Auslander-Reiten Theory on the homotopy category of projective modules

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Background

- (Auslander/Reiten, 1970's): defined almost split sequences in module categories; presented Auslander-Reiten formula; proved the existence theorem
- (Happel, 1987): defined Auslander-Reiten triangles (AR-triangles) in triangulated categories; characterized AR-triangles in $\mathbf{D}^b(\Lambda\text{-mod})$
- (Krause, 2000): proved the existence of AR-triangles in compactly generated triangulated categories (via Brown Representability)
- (Krause/L, 2006): presented AR-formula in $\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Inj})$; characterized AR-triangles in $\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Inj})$

Motivation

✠ (Jørgensen, 2005):

$$\left.\begin{array}{r}\Lambda \text{ coherent ring} \\ \forall_\Lambda M \text{ flat}, \operatorname{proj.dim.} M < \infty\end{array}\right\} \Rightarrow \mathbf{K}(\Lambda\text{-Proj}) \text{ compactly generated}$$

✠ (Neeman, 2008):

- Λ right coherent $\Rightarrow \mathbf{K}(\Lambda\text{-Proj})$ compactly generated
  - $X \in \mathbf{K}(\Lambda\text{-Proj})$ compact $\Leftrightarrow \begin{cases} X^n \text{ finitely generated}, \forall n, \\ X^n = 0 \text{ for } n \ll 0, \\ H^n(X^*) = 0 \text{ for } n \ll 0. \end{cases}$

(here $X^* = \operatorname{Hom}_\Lambda(X, \Lambda)$)

Main Results

k – commutative noetherian ring, complete, local

Λ – noetherian k-algebra,

E – injective envelope of $k/\mathfrak{m}$ as k-module, where $\mathfrak{m}$ denotes the unique maximal ideal of k,

$D = \mathrm{Hom}_k(-, E)\colon \mathrm{Mod}k \to \mathrm{Mod}k$,

induces $D\colon \Lambda\text{-}\mathrm{Mod} \to \Lambda^{\mathrm{op}}\text{-}\mathrm{Mod}$,

and duality $D\colon \Lambda\text{-}\mathrm{Art} \to \Lambda^{\mathrm{op}}\text{-}\mathrm{Noeth}$.

Theorem 1 Let Z and Y be complexes of projective Λ-modules, and suppose that Z is a compact object in $\mathbf{K}(\Lambda\text{-Proj})$. Then we have an isomorphism

$$D\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z, Y) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, \mathbf{p}DZ^*) \tag{1}$$

which is natural in Z and Y.

sketch of proof:

$$
\begin{aligned}
D\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z,Y) &\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj})}(\mathbf{p}Y^*,Z^*),E) \\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{D}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*,Z^*),E) \\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*,\mathbf{i}Z^*),E) \\
&\cong \mathrm{Hom}_k(H^0\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*,\mathbf{i}Z^*),E) \\
&\cong H^0\mathrm{Hom}_k(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*,\mathbf{i}Z^*),E) \\
&\cong H^0\mathrm{Hom}_k(\mathbf{i}Z^*\otimes_\Lambda \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*,\Lambda),E) \\
&\cong H^0\mathrm{Hom}_k(\mathbf{i}Z^*\otimes_\Lambda Y,E) \\
&\cong H^0\mathrm{Hom}_\Lambda(Y,\mathrm{Hom}_k(\mathbf{i}Z^*,E)) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Flat})}(Y,D\mathbf{i}Z^*) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y,j^*(D\mathbf{i}Z^*)) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y,\mathbf{p}D\mathbf{i}Z^*) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y,\mathbf{p}DZ^*)
\end{aligned}
$$

- $D\colon \Lambda^{\mathrm{op}}\text{-Inj} \longrightarrow \Lambda\text{-Flat}$
- $\mathrm{inc}\colon \mathbf{K}(\Lambda\text{-Proj}) \hookrightarrow \mathbf{K}(\Lambda\text{-Flat})$ has right adjoint j^*, and j^* preserves coproducts
- $\forall X$ complex of flat Λ-modules, $\exists$ two exact triangles

$$\begin{array}{ccccccc}
\mathbf{p}X & \xrightarrow{\alpha_X} & X & \xrightarrow{\beta_X} & \mathbf{a}X & \longrightarrow & \mathbf{p}X[1] \\
{\scriptstyle \exists!\,\xi_X}\downarrow & & \| & & \downarrow & & \downarrow \\
j^*(X) & \xrightarrow{\alpha'_X} & X & \xrightarrow{\beta'_X} & i^*(X) & \longrightarrow & j^*(X)[1].
\end{array}$$

define $\mathbf{r}\colon \mathbf{K}(\Lambda\text{-Flat}) \xrightarrow{\mathbf{p}|_{\mathbf{K}(\Lambda\text{-Flat})}} \mathbf{K}_{\mathrm{proj}}(\Lambda\text{-Mod}) \xrightarrow{\mathrm{inc}} \mathbf{K}(\Lambda\text{-Proj})$, (note that $\mathbf{r}(X) = \mathbf{p}X$), then ξ_X induces a morphism of triangle functors $\xi\colon \mathbf{r} \to j^*$

- X right bounded complex of flat modules $\Rightarrow \xi_X$ isomorphic

Proposition 2 Let Z be a compact object in $\mathbf{K}(\Lambda\text{-Proj})$ which is indecomposable. Then there exists an AR-triangle

$$\mathbf{p}DZ^*[-1] \to Y \to Z \to \mathbf{p}DZ^*. \tag{2}$$

Define

$$\mathbf{t} \colon \mathbf{K}(\Lambda\text{-Proj}) \xrightarrow{\mathrm{Hom}_\Lambda(-,\Lambda)} \mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj}) \xrightarrow{D} \mathbf{K}(\Lambda\text{-Inj}) \xrightarrow{\mathbf{p}} \mathbf{K}(\Lambda\text{-Proj}).$$

Then $\mathbf{t}$ restricts to an equivalence $\mathbf{t} \colon \mathbf{K}^c(\Lambda\text{-Proj}) \longrightarrow \mathcal{S}$,

$$\text{where } X \in \mathcal{S} \Leftrightarrow \begin{cases} X^n = 0 \text{ for } n \gg 0, \\ H^n(X) = 0 \text{ for } n \ll 0, \\ H^n(X) \text{ is artinian over } \Lambda, \forall n. \end{cases}$$

Theorem 3 Let M be a left artin Λ-module which is indecomposable and non-injective. Then there exists an Auslander-Reiten triangle

$$\mathbf{p}M[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} (\mathbf{p}DM)^* \xrightarrow{\gamma} \mathbf{p}M$$

in $\mathbf{K}(\Lambda\text{-Proj})$ which the functor Cok^1 (for complex $X = (X^n, d_X^n)$, $\mathrm{Cok}^1(X) := X^1/\mathrm{Im}d_X^0$) sends to an almost split sequence

$$0 \to M \xrightarrow{\overline{\alpha^1}} \mathrm{Cok}^1(Y) \xrightarrow{\overline{\beta^1}} \mathrm{Tr}DM \to 0$$

in the category of Λ-modules.

Relation with injective case

Recall:

$\forall Z' \in \mathbf{K}(\Lambda^{\mathrm{op}}\text{-}\mathrm{Inj})$, compact, indecomposable, $\exists$ an AR-triangle

$$\eta_{Z'}\colon\ D(\mathbf{p}Z')^*[-1] \xrightarrow{u} Y' \xrightarrow{v} Z' \xrightarrow{w} D(\mathbf{p}Z')^*.$$

$\forall Z \in \mathbf{K}(\Lambda\text{-}\mathrm{Proj})$, compact, indecomposable, $\exists$ an AR-triangle

$$\varepsilon_Z\colon\ \mathbf{p}DZ^*[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathbf{p}DZ^*.$$

Observation: there are one to one correspondence of these AR-triangles consider adjoint pairs

$$K(\Lambda\text{-Proj}) \underset{j^*}{\overset{\mathrm{inc}}{\rightleftarrows}} K(\Lambda\text{-Flat}) \underset{D}{\overset{D}{\rightleftarrows}} K(\Lambda^{\mathrm{op}}\text{-Inj})$$

Denote by Ψ and Φ the unit and counit of $(D \circ \mathrm{inc}, j^* \circ D)$.

The *Auslander category*

$$\mathcal{A} := \{X \in K(\Lambda\text{-Proj}) \mid \Psi_X \text{ isomorphic}\}$$

is a triangulated subcategory of $K(\Lambda\text{-Proj})$.

The *Bass category*

$$\mathcal{B} := \{Y \in K(\Lambda^{\mathrm{op}}\text{-Inj}) \mid \Phi_Y \text{ isomorphic}\}$$

is a triangulated subcategory of $K(\Lambda^{\mathrm{op}}\text{-Inj})$.

$\forall Z \in K(\Lambda\text{-Proj})$, compact, indecomposable,

$$\varepsilon_Z\colon\ \mathbf{p}DZ^*[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathbf{p}DZ^*$$

is an exact triangle in $\mathcal{A}$, hence an AR-triangle in $\mathcal{A}$.
Note that D induces a duality $\mathcal{A} \longrightarrow \mathcal{B}$, therefore

$$D(\varepsilon_Z)\colon DZ \xrightarrow{D\beta} DY \xrightarrow{D\alpha} D\mathbf{p}DZ^*[-1] \to DZ[1]$$

is an AR-triangle in $\mathcal{B}$.

On the other hand,

$$\eta_{D\mathbf{p}DZ^*[-1]}\colon DZ \xrightarrow{u} Y' \xrightarrow{v} D\mathbf{p}DZ^*[-1] \xrightarrow{w} DZ[1]$$

is an exact triangle in $\mathcal{B}$, hence an AR-triangle in $\mathcal{B}$.

Comparing with end terms we have $D(\varepsilon_Z) \cong \eta_{D\mathbf{p}DZ^*[-1]}$.

Thank you