Quiver-theoretic techniques for the Deligne-Simpson-Problem

Lecture Series given by

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Lecture Series

Lecture 1 (27/7). Deformed and multiplicative preprojective algebras and middle convolution. (§1.0 - §1.7)

Lecture 2 (28/7). Monodromy, logarithmic connections and parabolic bundles. (§1.8 - §2.5)

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1 DSP, preprojecitve algebras & middle convolutions

1.1 The Deligne Simpson Problem

In the following we let $k = \mathbb{C}$.

Problem. Given conjugacy classes $C_1, \ldots, C_k$ in $Gl_n(\mathbb{C})$, decide whether or not there is an irreducible solution to

$$A_1 \cdots A_k = I$$

with $A_i \in C_i$. In this context irreducibility means that no non-trivial proper subspace of $\mathbb{C}^n$ is stabilized by all A_i simultaneously.

Example. Given C_1, C_2, C_3 in $Gl_2(\mathbb{C})$ there is an irreducible solution iff

- none of the conjugacy classes is $\{\lambda \cdot I\}$
- the product of all 6 eigenvalues for the C_i is 1
- the product of 3 eigenvalues, one for each C_i, is never 1.

Exercise. Suppose $ABC = I$ is an irreducible solution with 2×2-matrices. Assume that λ, μ, ν are eigenvalues of A, B, C with $\lambda\mu\nu = 1$.

(i) Show that no two of A, B, C can have a common eigenvector

(ii) Let x, y, z be eigenvectors for λ, μ, ν. Show that $z = x + y$ after suitable rescaling.

(iii) Show that $(A - \lambda)Bz = -(B - \mu)(\lambda x)$

(iv) Show that $\mathrm{Im}(A - \lambda) = \mathrm{Im}(B - \mu)$ and that this is an invariant subspace, which is a contradiction to irreducibility.

Therefore note that irreducibility implies $\lambda\mu\nu \neq 1$.

Different authors obtained various results:

Deligne. If there is an irreducible solution then

$$\sum_i \dim C_i \geq 2n^2 - 2$$

with equality iff the solution is rigid (i.e. unique up to simultanious conjugation)

Simpson (1992). Studied case when C_i have generic eigenvalues one with distinct eigenvalues.

Katz (1996) "Rigid local systems", "Middle convolution"
Algorithm for computing all rigid irreducible solutions

Kostov Coined the name DSP, but he also wanted a construction of all irreducible solutions. He introduced the additive version $A_1 + \ldots + A_k = 0$ of the DSP and obtained many partial results.

Crawley-Boevey (CB) Found the link to root systems, obtained solution in the additive case and, together with Shaw, a sufficient condition for solutions in the multiplicative case. He also worked out the necessity, which has not been published yet.

1.2 Deformed preprojective algebras

Let Q be a quiver with vertex set I. We denote by kQ its path algebra.

Example.

$$\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet} \qquad Q$$

Then kQ has basis a, b, ba, e_1, e_2, e_3

The double of Q looks as follows and is denoted by $\bar{Q}$.

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "a"] & \overset{2}{\bullet} \arrow[l, shift left, "a^*"] \arrow[r, shift left, "b"] & \overset{3}{\bullet} \arrow[l, shift left, "b^*"] \end{tikzcd} \qquad \overline{Q}$$

We extend $\star$ to an involution on $\bar{Q}$ by $(a^\star)^\star = a$. Let

$$\varepsilon(c) = \begin{cases} 1 & c \text{ in } Q \\ -1 & c^\star \text{ in } Q \end{cases}$$

Definition. (CB+Holland)

Let $\lambda = (\lambda_i)_{i \in I} \in k^I$

The deformed preprojective algebra is defined by .

$$\Pi^\lambda(Q) := k\bar{Q}/(r)$$

with $r = \sum_{a \in Q} [a, a^\star] - \sum_{i \in I} \lambda_i e_i = \sum_{a \in Q} (aa^\star - a^\star a) - \sum_{i \in I} \lambda_i e_i = \sum_{a \in \bar{Q}} \varepsilon(a) aa^\star - \sum_{i \in I} \lambda_i e_i$.

Remark. The classical definition of the non-deformed case with $\lambda = 0$ is due to Gelfand+Ponomarev.

1.3 Multiplicative preprojective algebra

Let $k\bar{Q} \to L_Q$ be the universal localization of $k\bar{Q}$ obtained by adjoining an inverse for

$$1 + aa^\star \in k\bar{Q}$$

for all $a \in \bar{Q}$.

Definition. (CB+Shaw)

Fix an ordering $a_1 < a_2 < \ldots < a_m$ on the arrows of $\bar{Q}$. Fix a tuple of invertible elements $q = (q_i) \in (k^\times)^I$.

The multiplicative preprojective algebra is defined by $\Lambda^q := L_Q/(s)$ with

$$s = \overrightarrow{\prod_{a\in\bar{Q}}}(1 + aa^\star)^{\varepsilon(a)} - \sum_{i\in I} q_i e_i$$

where we denote by $\overrightarrow{\prod}$ the directed product induced by "<".

1.4 Representations

Definition. A representation of Q is given by vector spaces X_i $(i \in I)$ and linear maps $X_a : \; X_i \to X_j \; \forall a : i \to j$ in Q .

Representations of $Q \Longleftrightarrow kQ$-modules:

"$\Leftarrow$": Given a kQ-module X define $X_i = e_i X$, $i \in I$ and the linear maps X_a is given by multipliplication with a.

"$\Rightarrow$": $(X_i, X_a) \longrightarrow \bigoplus_{i\in I} X_i$, the kQ-action is given by the maps X_a.

$\Pi^\lambda(Q)$-modules $\longleftrightarrow$Representations of $\bar{Q}$ by vector spaces X_i and linear maps X_a $(a \in \bar{Q})$ satisfying

$$\sum_{\substack{a \in \bar{Q} \\ h(a) = i}} \varepsilon(a) X_a X_{a^\star} = \lambda_i \mathrm{id}_{X_i} \quad \forall i \in I$$

which we rewrite as

$$\sum_{\substack{a \in Q \\ h(a) = i}} X_a X_{a^\star} - \sum_{\substack{a \in Q \\ t(a) = i}} X_{a^\star} X_a = \lambda_i id_{X_i}$$

Λ^q-modules $\longleftrightarrow$ Representations with :

$(\mathrm{id}_{X_{h(a)}} + X_a X_{a^\star})$ is invertible $\forall a$ in $\bar{Q}$ and for all $i \in I$:

$$\overrightarrow{\prod_{\substack{a \in \bar{Q} \\ h(a) = i}}} (\mathrm{id}_{X_i} + X_a X_{a^\star})^{\varepsilon(a)} = q_i \mathrm{id}_{X_i}$$

1.5 Lemma

(i) If there is a representation of $\Pi^\lambda(Q)$ of dimension vector $\alpha = (\alpha_i) \in \mathbb{Z}^I$ ($\alpha_i = \dim X_i$), then $\lambda.\alpha := \sum_{i \in I} \lambda_i \alpha_i$ is zero

(ii) If there is a representation of $\Lambda^q(Q)$ of dimension vector $\alpha = (\alpha_i) \in \mathbb{Z}^I$ ($\alpha_i = \dim X_i$), then $q^\alpha := \prod_i q_i^{\alpha_i}$ is 1.

Proof. (i)

$$\begin{aligned} 0 &= \sum_{a \in Q} \operatorname{tr}(X_a X_{a^\star}) - \sum_{a \in Q} \operatorname{tr}(X_{a^\star} X_a) \\ &= \sum_{i \in I} \operatorname{tr}(\lambda_i \mathrm{id}_{X_i}) = \sum_{i \in I} \lambda_i \dim(X_i) \end{aligned}$$

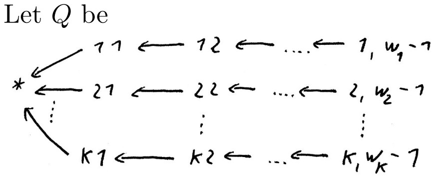
(ii) Since $\prod_{\substack{a \in \bar{Q} \\ h(a) = i}} \det((\mathrm{id}_{X_i} + X_a X_{a^\star})^{\varepsilon(a)}) = q_i$ and $\det(\mathrm{id} + \theta\phi) = \det(\mathrm{id} + \phi\theta)$ for arbitrary endomorphisms ϕ, θ, **(ii)** is obtained analogously to **(i)**.

□

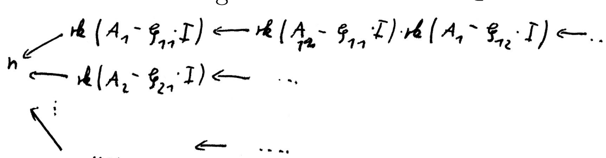
1.6 Link to the DSP

Given $C_1, \ldots, C_k$ in $Gl_n(k)$. Let w_i be the degree of the minimal polynomial for C_i and $\xi_{i,1}, \ldots, \xi_{i,w_i}$ be its roots.

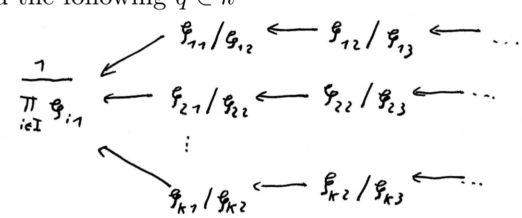
Let Q be



Consider the following dimension vector $\alpha \in \mathbb{Z}^I$



and the following $q \in k^I$



Lemma.

(i) There is a solution to $A_1 \cdot \ldots \cdot A_k = I$, $A_i \in \bar{C}_i$, where $\bar{C}_i$ denotes the closure of the conjugacy class C_i, iff there exists a representation of Λ^q of dimension α.

(ii) There is a solution to $A_1 \cdot \ldots \cdot A_k = I$, $A_i \in C_i$ iff there exists a "strict" representation of Λ^q of dimension α, i.e. all $X_a, X_{a^\star}$ have maximal rank.

(iii) There is an irreducible solution to $A_1 \cdot \ldots \cdot A_k = I$, $A_i \in \bar{C}_i$ iff there exists a simple representation of Λ^q of dimension α.

Similar results hold for solutions of $A_1 + \cdots + A_k = 0$ and representations of Π^λ, for suitable λ.

1.7 Reflection functors

If Q is a quiver, we denote by $(\cdot,\cdot)$ the bilinear form on $\mathbb{Z}^I$given by

$$(\alpha,\beta) := \sum_{i\in I} 2\alpha_i\beta_i - \sum_{\substack{a \in Q \\ a : i \to j}} \alpha_i\beta_j$$

If there is no loop at i, then we define $s_i : \mathbb{Z}^I \to \mathbb{Z}^I$ by

$$s_i(\alpha) = \alpha - (\alpha, \varepsilon[i])\varepsilon[i]$$

where $\varepsilon[i]$ is given by $\varepsilon[i]_j = \delta_{i,j}$. The Weyl group is the subgroup of the automorphism group generated by the s_i.

Theorem (CB+Holland, Rump). If i is loop free and $\lambda_i \neq 0$, then there exists an equivalence:

$$\Pi^\lambda(Q)\text{-modules} \leftrightarrow \Pi^{r_i(\lambda)}(Q)\text{-modules}$$

which acts as s_i on dimension vectors where r_i is given by the formula

$$r_i(\lambda).\beta := \lambda.s_i(\beta)$$

Theorem (Dettweiler+Reiter, CB+Shaw). If i is loop free and $q_i \neq 1$, there exists an equivalence

$$\Lambda^q(Q)\text{-modules} \leftrightarrow \Lambda^{u_i(q)}(Q)\text{-modules}$$

which acts as s_i on dimension vectors where u_i is given by the formula

$$u_i(q)^\beta := q^{s_i(\beta)}$$

1.8 Katz's Algorithm

We start with two lemmas which, in the setting of the DSP, were known to Katz.

Lemma. The simple modules for $\Lambda^q(Q)$ of dimension vector α are parametrized by a variety of dimension $2-(\alpha,\alpha)$ (perhaps empty or disconnected).

If $q_i = 1$ then $\Lambda^q(Q)$ has a simple $S[i]$ of dimension $\varepsilon[i]$.

Lemma. If there is a simple for $\Lambda^q(Q)$ of dimension vector α, then
$\alpha = \varepsilon[i]$ or $q_i \neq 1$ or $(\alpha, \varepsilon[i]) < 0$.

Katz's algorithm. Any rigid simple for $\Lambda^q(Q)$ can be reduced by a sequence of reflection functors to a simple $S[i]$ for some $\Lambda^{q'}(Q)$ for some i and q'.

1.9 Roots

Definition. Let Q be a quiver with vertex set I. We define a subset $F \subset \mathbb{N}^I$ by

$$F = \left\{\alpha \in \mathbb{N}^I : \alpha \neq 0, (\alpha, \varepsilon[i]) \leq 0 \ \forall i, \quad \alpha \text{ has connected support}\right\}$$

Then the roots are given by

$$\{w(\varepsilon[i]) : i \text{ loop free vertex}, w \in W\} \cup \{\pm w\alpha : \alpha \in F, w \in W\}$$

where the first set gives the real roots ($p(\alpha) = 0$) and the second one the imaginary roots ($p(\alpha) > 0$), where $p(\alpha) := 1 - \frac{1}{2}(\alpha, \alpha)$. We remark that the $\varepsilon[i]$ and F form fundamental regions for the real and imaginary roots, respectively.

Theorem. If there is a simple $\Lambda^q(Q)$ of dimension α, then α is a root.

Theorem.

Multiplicative version: There is a rigid simple for $\Lambda^q(Q)$ of dimension α iff

- α is a real root
- $q^\alpha = 1$
- There is no decomposition $\alpha = \beta + \gamma + \ldots$ as a sum of positive roots with $q^\beta = q^\gamma = \ldots = 1$

Additive version: There is a rigid simple for $\Pi^\lambda(Q)$ of dimension α iff

- α is a real root
- $q^\alpha = 1$
- There is no decomposition $\alpha = \beta + \gamma + \ldots$ as a sum of positive roots with $\lambda.\beta = \lambda.\gamma = \ldots = 0$.

2 Lifting, Riemann-Hilbert, parabolic bundles

Hereditary abelian category		Doubled version
kQ-modules	$\xrightarrow{\text{lifting}}$	$\Pi^\lambda(Q)$-modules$\leftrightarrow$additive DSP
	$\overset{\text{bad}}{\not\longrightarrow}$	$\Lambda^q(Q)$-modules $\leftrightarrow$DSP
parabolic bundles, or coherent sheaves on a weighted projective line	$\longrightarrow$	parabolic bundles + compatible ζ-connection $\to$ DSP, via. Riemann-Hilbert

2.1 Lifting representations

Theorem. A representation X of Q lifts to a representation of $\Pi^\lambda(Q)$ iff every indecomposable summand Y of X has $\lambda.\underline{\dim}Y = 0$.

Moreover, if it does lift, then the possible lifts are parametrized by $D\mathrm{Ext}^1_{kQ}(X,X)$.

Proof. "$\Rightarrow$" Take θ to be the projection onto Y and apply exercise 4 on the problem sheet.

"$\Leftarrow$": It suffices to show that if X is indecomposable and $\lambda.\underline{\dim}X = 0$, then X lifts. Consider

$$0 \to \bigoplus_{a:i\to j} kQe_j \otimes_k e_iX \to \bigoplus_i kQe_i \otimes_k e_iX \to X \to 0$$

where we set $P_1 = \bigoplus_{a:i\to j} kQe_j \otimes_k e_iX$, $P_0 = \bigoplus_i kQe_i \otimes_k e_iX$

Apply $\mathrm{Hom}_{kQ}(\cdot, X)$. This gives

$$0 \to \mathrm{End}_{kQ}(X) \to \mathrm{Hom}_{kQ}(P_0, X) \to \mathrm{Hom}_{kQ}(P_1, X) \to \mathrm{Ext}^1_{kQ}(X,X) \to 0$$

with $\mathrm{Hom}_{kQ}(P_0, X) = \oplus_i \mathrm{End}_k(X_i)$, $\mathrm{Hom}_{kQ}(P_1, X) = \oplus_{a:i\to j}\mathrm{Hom}_k(X_i, X_j)$. Also note that$\mathrm{Hom}_{kQ}(kQe_i \otimes Z, X) \simeq \mathrm{Hom}_k(Z, e_iX)$. Now dualize and use that $D\mathrm{Hom}_k(U,V) \simeq \mathrm{Hom}_k(V,U)$ via trace pairing. We obtain

$$0 \to D\mathrm{Ext}^1(X,X) \to \bigoplus_{a:i\to j} \mathrm{Hom}(X_j, X_i) \xrightarrow{\rho} \bigoplus_i \mathrm{End}_k(X_i) \xrightarrow{\tau} D\mathrm{End}_{kQ}(X) \to X \to 0$$

$\rho : (\Psi_a : X_j \to X_i) \mapsto (\sum_{h(a)=i} X_a\Psi_a - \sum_{t(a)=i} \Psi_a X_a)_i$, $\tau : \theta = (\theta_i) \mapsto f_\theta$ with $f_\theta(\Phi) = \sum_i \mathrm{tr}(\theta_i\Phi_i)$.

Now consider $\lambda \in k^I$ as an element in $\oplus_i\mathrm{End}_k(X_i)$. Obviously f_θ is zero on nilpotent elements and by construction also on multiples of the identity due to the assumption $\lambda.\underline{\dim}Y = 0$ for all indecomposable summands Y of X. By splitting the endomorphism ring into its nilpotent and identity part, f_θ equals 0, so that the theorem follows.

□

2.2 Application of Kac's Theorem

Corollary. There is a representation of $\Pi^\lambda(Q)$ of dimension α iff $\alpha = \beta+\gamma+\ldots$ for some positive roots $\beta, \gamma, \ldots$ with $\lambda.\beta = \lambda.\gamma = \ldots = 0$.

Consequence. Given $C_1, \ldots, C_k$ one can determine whether or not there is a solution to $A_1 + \ldots + A_k = 0$ with $A_i \in \bar{C}_i$.

2.3 Hilberts 21st problem

Let $X = \mathbb{CP}^1$(Riemann's sphere). Let $D = \{a_1, \ldots, a_k\} \subset X$. Consider a system of 1st order linear ordinary differential equations

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = A(x)\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

with $A(x) \in M_n(\mathbb{C}(x))$ nonsingular outside D.

We obtain a monochromy repr. $\rho : \Pi_1(X - D) \to Gl_n(\mathbb{C})$.

$\Pi_1(X-D)$

Problem. Can you get any representation ρ from some Fuchsian system (i.e. $A(x)$ has at worst simple poles at the a_i)?

For many years it was thought that work of Plemelj (1908) implied that the answer was yes. But Bolibruch (1989) discovered that the answer was actually no.

2.4 Riemann-Hilbert correspondence

Let $X = \mathbb{CP}^1$, $D = \{a_1, \ldots, a_k\}$, as above. Let E be a vector bundle on X. The notion of a Fuchsian system generalizes to that of a logarithmic (flat or integrable) connection on E:

$$\nabla : \mathcal{E} \to \Omega^1(\log \mathcal{D}) \otimes \mathcal{E}$$

with $\mathcal{E}$ the sheaf sections of E and $\mathcal{D}$ the sheaf of differential 1-forms $f(x)dx$ where $f(x)$ has at worst simple poles at the a_i. Check that$\nabla(fe) = df \otimes e + f\nabla(e)$ for $f \in \mathcal{O}_x(\mathcal{U})$, $e \in \mathcal{E}(\mathcal{U})$. Since $\dim_{\mathbb{C}} X = 1$, X is automatically flat!

Residues $\mathrm{Res}_{a_i}\nabla \in \mathrm{End}_k(E_{a_i})$, where E_{a_i} denotes the fibre of E at a_i, corresponds to $\mathrm{Res}_{X=a_i}A(x)$:

Theorem. Let $T = \{x \in \mathbb{C} : \ \mathrm{Re}\ x \in [0, 1)\}$, or any other transversal to Z in $\mathbb{C}$. Then monodromy gives an equivalence from the category of pairs $(\mathcal{E}, \nabla)$, such that all eigenvalues of all $\mathrm{Res}_{a_i}\nabla$ are in T, to the representations ρ of $\Pi_1(X-D)$.

Note that $\Pi_1(X-D) = \langle g_1, \dots, g_k : g_1 \cdot \ldots \cdot g_k = 1\rangle$, so $\rho(g_1)\cdot \ldots \cdot \rho(g_k) = 1$. In this non-resonant case $\rho(g_i)$ is conjugate to $e^{2\pi\sqrt{-1}\mathrm{Res}_{a_i}\nabla}$. We want to know about the existence of $(\mathcal{E}, \nabla)$ with $\mathrm{Res}_{a_i}\nabla$ prescribed up to conjugacy.

2.5 Parabolic bundles

Definition. A weighted projective line $\mathbb{X}$ consists of

- $X = \mathbb{CP}^1$,
- $D = (a_1, \dots, a_k)$, a_i distinct points and
- $w = (w_1, \dots, w_k)$, w_i positive integers , where $w_i = 1$ is the same as a_i unweighted

Definition. The category of (quasi) parabolic bundles on $\mathbb{X}$ has objects $\mathcal{E} = (E, E_{ij})$ where E is a vector bundle and the $E_{i,j}$ are flags of subspaces

$$E_{a_i} = E_{i,0} \supset E_{i,1} \supset \ldots \supset E_{i,w_i} = 0$$

The morphisms are given by vector bundle homomorphisms compatible with flags. Then one can draw an associated quiver

$$\begin{tikzcd}
 & 11 \arrow[dl] & 12 \arrow[l] & \cdots \arrow[l] & {1,w_1-1} \arrow[l] \\
\ast & 21 \arrow[l] & 22 \arrow[l] & \cdots \arrow[l] & {2,w_2-1} \arrow[l] \\
\vdots & & \vdots & & \vdots \\
 & k1 \arrow[uul] & k2 \arrow[l] & \cdots \arrow[l] & {k,w_k-1} \arrow[l]
\end{tikzcd}$$

where the dimension vector α of $\mathcal{E}$ is given by

$$\begin{tikzcd}
 & \dim E_{11} \arrow[dl] & \dim E_{12} \arrow[l] & \cdots \arrow[l] \\
\mathrm{rk}\, E & \dim E_{21} \arrow[l] & \cdots \arrow[l] & \\
 & \vdots & & \\
 & \cdots \arrow[uul] & \cdots \arrow[l] &
\end{tikzcd}$$

and the dimension type of $\mathcal{E}$ is $\alpha + (\deg E)\delta \in \mathbb{Z}^I \oplus \mathbb{Z}\delta$.

2.6 ζ-connections and lifting

Definition. Let $\mathbb{X}$ be a weighted projective line, $X = \mathbb{CP}^1$, $D = (a_1, \dots, a_k)$ and $w = (w_1, \dots, w_k)$. Let $\mathcal{E} = (E, E_{ij})$ be a parabolic bundle on $\mathbb{X}$. Let $\zeta = (\zeta_{ij})$ with $\zeta_{ij} \in \mathbb{C}$ for $1 \le i \le k$, $1 \le j \le w_i$.

A ζ-connection on $\mathcal{E}$ is a logarithmic connection on E with

$$(\mathrm{Res}_{a_i}\nabla - \zeta_{ij})(E_{i,j-1}) \subset E_{i,j} \; \forall i, j$$

Lifting Theorem (cf. Weil, 1938). $\mathcal{E}$ has a ζ-connection iff the dimension type $\alpha + d\delta$ of any indecomposable summand of $\mathcal{E}$ satisfies $d + \zeta \star [\alpha] = 0$ with

$$\zeta \star [\alpha] := \sum_i \zeta_{ij}\alpha_{ji} + \sum_{i,j}(\zeta_{i,j+1} - \zeta_{i,j})\alpha_{i,j}$$

Analogue of Kac's Theorem. The dimension types of indecomposable parabolic bundles are exactly the $\alpha + d\delta$ where α is a "strict" positive root, i.e. decreasing on the arms. Then given conjugacy classes $C_1, \ldots, C_k$, there exists a solution to $A_1 \cdot \ldots \cdot A_k = I$, $A_i \in \bar{C}_i$

$\overset{\text{Riemann-Hilbert}}{\Longleftrightarrow}$ there exists (E, ∇) with $\mathrm{Res}_{a_i}\nabla \in \bar{C}'_i$ for suitable C'_i,

$\Longleftrightarrow$ there exists a parabolic bundle of dimension vector α and with a ζ connection ∇ for suitable α, ζ.

$$\begin{tikzcd}
E_{a_i} = E_{i,0} \arrow[r, bend left, "\mathrm{Res}\,\nabla - \zeta_{i,1}\cdot I"] & E_{i,1} \arrow[l, hook'] \arrow[r, bend left, "\cdots"] & E_{i,2} \arrow[l, hook'] \arrow[r, bend left, "\cdots"] & \cdots \arrow[l, hook']
\end{tikzcd}$$

$\Longleftrightarrow$there exists a decomposition $\alpha = \beta + \gamma + \ldots$ with $\beta, \gamma, \ldots$ positive roots with $\zeta \star [\beta], \zeta \star [\gamma], \ldots \in \mathbb{Z}$.
That way we have also transformed the multiplicative version of the DSP into a problem which contains roots only.

3 Homological algebra and geometry for representations and a sufficient condition for existence of simples

The following strategy and arguments work for Π^λ as well as Λ^q. In the following we consider only Π^λ. For Λ^q one can use the published argument in [4].

3.1 Bimodule resolution

Lemma. For $\Pi = \Pi^\lambda(Q)$ there is an exact sequence of bimodules

$$P_2 \xrightarrow{f} P_1 \xrightarrow{g} P_0 \xrightarrow{\text{mult. \&add}} \Pi \to 0$$

where $P_2 = P_0 = \oplus_{i\in I}\Pi e_i \otimes e_i\Pi$ and $P_1 = \oplus_{\substack{a \in \bar{Q} \\ a : i \to j}} \Pi e_j \otimes e_i \Pi$ and we define f and g by

$$\begin{aligned}
f((p \otimes q)_i) &= \sum_{\substack{a \in \bar{Q} \\ h(a) = i}} \varepsilon(a)(pa^\star \otimes q)_a - \sum_{\substack{a \in \bar{Q} \\ t(a) = i}} \varepsilon(a)(p \otimes a^\star q)_a \\
g((p \otimes q)_{a:i\to j}) &= (pa \otimes q)_i - (p \otimes aq)_j
\end{aligned}$$

respectively. It is easily checked that $g \circ f = 0$.

The Koszul property For Q a non-Dynkin quiver f is injective for $\Pi = \Pi^0(Q)$.

In the case with no oriented cycles one adds the projective resolution of a simple kQ-module

$$0 \to 0 \to \bigoplus_{\substack{a \in Q \\ a : i \to j}} P[j] \to P[i] \to S[i] \to 0$$

and the Auslander-Reiten-sequences

$$0 \to \tau^{-n} P[i] \to \bigoplus_{\substack{a \in Q \\ a : i \to j}} \tau^{-(n+1)} P[j] \oplus \bigoplus_{\substack{a \in \bar{Q} \backslash Q \\ a : i \to j}} \tau^{-n} P[j] \to \tau^{-(n+1)} P[i] \to 0 \to 0$$

which exist and are exact for all $n > 0$ since Q is non-Dynkin, to give a projective resolution of the simple Π-module

$$0 \to \Pi e_i \to \bigoplus_{\substack{a \in \bar{Q} \\ a : i \to j}} \Pi e_j \to \Pi e_i \to S[i] \to 0$$

From these one can easily deduce that f is injective. The genaral case was described by Malkin, Ostrik and Vybornov.

3.2 The Calabi-Yau 2-property

Lemma. There exists an exact sequence

$$P_1 \simeq \mathrm{Hom}_{\Pi^e}(P_1, \Pi^e) \stackrel{\mathrm{Hom}(f,\Pi^e)}{\longrightarrow} \mathrm{Hom}_{\Pi^e}(P_2, \Pi^e) \simeq P_0 \to \Pi \to 0$$

Theorem.

(i) For $\Pi = \Pi^0(Q)$, Q a non-Dynkin quiver,

$$\mathrm{Ext}^n(M, N) \simeq D\mathrm{Ext}^{2n}(N, M)$$

for any finite dimensional Π-modules M, N.

(ii) In the general case we have

$$\dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N) + \dim \mathrm{Hom}(N, M) = (\underline{\dim} M, \underline{\dim} N)$$

3.3 The variety of representations

For a quiver Q with vertex set I and $\alpha \in \mathbb{N}^I$ let $\mathrm{Rep}(\bar{Q}, \alpha) = \bigoplus_{\substack{a\,:\,i \to j \\ \text{in } \bar{Q}}} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha_j})$, $Gl(\alpha) = \prod_{i \in I} Gl(k^{\alpha_i})$ and $\mathrm{End}(\alpha) = \mathrm{Lie}\, Gl(\alpha) = \oplus_{i \in I} \mathrm{End}(k^{\alpha_i})$. We define $\mu : \mathrm{Rep}(\bar{Q}, \alpha) \to \mathrm{End}(\alpha)$ by

$$x \mapsto \Big(\sum_{\substack{a \in \bar{Q} \\ h(a) = i}} \varepsilon(a) x_a x_{a^\star} \Big)_i$$

Then the image of μ is obviously contained in $\mathrm{End}(\alpha)_0 := \{(\theta_i) \in \mathrm{End}(\alpha) | \sum tr(\theta_i) = 0\}$. Given $\lambda \in k^I$, consider it as $(\lambda \cdot \mathrm{id}) \in \mathrm{End}(\alpha)$.

Lemma. *$Rep(\Pi^\lambda, \alpha) := \mu^{-1}(\lambda)$ is an affine variety. Every irreducible component has dimension at least*

$$\dim \mathrm{Rep}(\bar{Q}, \alpha) - \dim \mathrm{End}(\alpha)_0 = g + 2p(\alpha)$$

where $g = \dim \mathrm{End}(\alpha)_0 = \dim PGl(\alpha) = \alpha \cdot \alpha - 1$.

3.4 The moment map property

If $x \in \mathrm{Rep}(\bar{Q}, \alpha)$ then you get

$$\begin{tikzcd} D\mathrm{End}(\alpha) \arrow[r, "D(d\mu_x)"] \arrow[d, phantom, "\wr\wr"] \arrow[dr, phantom, "\circlearrowright"] & D\mathrm{Rep}(\bar{Q}, \alpha) \arrow[d, phantom, "\wr\wr"] \\ \mathrm{End}(\alpha) \arrow[r, "\varphi : \theta \mapsto {[\theta, x]}"] & \mathrm{Rep}(\bar{Q}, \alpha) \end{tikzcd}$$

with $D\mathrm{End}(\alpha) \simeq \mathrm{End}(\alpha)$ via trace pairing and $D\mathrm{Rep}(\bar{Q}, \alpha) \simeq \mathrm{Rep}(\bar{Q}, \alpha)$ by using ω with

$$\omega(x, y) := \sum_{a \in \bar{Q}} \varepsilon(a) \mathrm{tr}(x_a y_{a^\star})$$

We remark that the lower map is explicitly given by $\varphi : \theta = (\theta_i) \mapsto (\theta_j x_a - x_a \theta_i)_{a : i \to j \text{ in } \bar{Q}}$.

Lemma. The simples form a smooth open subset of $\mathrm{Rep}(\Pi^\lambda, \alpha)$ of dimension $g + 2p(\alpha)$ (but possibly empty or disconnected).

Proof. Let X be simple, so $\mathrm{End}(X) = k$. Thus $\dim \mathrm{Ker}(\varphi) = 1$, so by trace pairing $d\mu_x$ is onto $\mathrm{End}(\alpha)_0$ and therefore μ is smooth at x as a map to $\mathrm{End}(\alpha)_0$.

$\square$

3.5 Moduli space

Let $N(\lambda, \alpha) = \mathrm{Rep}(\Pi^\lambda, \alpha)//Gl(\alpha) = \mu^{-1}(\lambda)//Gl(\alpha)$, which can also be written as $\mathrm{Rep}(\bar{Q}, x)////_\lambda Gl(\alpha)$. This classifies semi-simple Π^λ-modules up to isomorphism. Consider the map $\mathrm{Rep}(\Pi^\lambda, \alpha) \xrightarrow{q} N(\lambda, \alpha)$.

Lemma. The simples in $N(\lambda, \alpha)$ form a smooth open subset of dimension $2p(\alpha)$ (possibly empty or disconnected).

We remark that this lemma is used by Katz's Algorithm.

Definition. (Le Bruyn+Procesi) A semi-simple representation is of type $(k_1, \beta^1, k_2, \beta^2, \ldots)$ if it is isomorphic to

$$\underbrace{S_1 \oplus \ldots \oplus S_1}_{k_1} \oplus \underbrace{S_2 \oplus \ldots S_2}_{k_2} \oplus \ldots$$

where the S_i are the non-isomorphic simples with dimension vectors β^i.

Theorem. The semi-simples of type $(k_1, \beta^1, \ldots)$ form a locally closed subset of $N(\lambda, \alpha)$ (maybe empty, etc.) of dimension $2p(\beta^1) + 2p(\beta^2) + \ldots$.

3.6 Fibres of the quotient map

In the following we consider the canonical quotient map $q : \mathrm{Rep}(\Pi^\lambda, \alpha) \longrightarrow N(\lambda, \alpha)$.

Theorem. If $x \in N(\lambda, \alpha)$ has type $(k_1, \beta^1, k_2, \beta^2, \ldots)$ then

$$\dim q^{-1}(x) \leq g + p(\alpha) - \sum_t p(\beta^t)$$

Idea. Induction, formula for $\dim \mathrm{Ext}^1$, and the following

Lemma. Let M, N be finite dimensional modules for Π^λ. Let $\alpha = \underline{\dim} M + \underline{\dim} N$. Then

$$\left\{(x, \theta, \phi) : x \in \mathrm{Rep}(\Pi^\lambda, \alpha),\ 0 \to N \xrightarrow{\theta} X \xrightarrow{\phi} M \to 0 \text{ exact}\right\}$$

is a variety of dimension $\dim Gl(\alpha) + \dim \mathrm{Ext}^1(M, N) - \dim \mathrm{Hom}(M, N)$.

Idea. Take the universal extension via the pushout

$$\begin{tikzcd}
0 \arrow[r] & N^\ell \arrow[r] \arrow[d, "\xi"'] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where $l := \dim \operatorname{Ext}^1(M, N)$ and a vector $\xi \in k^l$. Then choose basis for X, which is of dimension $\dim Gl(\alpha)$.

3.7 Sufficient condition for simples

Theorem. If α is apositive root with $\lambda.\alpha = 0$, $p(\alpha) > p(\beta) + p(\gamma) + \ldots$ for any decomposition $\alpha = \beta + \gamma + \ldots$ of positive roots $\beta, \gamma, \ldots$ with and $\lambda.\beta = \lambda.\gamma = \ldots = 0$, then there exists a simple for Π^λ of dimension α.

In the following we denote the condition of the theorem by †. By lifting $\operatorname{Rep}(\lambda, \alpha) \neq \emptyset$. It has dimension $\geq g + 2p(\alpha)$. However

$$\begin{aligned} \dim q^{-1}(\text{stratum of type } (k_1, \beta^1, \ldots)) &\leq (g + p(\alpha) - \sum p(\beta^t)) + \sum 2p(\beta^t) \\ &< g + 2p(\alpha) \end{aligned}$$

which is too small to fill up all the compositions so that under this hypothesis we obtain that $\operatorname{Rep}(\Pi^\lambda, \alpha)$ is irreducible.

4 Perpendicular categories and the necessity of the condition for simples

In this paragraph we show that the condition † follows from the existence of a simple Π^λ-module of dimension α.

Given matrices $C_1, \ldots, C_k$, we let w_i be the degree of their minimal polynomials with roots $\xi_{i,j}$. We have to construct Q and α for a given simple Π^λ-module.

Theorem.

(i) In the additive version we have that there exists a positive root α with $\xi \star [\alpha] = 0$ and $p(\alpha) > p(\beta) + p(\gamma) + \ldots$ for all decompositons $\alpha = \beta + \gamma + \ldots$ of positive roots $\beta, \gamma, \ldots$ with $\xi \star [\beta] = \ldots = 0$ iff there exists an irreducible solution to $A_1 + \ldots + A_k = 0$, $A_i \in C_i$.

(ii) In the multiplicative version we have that there exists a positive root α with $\xi^{[\alpha]} = 1$ and $p(\alpha) > p(\beta) + p(\gamma) + \ldots$ for all decompositons $\alpha = \beta + \gamma + \ldots$ of positive roots $\beta, \gamma, \ldots$ with $\xi^{[\beta]} = \ldots = 0$ iff there exists an irreducible solution to $A_1 \cdot \ldots \cdot A_k = I$, $A_i \in C_i$.

The notation $\xi \star [\alpha]$ was introduced in section 2.6. The notation $\xi^{[\alpha]}$ is the analogous expression involving products and powers.

4.1 Combinatorial part

Given a quiver Q with vertex set I consider the pairs $(\lambda, \alpha) \in k^I \times \mathbb{Z}^I$ and the equivalence relation "$\sim$" generated by $(\lambda, \alpha) \sim (r_i \lambda, s_i \alpha)$, whenever i is a loop free vertex with $\lambda_i \neq 0$ and r_i is given by the formula

$$r_i\lambda.\beta = \lambda.s_i\beta$$

Thus if $(\lambda, \alpha) \sim (\lambda', \alpha')$, then there is a composition of reflection functors Π^λ-mod $\longrightarrow \Pi^{\lambda'}$-mod , sending modules of dimension vector α to modules of dimension vector α' .

Definition. Given $\lambda \in k^I$, we define

$$\begin{aligned} F_\lambda \quad = \quad & \{\alpha \in \mathbb{Z}^I : \alpha \text{ positive root with } \lambda.\alpha = 0 \text{ such that for } (\lambda, \alpha) \sim (\lambda', \alpha'), \\ & i \text{ loopfree vertex with } \lambda_i = 0 \text{ then } (\alpha', \varepsilon[i]) \leq 0\} \end{aligned}$$

F_0 is the fundamental region defined above.

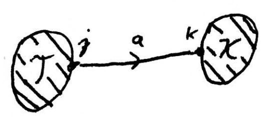
Lemma. If there is a simple for $\Pi^\lambda$of dimension vector α, then either $\alpha \in F_\lambda$ or $(\lambda, \alpha) \simeq (\lambda', \varepsilon[i])$ for some loopfree vertex i.

In the latter case S is a rigid simple obtained from the simple $S[i]$ for Π^λ by a sequence of reflection functors.

Theorem. Suppose $\alpha \in F_\lambda$, but † fails. Then after passing to an equivalent pair (λ, α), and passing to the support quiver of α, one of the following cases holds:

(I) Q is an extended Dynkin quiver with minimal positive imaginary root δ and $\lambda.\delta = 0$ and $\alpha = m\delta$ for some $m \geq 2$.

(II) $I = \mathcal{T}\dot{\cup}\mathcal{K}$ with only one arrow $a : j \to k$ from $\mathcal{T}$ to $\mathcal{K}$ linking the two parts of Q.



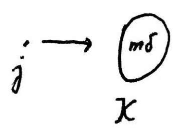
and $\alpha_j = \alpha_k = 1$, $\sum_{i\in\mathcal{T}} \lambda_i\alpha_i = 0$.

(III) $I = J\dot{\cup}\mathcal{K}$with $a : j \to k$. $Q|_\mathcal{K}$ is an extended quiver of Dynkin type with minimal positive imaginary root δ, $\delta_\mathcal{K} = 1$, $\alpha|_\mathcal{K} = m\delta$ $(m \geq 2)$, $\alpha|_J = 1$ and $\sum_{i\in\mathcal{K}} \lambda_i\delta_i = 0$.

4.2 Special and almost special cases

In the following we use Kostov's terminology. To prove the necessity of † it suffices to show there is no simple in cases **(I)**,**(II)** and **(III)**. **(II)** is easily checked by applying the trace function to $X_{a^*}X_a$, where a is the only edge from $\mathcal{T}$ to $\mathcal{K}$.

Similarly, in order to show that there is no simple of type (III), it suffices to show that there is none of type (III'), where J is replaced by a one element set $J = \{j\}$.



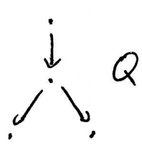
4.3 Perpendicular categories

Definition (Geigle+Lenzing, Schofield). Let $\mathcal{C}$ be a hereditary abelian category, M an object in $\mathcal{C}$. The perpendicular category $M^{\perp}$ is the full subcategory of $\mathcal{C}$ with objects $X \in \mathcal{C}$ such that $\mathrm{Hom}(M, X) = \mathrm{Ext}^1(M, X) = 0$.

We remark that $M^{\perp}$ is closed under kernels, images, cokernels, extensions, so it is itself a hereditary abelian category. If in addition $\mathcal{C}$ has finite dimensional Hom spaces and $\mathrm{Ext}^1(M, M) = 0$, then the inclusion $M^{\perp} \hookrightarrow \mathcal{C}$ has a left adjoint $l : \mathcal{C} \to M^{\perp}$.

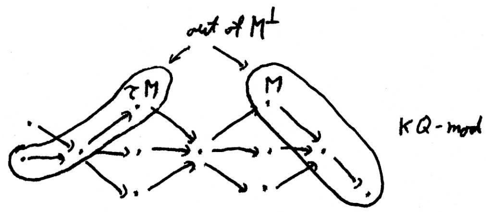
If $\mathcal{C} = kQ$-mod, $\mathrm{Ext}^1(M, M) = 0$ then $M^{\perp} \simeq kQ'$-mod for some quiver Q'.

Example. If Q is given by

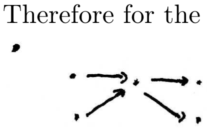


and we take M to be the indecomposable kQ-module of dimension vector

then the Auslander-Reiten-quiver looks as follows:



Therefore for the AR-quiver of $M^{\perp}$remains



and Q' is given by

Q'

In the general case, if Q is of extended Dynkin type and M a regular simple, then $M^{\perp}$ has tubular structure, with one wide tube made smaller and Q' is still of extended Dynkin type with one vertex less.

4.4 The multiplicative case

As seen above, for this case we need to lift from parabolic bundles on $\mathbb{X}$, which generalize to coherent sheaves on $\mathbb{X}$, defined by Geigle and Lenzing. We briefly mention the following theorem:

Theorem (Hübner+Lenzing). If M is an indecomposable parabolic bundle and $\operatorname{Ext}^1(M,M)=0$ then $M^{\perp} \simeq kQ$-mod for some quiver Q.

4.5 Preprojective Algebras

There is an abstract version of $\Pi^{\lambda}(Q)$ defined by $\Pi^{a}(A) := T_A \operatorname{Der}(A, A\otimes A)/(\Delta - a)$ with $\Delta(x) = x\otimes 1 - 1\otimes x$ and $a \in A$. Using this one can show:

Theorem. Suppose M is a module for kQ and $M^{\perp} \simeq kQ'$-mod. Then

$$\left\{X \text{ module for } \Pi^{\lambda}(Q),_{kQ} X \in M^{\perp}\right\} \simeq \Pi^{\lambda'}(Q')$$

for some λ'.

Theorem. Suppose M is an indecomposable bundle in $\mathbb{X}$ with $\operatorname{Ext}^1(M,M)=0$ so that $M^{\perp} \simeq kQ'$-mod.

Assume Q' has no connected component of shape

$\cdot \longrightarrow \cdot \longrightarrow \cdot \cdots \longrightarrow \cdot \longrightarrow \cdot$

Then

$$\left\{(\mathcal{E},\nabla),\ \mathcal{E} \text{ a coherent sheaf on } \mathbb{X},\ \mathcal{E}\in M^{\perp},\ \nabla\zeta-\text{connection on } \mathcal{E}\right\} \simeq \Pi^{\lambda'}(Q')\text{-mod}$$

for some λ'.

4.6 The Case (III')

In this section we prove there is no simple in the case **(III')**. We denote the quiver given by the component $\mathcal{K}$ by Q'.

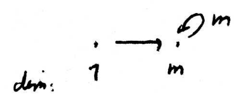
Consider

$$\operatorname{Rep}(\Pi^{\lambda}(Q),\alpha) \xrightarrow{\rho} \operatorname{Rep}(Q,\alpha) \xrightarrow{\sigma} \operatorname{Rep}(\tilde{Q}, m\delta)$$

Step 1 $\operatorname{Rep}(\Pi^{\lambda}(Q),\alpha)$ has $m+1$ irreducible components and all of them dominate $\operatorname{Rep}(\tilde{Q}, m\delta)$.

Step 2 Choose a regular simple module M for $k\tilde{Q}$. Then $M^{\perp}$ defines an open subset of $\mathrm{Rep}(\tilde{Q}, m\delta)$. Thus, if there is a simple for $\Pi^{\lambda}(Q)$ of dimension vector α, there is one whose image is in $\mathrm{Rep}(\tilde{Q}, m\delta)$. Then ${}_{kQ}S \simeq ({}_{kQ}M)^{\perp}$, so S corresponds to a simple for $\Pi^{\lambda'}(Q')$ for some smaller quiver $\tilde{Q}'$.

If one keeps shrinking by Step 1 and 2, one eventually obtains for Π^0:



In this quiver such simples do not exist as one checks the following:

Lemma. If A, B are $m \times m$ matrices, $m \geq 2$, and $\mathrm{rk}(AB - BA) \leq 1$ then A and B have a common invariant subspace.

5 Literature

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