiver? What are the indecomposables over $\mathbb{C}$? Classify all representations of $\hat{A}_0$ over $\mathbb{C}$. What are the indecomposable representations of $\hat{A}_0$ over $\mathbb{F}_q$? What are the absolutely indecomposable ones? Classify all representations of $\hat{A}_0$ over $\mathbb{F}_q$. Kac denominator formula?

8. Same question with $\hat{A}_1$ instead of $\hat{A}_0$.

9. Describe the Jordan normal form of a complex *nxn* matrix. Prove that the set of possible Jordan normal forms for an *nxn* matrix over $\mathbb{F}_q$ can be parametrized by maps from $\nu : \Phi' \to \mathcal{P}$, such that $\sum_{f \in \Phi'} \deg(f)|\nu(f)| = n$. Here Φ' is the set of all monic irreducible polynomials over $\mathbb{F}_q$, and $\mathcal{P}$ is the set of all partitions (of all positive integers n). (See slide 19 for a hint.)