Recollements and tilting objects

Lidia Angeleri Hügel
Università di Verona, Italy
(joint work with Steffen König and Qunhua Liu)

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Ring epimorphisms

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1. λ is a ring epimorphism.
2. $\lambda_* : \mathrm{Mod}\text{-}S \rightarrow \mathrm{Mod}\text{-}R$ is a full embedding.

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Then λ is said to be a homological ring epimorphism.

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(T3) there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where T_0, T_1 belong to Add T.

Classifying tilting modules

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Two tilting modules T and T' are equivalent if $\operatorname{Gen} T = \operatorname{Gen} T'$.

Classifying tilting modules

Theorem (Bazzoni–Herbera 2005). Every tilting class is of the form

$$\mathrm{Gen}\, T = \mathsf{Ker}\, \mathsf{Ext}^1_R\,(\mathcal{U}, -)$$

where $\mathcal{U} \subset \mathrm{mod}\text{-}R$ be a set of R-modules of pdim 1.

Tilting modules arising from universal localization.

Let now $\mathcal{U} \subset \mathrm{mod}\text{-}R$ be a set of R-modules of pdim 1.
For each $U \in \mathcal{U}$, fix a projective resolution in $\mathrm{mod}\text{-}R$

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and set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$.

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Theorem (Schofield). There is $\lambda \colon R \to R_{\mathcal{U}}$ such that

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$\lambda \colon R \to R_{\mathcal{U}}$ is a ring epimorphism with $\text{Tor}_1^R(R_{\mathcal{U}}, R_{\mathcal{U}}) = 0$,
the universal localization of R at $\mathcal{U}$.

Tilting modules arising from universal localization.

A classification result. Over a *Dedekind domain*, every tilting module is equivalent to a module of the form

$$R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

where $\mathcal{U} = \{R/\mathfrak{m} \mid \mathfrak{m} \in \mathfrak{P}\}$ and $\mathfrak{P}$ is a set of maximal ideals of R

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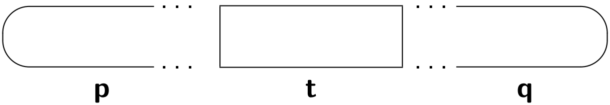
Similar results also for
commutative 1-Gorenstein rings, HNP–rings ...

Further tilting modules.

Example. Let R be a (connected) hereditary finite dimensional algebra. The Auslander-Reiten-quiver of R is of the form



p is the preprojective component
q is the preinjective component
t is a family of regular components.

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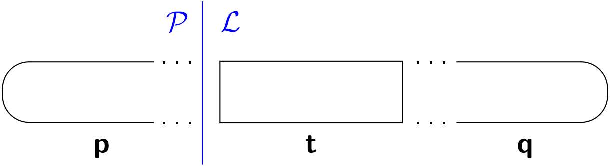
There is a torsion pair $(\mathcal{P}, \mathcal{L})$ maximal w.r.t. $\mathbf{p} \subset \mathcal{P}$ e $\mathbf{t} \subset \mathcal{L}$



with a large tilting module $L \in \mathrm{Mod}\text{-}R$ such that $\mathrm{Gen} L = \mathcal{L}$
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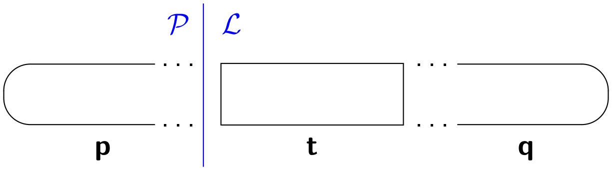

with a large tilting module $L \in \text{Mod-}R$ such that $\text{Gen}L = \mathcal{L}$ (Lukas 1991, Kerner–Trlifaj 2005).

Note: L is **not** equivalent to a tilting module of the form $S \oplus S/R$ for some injective ring epimorphism $R \to S$.

Constructing recollements from tilting modules

Theorem (A-Archetti 2008). For every tilting module T there are an exact sequence

$$0 \to R \to T_0 \to T_1 \to 0$$

and a set $\mathcal{U}$ a set of R-modules of pdim 1 such that

1. $T_0, T_1 \in \mathrm{Add}\, T$,
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2. $\mathrm{Gen}\, T = \mathrm{KerExt}^1_R(\mathcal{U}, -)$,
3. the perpendicular category $T_1^{\perp} = \bigcap_{i \geq 0} \mathrm{KerExt}^i_R(T_1, -)$ coincides with the essential image of the functor

$$\lambda_* : \mathrm{Mod}\text{-}R_{\mathcal{U}} \to \mathrm{Mod}\text{-}R$$

induced by the universal localization λ at $\mathcal{U}$.

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Let T and T_1 be as above. Consider

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Note: $\mathcal{Y}$ is closed under small coproducts.

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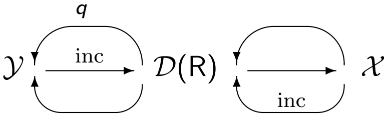
Theorem (A–König–Liu 2008). Every tilting module T of projective dimension one induces a recollement



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- T_1 is an exceptional generator of $\mathcal{X}$.
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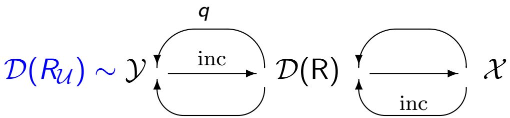

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$$\mathcal{D}(R_{\mathcal{U}}) \sim \mathcal{Y} \;\; \overset{q}{\underset{}{\leftarrow}} \!\!\!\!\! \xrightarrow{\text{inc}} \!\!\!\!\! \leftarrow \;\; \mathcal{D}(\mathrm{R}) \;\; \leftarrow \!\!\!\!\! \xrightarrow{} \!\!\!\!\! \underset{\text{inc}}{\leftarrow} \;\; \mathcal{X} \sim \mathcal{D}(V)$$

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  In this case λ_* induces an equivalence $\mathcal{D}(R_{\mathcal{U}}) \sim \mathcal{Y}$.
- $\mathcal{X} \sim \mathcal{D}(V)$ with $T_1 \mapsto V_V$ for some ring V $\Leftrightarrow$ $T \in \mathrm{mod}\text{-}R$ up to equivalence.

Example 1

Over the *Kronecker-algebra*

$$\bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet$$

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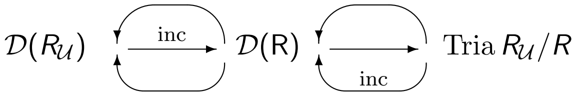

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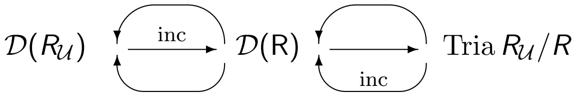

and the infinite dimensional tilting modules, up to equivalence, are:
- the tilting module L with $\mathrm{Gen} L = \mathcal{L}$
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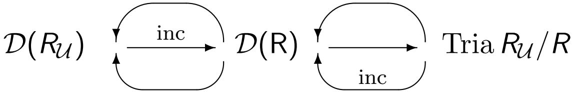
Over a Prüfer domain every tilting module T induces a recollement



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and T is of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R \;\Leftrightarrow\; \mathrm{pd} R_{\mathcal{U}} \leq 1$.

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Moreover

$$\{\text{equivalence classes of tilting modules}\} \overset{1-1}{\leftrightarrow} \{\text{perfect Gabriel topologies}\}$$

$$T \mapsto (R \to R_{\mathcal{U}})$$

(Bazzoni-Eklof-Trlifaj, Salce 2005).

Example 3

Over the quasi-hereditary algebra $R = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1\,3 \\ 2 \end{matrix} \oplus \begin{matrix} 3 \\ 2 \end{matrix}$ consider the characteristic tilting module

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Here $\lambda : R \to R_{\mathcal{U}}$, the universal localization at $\mathcal{U} = \{ \begin{array}{c}2\\1\end{array} \}$,

is **not** a homological epimorphism.