Problems for Hausel's Lecture, Friday

1. (a) Prove that a non-invertable endomorphism of an indecomposable representation of a quiver is nilpotent.
   (b) Prove that if ϕ_1 and ϕ_2 are non-invertable endomorphisms of an indecomposable representation Z of a quiver then so is their sum. (This means that $\mathrm{End}(Z)$ is a *local* ring.)

2. Let X, Y be two representations of a quiver Γ. Define the *radical* by

$$\mathrm{Rad}(X,Y) = \left\{ \phi \in \mathrm{Hom}(X,Y) \;\middle|\; \begin{array}{c} \tau\phi\sigma \text{ is non-invertible for every pair } Z \xrightarrow{\sigma} X \\ \text{and } Y \xrightarrow{\tau} Z \text{ with } Z \text{ indecomposable} \end{array} \right\}$$

   prove that

   (a) $\mathrm{Rad}(X,Y)$ is a subspace of $\mathrm{Hom}(X,Y)$
   (b) $\mathrm{Rad}(X, Y_1 \oplus Y_2) = \mathrm{Rad}(X, Y_1) \oplus \mathrm{Rad}(X, Y_2)$
   (c) $\mathrm{Rad}(X_1 \oplus X_2, Y) = \mathrm{Rad}(X_1, Y) \oplus \mathrm{Rad}(X_2, Y)$
   (d) If X and Y are indecomposable, then $\mathrm{Hom}(X,Y) \setminus \mathrm{Rad}(X,Y)$ equals the set of isomorphisms $X \to Y$.

3. By considering for a representation X and indecomposable representation Y the quantity

$$\frac{\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Rad}(X,Y)}{\dim \mathrm{Hom}(Y,Y) - \dim \mathrm{Rad}(Y,Y)}$$

   show that X can be uniquely written as a direct sum of indecomposables.
   (elementary Krull-Schmidt from arXiv:0804.1428)

4. Let finite group G act linearly on the finite vector spaces V and W. Let S denote a slice of the action on V, i.e. $S \subset V$ such that $|[X] \cap S| = 1$ for all $[X] \in V/G$. Show that the orbits on $V \oplus W$ correspond to the choice of $X \in S$ and a $C_G(X)$ orbit on W. Using the Fourier transform idea of Kraft-Riedtmann on slide 35 deduce that if G acts on W^* by the dual representation then

$$|(V \oplus W)/G| = |(V \oplus W^*)/G|.$$

5. Consider the Nakajima quiver varieties for the quiver $\hat{A}_0$. Show that for $v, w \in \mathbb{N}$ the moment map

$$\mu_{v,w} : \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^v) \oplus \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^v) \oplus \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^w) \oplus \mathrm{Hom}(\mathbb{C}^w, \mathbb{C}^v) \to \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^v)$$

   is given by $(A, B, I, J) \mapsto AB - BA + IJ$. So that

$$\mathcal{M}_{v,w} = \{A, B, I, J | AB - BA + IJ = Id_v\} // \mathrm{GL}_v$$

   is the *twisted ADHM* space of instantons on $\mathbb{C}^2$. Prove that the generating function of the Betti numbers of quiver varieties in this case simplifies to:

$$\sum_{v=0}^{\infty} P_c(\mathcal{M}(v,w); q^{1/2}) q^{-vw} X^v = \prod_{i=1}^{\infty} \prod_{l=1}^{k} \frac{1}{(1 - X^i q^l)}.$$

   This is originally due to (Nakajima-Yoshioka 2004) and has been obtained via Fourier transform in math.AG/0511163. (One expects modular properties of the generating functions of Betti numbers of Nakajima quiver varieties for affine quivers, but apart from $\hat{A}_0$ above, and $\hat{A}_1$ in hep-th/0603162 no other cases have been carefully studied.)