Kac's conjectures on quiver representations via arithmetic harmonic analysis

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(Modified) Plan for four lectures

Lecture 1: Representations of quivers; Kac's conjectures

Lecture 2: Arithmetic and cohomology of varieties

Lecture 3: Affine GIT and symplectic quotients

Lecture 4: Betti numbers of Nakajima's quiver varieties; proof of Kac Conjecture 1

(Lecture 5: Cohomology of character varieties; attack on Kac Conjecture 2)

(Lecture 6: Topology of Hitchin map and arithmetic of character variety; another attack on Kac Conjecture 2)

Quivers and their representations

- a *quiver* Γ is an oriented and connected graph with vertices $I = (1, \ldots, n)$ and arrows or oriented edges $E \subset I \times I$, (possibly multiple edges and edge-loops)
- denote $a = (t(a), h(a)) \in E$ the *tail* and *head* of the arrow a
- $\mathbb{K}$ field; (either $\mathbb{C}$ or $\mathbb{F}_q$)
- a *representation* ρ of Γ is a collection of finite dimensional $\mathbb{K}$-vector spaces $\{V_i\}_{i \in I}$ and homomorphisms $\rho_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$ for every $a \in E$
- $\dim \rho = (\dim V_1, \ldots, \dim V_n) \in \mathbb{N}^I$ is the *dimension* of ρ

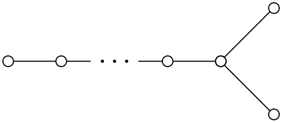
Examples of quivers

- *finite* quivers of type A_n,D_n,E_6, E_7, E_8
- *affine* or quivers of type $\hat{A}_n$,$\hat{D}_n$,$\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$
- finite and affine quivers are called *tame*
- all other quivers are called *wild*
- *polygon* quiver V_m (usually with dimension vector $(2,1,\ldots,1)$)
- *loop* quiver S_g
- *star-shaped* and more generally *comet-shaped* quivers

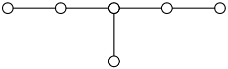
Examples of quivers: Finite quivers

A_n ($n \geq 1$ vertices)

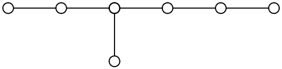
D_n ($n \geq 4$ vertices)



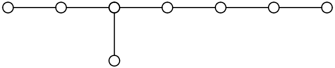
E_6



E_7

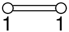


E_8

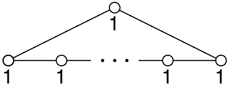


Examples of quivers: Affine quivers

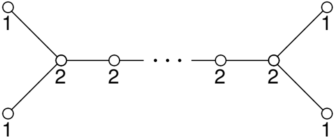
$\hat{A}_1$



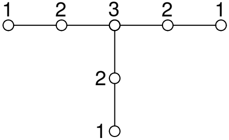
$\hat{A}_n$ $(n+1 \geq 3$ vertices)



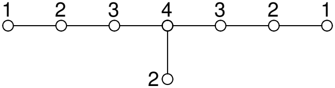
$\hat{D}_n$ $(n+1 \geq 5$ vertices)



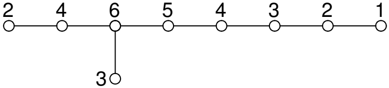
$\hat{E}_6$



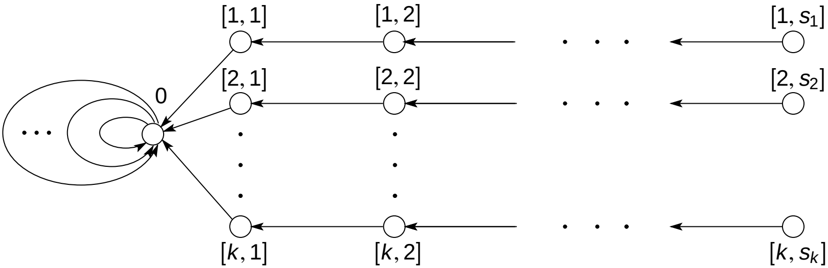
$\hat{E}_7$



$\hat{E}_8$



Examples of quivers: Comet-shaped quivers



- star-shaped if number of loops on central vertex $g = 0$
- V_k is when $g = 0$ and $s_i = 1$
- S_g is when $k = 0$; $S_1 = \hat{A}_0$ is the only tame quiver
- the tame comet-shaped quivers are all the finite quivers and $\hat{A}_0, \hat{D}_4, \hat{E}_6, \hat{E}_7, \hat{E}_8$

Classifying quiver representations

- two representations ρ_1 on $\{V_i^1\}_{i\in I}$ and ρ_2 on $\{V_i^2\}_{i\in I}$ can be added $\rho_1 \oplus \rho_2$ on $\{V_i^1 \times V_i^2\}_{i\in I}$ in the obvious way
- a non-trivial quiver representation is *indecomposable* if it cannot be written as a direct sum of non-trivial quiver representations
- every representation of a quiver is the direct sum of indecomposables;
  this decomposition is unique $\rightsquigarrow$
  indecomposable representations are building blocks for all representations
- Problem: classify indecomposables!
- Call the dimension of an indecomposable representation in $\mathbb{N}^I$ a *positive root*. Denote $\Delta_+ \subset \mathbb{N}^I$ set of positive roots
- Determine $\Delta_+ \subset \mathbb{N}^I$!

Weyl group

- let $\alpha_i(j) = \delta_{ij}$ *simple root*;
  $(\alpha_i; \alpha_j) = \delta_{ij} - \frac{1}{2}(b_{ij} + b_{ji})$ symmetric bilinear form on $\mathbb{Z}^I$
  b_{ij} is number of arrows from i to j
- $(i, i) \notin E \Leftrightarrow (\alpha_i, \alpha_i) = 1$
  then α_i *fundamental root*;
  $\Pi \subset \mathbb{N}^I$ set of fundamental roots
- For a fundamental root α_i define
  $r_{\alpha_i} : \mathbb{Z}^I \to \mathbb{Z}^I$ by $r_{\alpha_i}(\lambda) = \lambda - 2(\lambda, \alpha_i)\alpha_i$
  $r_{\alpha_i}^2 = Id$ reflection
- Let $W := \langle r_\alpha \rangle_{\alpha \in \Pi} \leq Aut(\mathbb{Z}^I)$ be the Weyl group of Γ
- Extend action of W to $\mathbb{Z}^I \oplus \mathbb{Z}\rho$ by $r_{\alpha_i}(\rho) = \rho - \alpha_i$ and define
  $s(w) = \rho - w(\rho) \in \mathbb{N}^I \setminus \{0\}$

Kac denominator formula

- Assume Γ loopless. For $\alpha = \sum k_i \alpha_i \in \mathbb{N}^I$ write $X^\alpha := X_1^{k_1} \cdot \cdots \cdot X_n^{k_n}$, expand to get

$$\sum_{w \in W} \det(w) X^{s(w)} = \prod_{\alpha \in \mathbb{N}^I} (1 - X^\alpha)^{m_\alpha}$$

Kac denominator formula; $m_\alpha \in \mathbb{Z}$ *multiplicity* of α

- (Kac 1974) proves for the weight decomposition of the Kac-Moody algebra $\mathfrak{g}(\Gamma) = \oplus_{\alpha \in \mathbb{N}^I} \mathfrak{g}(\Gamma)_\alpha$ that $\dim(\mathfrak{g}(\Gamma)_\alpha) = m_\alpha \geq 0$

Theorem (Kac 1974)

Let $L(\mathbf{w})$ be an irreducible representation of $\mathfrak{g}(\Gamma)$ of highest weight $\Lambda \in P$. Let $L(\Lambda) = \oplus_{\alpha \in \mathbb{N}^I} L(\Lambda)_{\Lambda - \alpha}$ denote its weight space decomposition. Then the Weyl-Kac character formula *holds:*

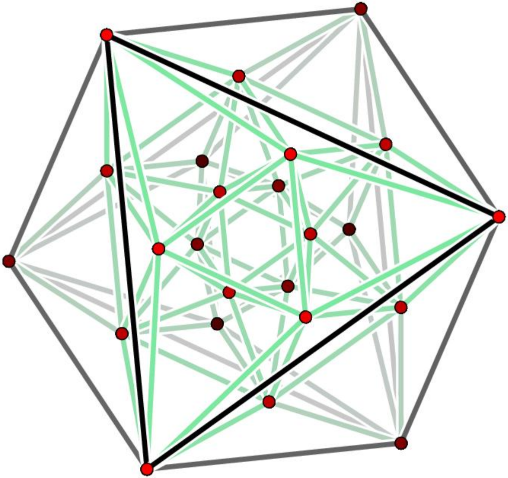
$$\sum_{\alpha \in \mathbb{N}^I} \dim\left(L(\Lambda)_{\Lambda-\alpha}\right) X^\alpha = \frac{\sum_{w \in W} \det(w) X^{\Lambda+\rho-w(\Lambda+\rho)}}{\prod_{\alpha \in \mathbb{N}^I} (1 - X^\alpha)^{m_\alpha}}$$

Example: A_2 root system

- Let Γ be A_2 quiver
- Up to isomorphism there are three indecomposable representations of dimension vectors $(1,0)$, $(0,1)$ and $(1,1)$
- $(,)$ is positive definite on $\mathbb{Z}^2$
- $r_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$; $(r_1 r_2)^3 = 1$
- Weyl group
  $S_3 = \{r_1, r_2 | r_1^2 = r_2^2 = (r_1 r_2)^3 = 1\} = \{1, r_1, r_2, r_1 r_2, r_2 r_1, r_1 r_2 r_1\}$
  is finite
- Weyl (= finite Kac) denominator formula gives
  $1 - X_1 - X_2 + X_1 X_2^2 + X_2 X_1^2 - X_1^2 X_2^2 = (1 - X_1)(1 - X_2)(1 - X_1 X_2)$
- thus all three positive roots appear with multiplicity one

Example: D_4 root system

There are 24 roots in $\mathbb{N}^4$ in the D_4 root system.
They form the regular 24-cell.



Kac's theorem

Theorem (Kac 1982)

Assume $\mathbb{K} = \mathbb{C}$

- Δ_+ *is independent of the orientation of* Γ
- $\alpha \in \Delta_+, w \in W \Rightarrow w(\alpha)$ *or* $-w(\alpha) \in \Delta_+$
- *When* Γ *is loopless* $\alpha \in \Delta_+ \Leftrightarrow m_\alpha > 0$.

- $m_\alpha > 0$ is independent of the orientation on Γ
- fundamental roots have $m_{\alpha_i} = 1$
- $|\Delta_+| < \infty \Leftrightarrow |W| < \infty \Leftrightarrow (,)$ is pos. def., $\Leftrightarrow \Gamma$ is finite (Gabriel, 1972)
- Kac's proof proceeds by
  1. constructing a complex algebraic variety $\mathcal{Z}(\Gamma, \alpha)$ parametrizing indecomposable representations of Γ to $\mathbb{C}$ of dimension α modulo isomorphism.
  2. showing that $\mathcal{Z}(\Gamma, \alpha)$ can be defined over $\mathbb{Z}$
  3. counting the points of $\mathcal{Z}(\Gamma, \alpha)$ over a finite field $\mathbb{F}_q$
  4. finding that the count is independent of the orientation

Examples of quiver representations over $\mathbb{K} = \mathbb{F}_q$

- Let S_g be the quiver on one vertex and g loops. Classifying representations of S_g of dimension (d) is classifying the isomorphism classes of g tuples of $d \times d$ matrices.
- Reps of S_1 classified by Jordan normal form. Representations of S_g for $g > 1$ are *wild*
- $\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \sim \begin{pmatrix} x + \sqrt{\epsilon} y & 0 \\ 0 & x - y\sqrt{\epsilon} \end{pmatrix}$, with $x \in \mathbb{F}_q$, $y \in \mathbb{F}_q^\times$, $\mathbb{F}_q^\times = \langle \epsilon \rangle$ and $\sqrt{\epsilon} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ (q odd) is indecomposable over $\mathbb{F}_q$ but not indecomposable over $\overline{\mathbb{F}_q}$
- an absolutely indecomposable representation of S_1 of dimension (2) is $\sim \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ with $x \in \mathbb{K}$. Up to isomorphism there are q absolutely indecomposable representations of S_1 of dimension (d) over $\mathbb{F}_q$.

The A-polynomial

- $\mathbb{K} = \mathbb{F}_q$; a representation of a quiver over $\mathbb{K}$ is *absolutely indecomposable* if it is indecomposable over $\overline{\mathbb{K}}$
- $\alpha \in \mathbb{N}^I$ a dimension vector;
  $\{V_i\}_{i \in I}$ such that $\dim V_i = \alpha(i)$;
  $\mathbb{V}_\alpha := \bigoplus_{a \in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)})$;
  $\mathrm{G}_\alpha := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$; clearly G_α acts on $\mathbb{V}_\alpha$
- $A_\Gamma(\alpha, q) := |\{\rho \in \mathbb{V}_\alpha | \rho \text{ is abs. indec.}\}/\mathrm{G}_\alpha|$

Theorem (Kac, 1982)

- $A_\Gamma(\alpha, q) \in \mathbb{Z}[q]$ *is either 0 or monic of degree*$=1-(\alpha,\alpha)$
- $A_\Gamma(\alpha, q)$ *is independent of the orientation of* Γ
- $A_\Gamma(\alpha, q) \neq 0 \Leftrightarrow \alpha \in \Delta_+$
- $A_\Gamma(\alpha, q) = A_\Gamma(w(\alpha), q)$, *when* $w \in W$ *and* $\alpha, w(\alpha) \in \mathbb{N}^I$
- $A_\Gamma(\alpha, q) = 1 \Leftrightarrow \ \alpha = w(\alpha_i)$ *for some* $w \in W$ *and* $\alpha_i \in \Pi$

Positive roots with $A_\Gamma(\alpha, q) = 1$ are called *real roots*
the rest, when $\deg(A_\Gamma(\alpha, q)) > 0$ are *imaginary roots*
$\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$

Kac's conjectures

Conjecture (Kac, 1982)

1. *When Γ is loopless, the constant term $A_\Gamma(\alpha, 0) = m_\alpha$*
2. *$A_\Gamma(\alpha, q) \in \mathbb{N}[q]$, i.e. the coefficients of $A_\Gamma(\alpha, q)$ are ≥ 0.*

Both conjectures were known to Kac for finite and affine quivers and for the "polygon"-quiver V_m with dimension vector $(2, 1, \ldots, 1)$.

Theorem (Crawley-Boevey,Van den Bergh 2004)

Both conjectures hold true for any quiver with α indivisible; i.e. $gcd(\alpha(i)) = 1$

Every quiver supports infinitely many divisible dimension vectors $\rightsquigarrow$ both conjectures remained open for any wild quiver

We prove Conjecture 1 in these lectures.

Hua's formula

Theorem (Hua, 2000)

Fix quiver Γ. Let $A_\Gamma(\alpha, q) = \sum t_i^\alpha q^i$, then:

$$\prod_{\alpha \in \mathbb{N}^n} \prod_{j=0}^{\infty} \prod_{i=0}^{\infty} (1 - q^{i+j} X^\alpha)^{t_i^\alpha} =$$

$$\sum_{\mathbf{v} \in \mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda \in \mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j) \in \mathcal{E}} q^{\langle \lambda^i, \lambda^j \rangle}}{\prod_{i \in I} \left(q^{\langle \lambda^i, \lambda^i \rangle} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - q^{-j}) \right)},$$

where $\mathcal{P}(\mathbf{v})$ is the set of n-tuples of partitions $(\lambda^1, \ldots, \lambda^n)$, with $|\lambda^i| = \mathbf{v}_i$, and for two partitions $\langle \nu, \mu \rangle = \sum_{ij} \min(\nu_i, \mu_j)$.

Thus Conjecture 1 would follow by showing that the combinatorial RHS when $q = 0$ reduces to the combinatorial LHS of

$$\sum_{w \in W} \det(w) X^{s(w)} = \prod_{\alpha \in \mathbb{N}^n} (1 - X^\alpha)^{m_\alpha}.$$

Remarks on problem session

- Read about the $\hat{A}_1$ root system at `http://sbseminar.wordpress.com/2008/11/02/`

$$\prod_{m=1}^{\infty}\left(1-X^{i-1}Y^{i}\right)\left(1-X^{i}Y^{i-1}\right)\left(1-X^{i}Y^{i}\right)=\sum_{i\in\mathbb{Z}}^{\infty}(-1)^{i}X^{i(i-1)/2}Y^{i(i+1)/2}.$$

- (Macdonald 1972) found the infinite product formulas for affine root systems, (Kac 1974) reproved it and explained the appearance of imaginary roots in terms of the Kac denominator formula for the affine Kac-Moody algebras $\rightsquigarrow$ sometimes affine Kac denominator formula is referred to as Macdonald-Kac formula

Theorem

Let Γ be a quiver of tame type, $\alpha \in \mathbb{N}^{I} \setminus \{0\}$ then
α is decomposable $\Leftrightarrow (\alpha,\alpha) > 1$
$\alpha \in \Delta_{+}^{re} \Leftrightarrow (\alpha,\alpha) = 1$
$\alpha \in \Delta_{+}^{im} \Leftrightarrow (\alpha,\alpha) \leq 0$

Jordan normal form over $\mathbb{F}_q$

- let Φ' denote all monic irreducible polynomials over $\mathbb{F}_q$
- let $f \in \Phi'$ in the form $f = t^d + a_{d-1}t^{d-1} + \cdots + a_0$
  $d \times d$ *companion matrix* $J(f)$ and $dm \times dm$ matrix $J_m(f)$ are given by

$$J(f) := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{d-1} \end{bmatrix} \qquad J_m(f) := \begin{bmatrix} J(f) & 1 & 0 & \dots & 0 \\ 0 & J(f) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & J(f) \end{bmatrix}.$$

- Up to isomorphism, indecomposable representations of S_1 over $\mathbb{F}_q$ are of the form $J_m(f)$ for $f \in \Phi'$ and $m > 0$
- thus representations of S_1 of dimension n are classified by $\nu : \Phi' \to \mathcal{P}$ such that $\sum_{f \in \Phi'} \deg(f)|\nu(f)| = n$
- $\mathrm{GL}_n(\mathbb{F}_q)/\mathrm{GL}_n(\mathbb{F}_q)$ are parametrized by $\nu : \Phi \to \mathcal{P}$ such that $\sum_{f \in \Phi'} \deg(f)|\nu(f)| = n$, where $\Phi = \Phi' \setminus \{t\}$

Ingredients into Hua's formula

- Burnside orbit counting formula:
  finite group G acts on set X
  $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \sum_{[g]\in G/G}\frac{|X_g|}{|C_g|}$, where
  $X_g = \{x \in X | gx = x\}$
- Count orbits of G_α on $\mathbb{V}_\alpha$, i.e. find $M_\Gamma(\alpha, q) := |\mathbb{V}_\alpha/\mathrm{G}_\alpha|$
- $\lambda, \mu \in \mathcal{P}$ then $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ then
  $\langle\lambda, \mu\rangle = \sum_{ij}\min(\lambda_i, \mu_j)$
- the cardinality of the centralizer of $J_\lambda(f) = \oplus J_{\lambda_i}(f) \in \mathrm{GL}_{d|\lambda|}(\mathbb{F}_q)$
  $|C_{J_\lambda(f)}| = q^{d\langle\lambda,\lambda\rangle}\prod_k\prod_i^{m_k(\lambda)}(1-q^{-i})$
- let $J_\lambda(f) \in \mathrm{GL}_m(\mathbb{F}_q)$ and $J_\mu(g) \in \mathrm{GL}_n(\mathbb{F}_q)$ then
  $|\{M \in \mathrm{Mat}_{m\times n}(\mathbb{F}_q) \,|\, J_\lambda(f)M = MJ_\mu(g)\}| = \begin{cases} q^{\deg(f)\langle\lambda,\mu\rangle} & \text{if } f = g \\ 1 & \text{ow} \end{cases}$
- Krull-Schmidt $\Rightarrow \sum_{\alpha\in\mathbb{N}^I} M_\Gamma(\alpha, q)X_\alpha = \prod_{\alpha\in\mathbb{N}^I}(1-X^\alpha)^{-I_\Gamma(\alpha,q)}$,
  where $I_\Gamma(\alpha, q) := |(\mathbb{V}_\alpha/\mathrm{G}_\alpha)^{indec.}|$
- inclusion-exclusion + Möbius $\Rightarrow$
  $A_\Gamma(r\alpha, q) = \sum_{d|r}\frac{1}{d}\sum_{k|d}\mu(k)I_\Gamma(\frac{d}{k}\alpha, q^k)$ where α indivisible

Étale cohomology

- X variety defined over $\mathbb{Z}$
- (Grothendieck 1958) constructs étale cohomology $H_c^k(X(\overline{\mathbb{F}}_q);\mathbb{Q}_\ell)$
- $\mathrm{Frob}_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ by $x \mapsto x^q$ *Frobenius automorphism* $\rightsquigarrow$
  $\mathrm{Frob}_q : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q) \rightsquigarrow$
  $\mathrm{Frob}_q : H_c^k(X(\overline{\mathbb{F}}_q);\mathbb{Q}_\ell) \to H_c^k(X(\overline{\mathbb{F}}_q);\mathbb{Q}_\ell)$
- as $(\overline{\mathbb{F}}_q)^{\mathrm{Frob}_q} = \mathbb{F}_q$ Grothendieck-Lefschetz fixed point theorem $\rightsquigarrow$

$$|X(\mathbb{F}_q)| = |X(\overline{\mathbb{F}}_q)^{\mathrm{Frob}_q}| = \sum_{i=0}^{2n}(-1)^i \mathrm{tr}(Frob_q : H_c^i(X,\mathbb{Q}_\ell) \to H_c^i(X,\mathbb{Q}_\ell))$$

- as $\mathrm{Frob}_{q^k} = (\mathrm{Frob}_q)^k \rightsquigarrow$
  $|X(\mathbb{F}_{q^k})| = \lambda_1^k + \lambda_2^k + \cdots + \lambda_N^k$, where $\lambda_i \in \overline{\mathbb{Q}}_\ell$ eigenvalues of $Frob_q$
- (Deligne 1974) proved Weil's Riemann hypothesis: eigenvalues of Frob_q have absolute value $q^{i/2}$ for $i \in \mathbb{N}$

Weight filtration

- Jordan decomposition of $Frob_q$ on $H_c^k \Rightarrow$ weight filtration $W_l \subset H_c^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2}$ $i \leq l$
- comparison theorem: $H_c^*(X(\mathbb{C});\mathbb{C}) \cong H_c^*(X(\overline{\mathbb{F}}_q),\mathbb{Q}_\ell)\otimes\mathbb{C}$
- (Deligne 1974) constructs weight filtration on $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_k = H_c^k(X(\mathbb{C});\mathbb{Q})$ which is functorial
- when $W_{k-1} \cap H_c^k(X;\mathbb{Q}) = 0$
  the weight filtration is *pure*;
  e.g. when X is smooth projective;
  or when $X \subset \overline{X}$, with $\overline{X}$ smooth projective and injects on H_c^*
  e.g. when X is a symplectic quiver variety; a Nakajima quiver variety,$\mathcal{M}_{\mathrm{DR}}$ moduli space of flat connections and $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of Higgs bundles on a Riemann surface
- weight filtration is *not* pure or *mixed* e.g. for $X = \mathrm{GL}_n$ or for $\mathcal{M}_{\mathrm{B}}$ the character variety of representations of the fundamental group of a Riemann surface to GL_n

Example

- Take $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \cong \{(x,y) \in \mathbb{C}^2 | xy = 1\}$
  $H^2_c(X; \mathbb{C}) \cong \mathbb{C}$, $H^1_c(X, \mathbb{C}) \cong \mathbb{C}$
- $X(\overline{\mathbb{F}}_q) = \overline{\mathbb{F}}_q^\times$
- $$\begin{array}{rccc} Frob_q & : \overline{\mathbb{F}}_q^\times & \to & \overline{\mathbb{F}}_q^\times \\ & x & \mapsto & x^q \end{array}$$
- $X(\overline{\mathbb{F}_q})^{Frob_q} = X(\mathbb{F}_q) = \mathbb{F}_q \setminus \{0\}$, thus $|X(\overline{\mathbb{F}_q})^{Frob_q}| = q - 1$
- Grothendieck-Lefschetz$\Rightarrow$
  $|X(\overline{\mathbb{F}_q})^{Frob_q}| = \sum_{i=0}^{2} (-1)^i \mathrm{tr}(Frob_q : H^i_c(X, \mathbb{Q}_\ell) \to H^i_c(X, \mathbb{Q}_\ell))$
- thus $1 = Frob_q : H^1_c(X; \mathbb{Q}_\ell) \to H^1_c(X, \mathbb{Q}_\ell))$ and
  $q \cdot \_ = Frob_q : H^2_c(X; \mathbb{Q}_\ell) \to H^2_c(X, \mathbb{Q}_\ell))$
- $\Rightarrow 0 = W_1(H^2_c(X(\mathbb{C}), \mathbb{Q}))$ and
  $W_0(H^1_c(X(\mathbb{C}); \mathbb{Q})) = H^1_c(X(\mathbb{C}), \mathbb{Q})$
- weight filtration is mixed on $H^1(X(\mathbb{C}), \mathbb{Q})$

Arithmetic and topological content of the E-polynomial

- For a complex variety $X(\mathbb{C})$ define *E-polynomial* $E(X;q)=\sum \dim(W_i/W_{i-1}(H_c^k(X)))(-1)^k q^{\frac{i}{2}}$
- basic properties:
  additive - if $X_i \subset X$ locally closed s. t. $\dot{\cup} X_i = X$ then $E(X;q)=\sum E(X_i;q)$
  multiplicative - $F \to E \to B$ locally trivial in the Zariski topology $E(E;q)=E(B;q)E(F;q)$
- when weight filtration is pure
  $E(X;q)=\sum \dim(H_c^k(X))(-q^{1/2})^k$ is the Poincaré polynomial
- if all eigenvalues λ_i of Frob_q on $H_c^*(X(\overline{\mathbb{F}}_q);\mathbb{Q}_\ell)$ are integer powers of q, then $|X(\mathbb{F}_{q^n})|=\sum \lambda_i^n$ is a *polynomial* in q^n and $=E(X;q)$

Theorem (Katz 2006)

If X is a variety defined over $\mathbb{Z}$ and $\#\{X(\mathbb{F}_q)\}=E(q)$ is a polynomial in q, then $E(M;q)=E(q)$.

e.g. if $E(q)\in\mathbb{Q}[q] \overset{Katz}{\Rightarrow}$ E(q)$\in\mathbb{Z}[q]$ proves Kac's result that $A_\Gamma(\alpha,q)=\#\{\mathcal{Z}(\Gamma,\alpha)(\mathbb{F}_q)\}\in\mathbb{Z}[q]$

Remarks on questions

- Let $X = \mathbb{C}^2 \setminus \{0\}$ smooth, $\pi := X \to \mathbb{P}^1$ by $\pi(x, y) \mapsto [x : y]$ is a geometric quotient by the group action of $\mathbb{C}^\times$ by $x \mapsto \lambda x \rightsquigarrow$ principal bundle locally trivial in the Zariski topology
  $E(X; q) = E(\mathbb{P}^1; q)E(\mathbb{C}^\times; q) = (q + 1)(q - 1) = q^2 - 1$ but
  $P_c(\mathbb{P}^1; t) = 1 + t^2$, $P_c(\mathbb{C}^\times; t) = t + t^2$ and $P_c(X; t) = t + t^4$ but
  $(1 + t^2)(t + t^2) = (t + t^2 + t^3 + t^4) \neq t + t^4$!
  cohomology is <u>not</u> multiplicative (and not additive either)
- hint for question 2 on Problem list 1:
  define an ordering $\le$ on $\mathbb{N}^I$ such that if $\gamma_i \le \beta_i$ for all i then $\gamma \le \beta$;
  find the smallest non-trivial term in $F := 1 + \sum_{\alpha \in \mathbb{N}^I \setminus \{0\}} a_\alpha X^\alpha$ say X^γ with $\gamma \in \mathbb{N}^I \setminus \{0\}$
  then show that $F(1 - X^\gamma)^{a_\gamma}$ has no non-trivial terms X^β for $\beta \le \gamma$.
  Proceed with $F(1 - X^\gamma)^{a_\gamma}$.

Examples

- $E(\mathbb{C}^{\times}; q) = q - 1$; $E(\mathbb{C}^{\times}; q) = E(\mathbb{C}; q) + E(\{0\}; q) = q$
- Let $\mathcal{U}$ be the variety $x_1 y_1 + x_2 y_2 = 1$ in $\mathbb{C}^2 \times \mathbb{C}^2$.
- the number of solutions of the equation $x_1 y_1 + x_2 y_2 = 1$ in $\mathbb{F}_q$ is $2(2q-1)(q-1) + (q-2)(q-1)^2 = (q-1)(q^2+q)$ because
  - $(2q-1)(q-1)$ when $x_1 y_1 = 0$
  - $(q-1)(2q-1)$ when $x_1 y_1 = 1$
  - $(q-1)^2$ in the other $q-2$ cases.
- $\Rightarrow$ the number of points on $\mathcal{U}(\mathbb{F}_q)$ is $(q-1)(q^2+q)$, $\overset{\mathit{Katz}}{\Rightarrow} E(\mathcal{U}, q) = (q-1)(q^2+q)$

Affine GIT quotients

- Let M be a complex affine variety i.e. $M = \mathrm{Spec}(\mathbb{C}[M])$, where $\mathbb{C}[M]$ is a finitely generated $\mathbb{C}$-algebra without nilpotents
- G is a complex reductive group $\Leftrightarrow G = K_{\mathbb{C}}$ is a complexification of its maximal compact subgroup $K \subset G$ (i.e. $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$)
- G acts on M, then the invariants $\mathbb{C}[M]^G$ form a finitely generated $\mathbb{C}$-algebra without nilpotents
- define $M/\!/G := \mathrm{Spec}(\mathbb{C}[M]^G)$ the quotient map $\pi : M \to M/\!/G$ arises via the embedding $\mathbb{C}[M]^G \subset \mathbb{C}[M]$
- $M/\!/G$ parametrizes closed orbits of G (good quotient)
- when G acts freely $M/\!/G$ is identified with the orbit space (geometric quotient)
- when G acts freely and M is additionally non-singular $\Rightarrow M/\!/G$ is non-singular and $M \to M/\!/G$ is a principal bundle

Example of an affine GIT quotients

- Let $M = \mathbb{C}^n$ then $\mathbb{C}[M] = \mathbb{C}[z_1, \ldots, z_n]$
- the circle group $\mathrm{G} = \mathrm{GL}_1 = \mathbb{C}^\times$ is reductive as it is the complexification $\mathrm{U}(1) \subset \mathrm{GL}_1$
- Let $\mathrm{G} = \mathrm{GL}_1 = \mathbb{C}^\times$ act on $\mathbb{C}^n$ by multiplication $x \mapsto \lambda x$
- then $\lambda \in \mathrm{GL}_1$ acts on $\mathbb{C}[z_1, \ldots, z_n]$ as $z_i \mapsto \lambda z_i$
- thus $\mathbb{C}[M]^{\mathrm{G}} = \mathbb{C} \rightsquigarrow \mathbb{C}^n /\!/ \mathbb{C}^\times = \{0\}$ is a point
- there is only one closed orbit of $0 \in \mathbb{C}^n$

Affine symplectic quotients

- M non-singular affine variety
- $\omega \in H^0(M; \Lambda^2(T^*M))$ is *symplectic* $\Leftrightarrow$ it is nowhere degenerate and $d\omega = 0$
- $X \in H^0(M; TM)$ vector field is *Hamiltonian* $\Leftrightarrow$ there exists algebraic function $f : M \to \mathbb{C}$ such that for every $Y \in H^0(M; TM)$ $\omega(X, Y) = df(Y)$
- in particular $df(X) = 0 \Leftrightarrow X$ is tangent to the level sets of f (conservation of energy)
- an action of an algebraic group G on (M, ω) is *Hamiltonian* $\Leftrightarrow$ if the vector fields $X_v \in H^0(M; TM)$ induced by any one parameter subgroup G_v for $v \in \mathfrak{g}$ are simultaneously Hamiltonian $\Leftrightarrow$ there is a map $\mu : M \to \mathfrak{g}^*$ such that $\langle T\mu(Y), v\rangle = \omega(X_v, Y)$
- μ is called a *moment map*; it is G-equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$
- assume complex reductive group G acts on a symplectic affine variety M with moment map μ then the *complex symplectic quotient* at level $\xi \in (\mathfrak{g}^*)^{\mathrm{G}}$ is $M////_{\xi}\mathrm{G} := \mu^{-1}(\xi)//\mathrm{G}$

Example

- Let $M := T^*\mathbb{C}^2 = \mathbb{C}^4$; $\mathbb{C}[M] = \mathbb{C}[x_1, x_2, y_1, y_2]$; symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$
- $\lambda \in \mathbb{C}^\times$ acts symplectically on M by $(x_1, x_2, y_1, y_2) \mapsto (\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2)$
- the vector field $X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}$ is the Hamiltonian vector field of $f : M \to \mathbb{C}$ given by $f(x_1, x_2, y_1, y_2) = x_1 y_1 + x_2 y_2$ because $df = y_1 dx_1 + x_1 dy_1 + y_2 dx_2 + x_2 dy_2 = \omega(X_1, .)$
- The moment map $\mu : M \to \mathfrak{g}^*$ is just $\mu = f$
- the level set $\mu^{-1}(1)$ is $\mathcal{U} = \{x_1 y_1 + x_2 y_2 = 1\}$ non-singular acted upon freely by $\mathbb{C}^\times \rightsquigarrow X := M////_1 \mathrm{G} = \mu^{-1}(1)//\mathbb{C}^\times$ is a non-singular symplectic affine surface
- the map $X \to \mathbb{P}^1$ induced by $(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2)$ makes it a fibration with fibers $\cong \mathbb{A}^1$
  $\Rightarrow$ weight filtration on $H_c^*(X)$ is pure
- $\mathcal{U} \to X$ is GL_1-principal bundle, and so
  $|X(\mathbb{F}_q)| = \frac{|\mathcal{U}(\mathbb{F}_q)|}{|\mathrm{GL}_1(\mathbb{F}_q)|} = q^2 + q \overset{Katz}{\Rightarrow} E(X; q) = q^2 + q$
  $\Rightarrow$ by purity $P_c(X; t) = t^2 + t^4$

Linear symplectic quotients

- G complex reductive group; $\mathbb{V}$ finite dimensional complex vector space
- assume G acts on $\mathbb{V}$ linearly via the representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$, with derivative the Lie algebra homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$
- symplectic structure on $\mathbb{M} := \mathbb{V} \times \mathbb{V}^*$ given by $\omega((v_1, w_1), (v_2, w_2)) = w_1(v_2) - w_2(v_1)$
- G acts on $\mathbb{V} \times \mathbb{V}^*$symplectically via the representation $\rho \oplus \rho^*$ where $\rho^* : \mathrm{G} \to \mathrm{GL}(\mathbb{V}^*)$ is the dual representation
- this action is Hamiltonian with moment map $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$defined by $\mu(v, w)(X) = \langle \varrho(X)v, w\rangle$
- for $\xi \in (\mathfrak{g}^*)^{\mathrm{G}}$ we have the *linear symplectic quotient* $\mathbb{M}////_{\xi}\mathrm{G} = \mu^{-1}(\xi)//\mathrm{G}$

Symplectic Quiver varieties

- For a quiver Γ and dimension vector α let $\{V_i\}_{i\in I}$ be a collection of finite dimensional vector spaces of dimension α
- $\mathbb{V}_\alpha = \bigoplus_{a\in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)})$
- $\mathrm{G}_\alpha = \bigtimes_{i\in I} \mathrm{GL}(V_i)/\mathrm{GL}_1$, where $\mathrm{GL}_1 = (\lambda,\ldots,\lambda)_{\lambda\in\mathrm{GL}_1} < \bigtimes_{i\in I} Z(\mathrm{GL}(V_i)) < \bigtimes_{i\in I} \mathrm{GL}(V_i)$
- its Lie algebra $\mathfrak{g}_\alpha = \{X_i \in \mathfrak{gl}(\mathbb{V}_i) | \sum_i \mathrm{tr}(X_i) = 0\} \subset \bigtimes_i \mathfrak{gl}(\mathbb{V}_i)$
- action $\rho : \mathrm{G}_\alpha \to \mathrm{GL}(\mathbb{V}_\alpha)$ from left and right
- for a *generic* $\xi \in (\mathfrak{g}_\alpha^*)^{\mathrm{G}_\alpha}$ define the quiver variety by

$$\mathcal{M}_\alpha = \mathbb{V}_\alpha \times \mathbb{V}_\alpha^* ////_\xi \mathrm{G}_\alpha$$

- if $\alpha \in \mathbb{N}^I$ is indivisible ($\gcd(\alpha) = 1$) then $\mathcal{M}_\alpha$ is non-singular, while if α is divisible ($\gcd(\alpha) > 1$) $\mathcal{M}_\alpha$ has singular points (when non-empty).
- when non-empty $\dim \mathcal{M}_\alpha = 2 - 2(\alpha,\alpha)$
- (Crawley-Boevey, Van den Bergh 2004) when α indivisible $|\mathcal{M}_\alpha(\mathbb{F}_q)| = q^{1-(\alpha,\alpha)} A_\Gamma(\alpha,q)$ & $H_c^*(\mathcal{M}_\alpha;\mathbb{Q})$ is pure $\rightsquigarrow$ $q^{1-(\alpha,\alpha)} A_\Gamma(\alpha,q) = P_c(\mathcal{M}_\alpha, q^{1/2}) \in \mathbb{N}[q]$
  $\rightsquigarrow$ Kac's Conjecture 2 when α indivisible

Example: Affine ALE spaces

- Γ affine quiver
- δ minimal positive imaginary root
- Then δ is indivisible and $2 - 2(\delta, \delta) = 2$
- $\rightsquigarrow$ for generic $\xi \in (\mathfrak{g}_\delta^*)^{G_\delta}$
  $\mathcal{M}_\xi(\delta)$ is non-singular surface *affine ALE space*
  (Kronheimer 1990)
- while $\mathcal{M}_0(\delta) = \mathbb{C}^2 /\!/ H$, where $H < \mathrm{SL}_2$ is a finite subgroup corresponding to Γ via the McKay correspondence
- previous example corresponded to $\hat{A}_1$ quiver

Nakajima quiver varieties

- $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ and $dim(V_i) = \mathbf{v}_i$ and $\dim(W_i) = \mathbf{w}_i$ then $\mathrm{G}_{\mathbf{v}} = \times_{i\in I}\mathrm{GL}(V_i)$ naturally acts on $\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{(i,j)\in E}\mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I}\mathrm{Hom}(W_i, V_i)$ the corresponding holomorphic symplectic quotient

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}_{\mathbf{v},\mathbf{w}}(\mathbf{1}_{\mathbf{v}})//\mathrm{G}_{\mathbf{v}}$$

  is the affine *Nakajima quiver variety*
- always non-singular of dimension $2d_{\mathbf{v},\mathbf{w}} = 2\left(\sum_{(i,j)\in E}\mathbf{v}_i\mathbf{v}_j + \sum_{i\in I}\mathbf{v}_i(\mathbf{w}_i - \mathbf{v}_i)\right)$
- Crawley-Boevey's trick: to a quiver Γ with two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I \rightsquigarrow \Gamma_{\mathbf{w}}$ which has $2n$ vertices $I' = \{1, \ldots, n, *\}$ with the same oriented arrows on $I \subset I'$ and $\mathbf{w}_i$ arrows from $*$ to i. Then one can identify $\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathcal{M}^{\Gamma_{\mathbf{w}}}_{(\mathbf{v},1)}$, $(\mathbf{v}, 1)$ is clearly indivisible so $P_c(\mathbb{M}_{\mathbf{v},\mathbf{w}}; q^{1/2}) = q^d A_{\Gamma_{\mathbf{w}}}((\mathbf{v}, 1), q)$

Fourier transform on a finite vector space

- V finite dimensional vector space over $\mathbb{F}_q$
- $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ non-trivial additive character
- $f : V \to \mathbb{C}$ its Fourier transform $\hat{f} : V^* \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in V} f(X)\Psi(\langle X, Y \rangle).$$

- Fourier inversion formula: $\hat{\hat{f}}(x) = |V| f(-x)$
- first application (Kraft-Riedtmann 1985)
  finite group G acts on V then
  $\mathcal{F} : \mathbb{C}^V \to \mathbb{C}^{V^*}$ given by $\mathcal{F}(f) = \hat{f}$ is a linear map, an isomorphism by Fourier Inversion and G equivariant by definition
  $\Rightarrow |V/G| = \dim((\mathbb{C}^V)^G) = \dim((\mathbb{C}^{V^*})^G) = |V^*/G|$
  $\Rightarrow |\mathbb{V}_\alpha^\Gamma(\mathbb{F}_q)/\mathrm{G}_\alpha| = |\mathbb{V}_\alpha^{\Gamma'}/\mathrm{G}_\alpha|$,
  where Γ' is obtained from Γ by reversing one arrow $\Rightarrow$
  $M_\Gamma(\alpha, q)$ is independent of the orientiation on Γ $\Rightarrow$
  $A_\Gamma(\alpha, q)$ is independent of the orientation of the arrows (without Kac-Stanley-Hua combinatorics)

Fourier transform on $\mathfrak{g}^*$

- Recall G acts on $\mathbb{V}$, with derivative $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$, inducing action on $\mathbb{M} := \mathbb{V} \times \mathbb{V}^*$, Hamiltonian with moment map $\mu : \mathbb{M} \to \mathfrak{g}^*$, given by $\mu(v, w)(X) = \langle \varrho(X)v, w\rangle$
- For $\xi \in \mathfrak{g}^*(\mathbb{F}_q)$ the count function of the moment map $\mu : \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}^*(\mathbb{F}_q) \to \mathfrak{g}^*(\mathbb{F}_q)$
$\#_\mu(\xi) := \#\{(v, w) \in \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}^*(\mathbb{F}_q) | \mu(v, w) = \xi\} = \sum_{(v,w)\in\mathbb{M}} \delta_{\mu(v,w)}(\xi)$
- $\widehat{\#}_\mu(x) = \sum_{(v,w)\in\mathbb{M}} \widehat{\delta}_{\mu(v,w)}(x) = \sum_{(v,w)\in\mathbb{M}} \Psi(\langle \mu(v, w), x\rangle) = \sum_{(v,w)\in\mathbb{M}} \Psi(\langle \varrho(x)v, w\rangle) = \sum_{v\in\mathbb{V}} \sum_{w\in\mathbb{V}^*} \Psi(\langle \varrho(x)v, w\rangle) = |\mathbb{V}| \sum_{v\in\mathbb{V}} \delta_0(\rho(x)v) = |\mathbb{V}| a_\rho(x)$,
where $a_\varrho(x) = |\ker \varrho(x)|$

Proposition (Hausel, 2006)

$$\hat{\#}_\mu(x) = |\mathbb{V}| a_\varrho(x) \quad \overset{\substack{\textit{Fourier} \\ \textit{inversion}}}{\Longrightarrow} \quad \#_\mu = \frac{|\mathbb{V}|}{|\mathfrak{g}|} \hat{a}_\varrho$$

Example

- $\varrho : \mathfrak{gl}(1) \to \mathfrak{gl}(\mathbb{C}^2)$ by $(\alpha) \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
  $a_\varrho : \mathbb{F}_q \to \mathbb{C}$ is $a_\varrho(\alpha) = 1$ unless $\alpha = 0$ when $a_\varrho(0) = q^2$
  $a_\varrho = 1 + (q^2 - 1)\delta_0$ and so
  $\hat{a}_\mu = q\delta_0 + (q^2 - 1)$.
  Now $\mu : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathfrak{gl}(1)^*$ is given by $x_1 y_1 + x_2 y_2$.
  Recall $\mathcal{U} = \mu^{-1}(1)$. Indeed
  $\#\mathcal{U}(\mathbb{F}_q) = \#_\mu(1) = \frac{q^2}{q}\hat{a}_\rho(1) = q(q^2 - 1) = (q - 1)(q^2 + q)$

Fourier transform for Nakajima quiver varieties

- we start counting points on $\mathcal{M}(\mathbf{v},\mathbf{w})(\mathbb{F}_q)$ by Fourier transform.
- Recall $\varrho_{\mathbf{v},\mathbf{w}} : \mathfrak{g}_{\mathbf{v}} \to \mathfrak{gl}(\mathbb{V}_{\mathbf{v},\mathbf{w}})$ $a_{\varrho_{\mathbf{v},\mathbf{w}}} = |\ker(\varrho_{\mathbf{v},\mathbf{w}})|$
  $\mathcal{V}(\mathbf{v},\mathbf{w}) := \mu_{\mathbf{v},\mathbf{w}}^{-1}(\mathbf{1}_{\mathbf{v}})$
  $\Phi(\mathbf{w}) := \sum_{\mathbf{v}\in\mathbb{N}^I} |\mathcal{M}(\mathbf{v},\mathbf{w})(\mathbb{F}_q)| \frac{|\mathfrak{g}_{\mathbf{v}}|}{|\mathbb{V}_{\mathbf{v},\mathbf{w}}|} X^{\mathbf{v}} =$
  $\sum_{\mathbf{v}\in\mathbb{N}^I} \frac{|\mathcal{V}(\mathbf{v},\mathbf{w})(\mathbb{F}_q)|}{|\mathrm{G}_{\mathbf{v}}(\mathbb{F}_q)|} \frac{|\mathfrak{g}_{\mathbf{v}}|}{|\mathbb{V}_{\mathbf{v},\mathbf{w}}|} X^{\mathbf{v}} = \sum_{\mathbf{v}\in\mathbb{N}^I} \sum_{x\in\mathfrak{g}_{\mathbf{v}}} \frac{a_{\varrho_{\mathbf{v},\mathbf{w}}}(x)\Psi(\mathrm{tr}_{\mathbf{v}}(x))}{|\mathrm{G}_{\mathbf{v}}(\mathbb{F}_q)|} X^{\mathbf{v}} =$
  $\sum_{\mathbf{v}\in\mathbb{N}^I} \sum_{[x]\in\mathfrak{g}_{\mathbf{v}}/\mathrm{G}_{\mathbf{v}}} \frac{a_{\varrho_{\mathbf{v},\mathbf{w}}}(x)\Psi(\mathrm{tr}_{\mathbf{v}}(x))}{|C_x|} X^{\mathbf{v}}.$
  $\Phi_{nil}(\mathbf{w}) := \sum_{\mathbf{v}\in\mathbb{N}^I} \sum_{[x]\in(\mathfrak{g}_{\mathbf{v}}/\mathrm{G}_{\mathbf{v}})^{nil}} \frac{a_{\varrho_{\mathbf{v},\mathbf{w}}}(x)}{|C_x|} X^{\mathbf{v}}$ and
  $\Phi_{reg} := \sum_{\mathbf{v}\in\mathbb{N}^I} \sum_{[x]\in(\mathfrak{g}_{\mathbf{v}}/\mathrm{G}_{\mathbf{v}})^{reg}} \frac{a_{\varrho_{\mathbf{v},\mathbf{0}}}(x)\Psi(\mathrm{tr}_{\mathbf{v}}(x))}{|C_x|} X^{\mathbf{v}}.$
  we notice $\Phi(\mathbf{w}) = \Phi_{reg}\Phi_{nil}(\mathbf{w})$ but $\Phi(\mathbf{0}) = 1 \Rightarrow \Phi_{reg} = \frac{1}{\Phi_{nil}(\mathbf{0})}$
  $\Phi(\mathbf{w}) = \frac{\Phi_{nil}(\mathbf{w})}{\Phi_{nil}(\mathbf{0})}$
- we find combinatorially $\Phi_{nil}(\mathbf{w})$ it is a rational function in $q \Rightarrow$ so is $|\mathcal{M}(\mathbf{v},\mathbf{w})(\mathbb{F}_q)| \Rightarrow$ it is a polynomial $\overset{Katz}{\Rightarrow}$ it is $E(\mathcal{M}(\mathbf{v},\mathbf{w}); q)$
- we show that the weight filtration on $\mathcal{M}(\mathbf{v},\mathbf{w})(\mathbb{C})$ is pure by finding a compactification $\overline{\mathcal{M}(\mathbf{v},\mathbf{w})}$ which is an orbifold and surjects on cohomology $\Rightarrow$
  $E(\mathcal{M}(\mathbf{v},\mathbf{w}); q) = P_c(\mathcal{M}(\mathbf{w},\mathbf{w}); q^{1/2})$

Betti numbers of Nakajima quiver varieties

Theorem (Hausel 2006)

For any quiver Γ, and $\mathbf{w} \in \mathbb{N}^I$ the Betti numbers Nakajima quiver varieties are:

$$\sum_{\mathbf{v}\in\mathbb{N}^I} \sum_i \dim(H_c^{2i}(\mathcal{M}(\mathbf{v},\mathbf{w})))q^{i-d(\mathbf{v},\mathbf{w})}X^{\mathbf{v}} =$$

$$= \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} q^{\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}},$$

where $2d(\mathbf{v},\mathbf{w}) = 2\sum_{(i,j)\in E}\mathbf{v}_i\mathbf{v}_j + 2\sum_{i\in I}\mathbf{v}_i(\mathbf{w}_i - \mathbf{v}_i)$ *is the dimension of* $\mathcal{M}(\mathbf{v},\mathbf{w})$, $X^{\mathbf{v}} = \prod_{i\in I} T_i^{\mathbf{v}_i}$ *and* $\langle\lambda,\mu\rangle = \sum_{i,j}\min(\lambda_i,\mu_j)$

Note that the denominator is the LHS of Hua's formula!
Need to relate it to the Kac denominator formula.

Weyl-Kac character formula for Nakajima quiver varieties

Theorem (Kac 1974)

Let $L(\mathbf{w})$ be an irreducible representation of $\mathfrak{g}(\Gamma)$ of highest weight $\Lambda \in P$. Let $L(\Lambda) = \oplus_{\alpha\in\mathbb{N}^I} L(\Lambda)_{\Lambda-\alpha}$ denote its weight space decomposition. Then

$$\sum_{\alpha\in\mathbb{N}^I} \dim\left(L(\Lambda)_{\Lambda-\alpha}\right) X^{\alpha} = \frac{\sum_{w\in W} \det(w) X^{\Lambda+\rho-w(\Lambda+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^{\alpha})^{m_\alpha}}$$

Theorem (Nakajima 1998)

Fix $\mathbf{w} \in \mathbb{N}^I$ then there is an irreducible representation of the Kac-Moody algebra $\mathfrak{g}(\Gamma)$ of highest weight $\Lambda_{\mathbf{w}}$ on $\bigoplus_{\mathbf{v}\in\mathbb{N}^I} H_c^{2d_{\mathbf{v},\mathbf{w}}}(\mathcal{M}(\mathbf{v},\mathbf{w}))$, in particular

$$\sum_{\mathbf{v}\in\mathbb{N}^I} \dim\left(H^{2d_{\mathbf{v},\mathbf{w}}}(\mathcal{M}(\mathbf{v},\mathbf{w})\right) X^{\mathbf{v}} = \frac{\sum_{w\in W} \det(w) X^{\Lambda_{\mathbf{w}}+\rho-w(\Lambda_{\mathbf{w}}+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^{\alpha})^{m_\alpha}}$$

Proof of Kac's Conjecture 1

Weyl-Kac-Nakajima formula + our main formula $\rightsquigarrow$

$$\frac{\sum_{w\in W} \det(w) X^{\Lambda_{\mathbf{w}}+\rho-w(\Lambda_{\mathbf{w}}+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^\alpha)^{m_\alpha}} = \left(\frac{\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} q^{\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}}{\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}} \right)_{q=0}$$

In the special case when $\mathbf{w} = m\mathbf{1}$, i.e. $\Lambda_{\mathbf{w}} = m\rho$ and $m \to \infty$

$$\prod_{\alpha\in\mathbb{N}^I}(1-X^\alpha)^{m_\alpha} = \left(\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)} \right)_{q=0}$$

$$\overset{\text{Hua}}{=} \left(\prod_{\alpha\in\mathbb{N}^n} \prod_{j=0}^{\infty} \prod_{i=0}^{\infty} (1-q^{i+j}X^\alpha)^{t_i^\alpha} \right)_{q=0} = \prod_{\alpha\in\mathbb{N}^n}(1-X^\alpha)^{t_0^\alpha}$$

Theorem (Hausel 2006)

$A_\Gamma(\alpha, 0) = m_\alpha$