Degenerations in the additive categories of almost cyclic coherent Auslander-Reiten components

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Fact

$\leq_{\mathrm{deg}}$ *is a partial order on* $\mathrm{mod}_A(d)$.

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- $M \leq_{\text{ext}} N \Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences

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in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some $1 \leq s \in \mathbb{N}$

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Remark

The converse implication is not true in general even for very simple representation-finite algebras as in the following Riedtmann's example.

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Then kQ/I is an algebra of finite type and M, N are nonisomorphic and indecomposable. Moreover, $M <_{\text{deg}} N$, but $M \not<_{\text{ext}} N$.

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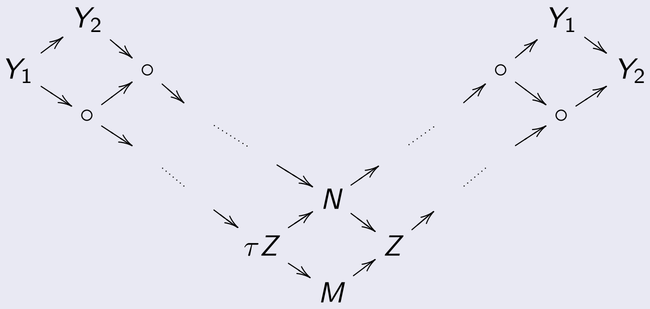
  2. For each injective module I in $\mathcal{C}$ there is an infinite sectional path $\cdots \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ in $\mathcal{C}$.

Möbius and coil configurations

A full translation subquiver of Γ_A of the form



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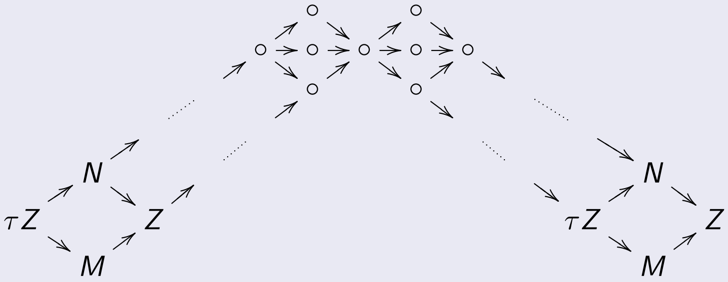

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Proposition

Let A be an algebra and $\mathcal{C}$ a component in Γ_A which contains a Möbius configuration or a coil configuration. Then there exist indecomposable modules M and N in $\mathcal{C}$ such that $[M] = [N]$ and $M <_{\text{deg}} N$.

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LEMMA

Let A be an algebra and

$$0 \to M_1 \xrightarrow{[f_1, u_1]^t} N_1 \oplus M_2 \xrightarrow{[u_2, f_2]} N_2 \to 0$$

$$0 \to M_2 \xrightarrow{[f_2, v_1]^t} N_2 \oplus M_3 \xrightarrow{[v_2, f_3]} N_3 \to 0$$

be short exact sequences in $\operatorname{mod} A$. Then the sequence

$$0 \to M_1 \xrightarrow{[f_1, v_1 u_1]^t} N_1 \oplus M_3 \xrightarrow{[-v_2 u_2, f_3]} N_3 \to 0$$

is exact.

PROPOSITION (Riedtmann)

If

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The same conclusion holds for an exact sequence

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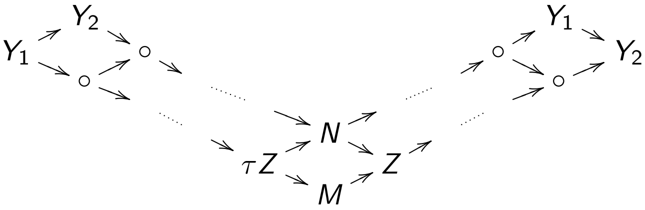
Assume first that $\mathcal{C}$ admits a Möbius configuration.



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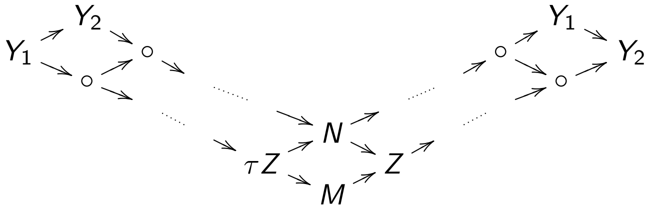

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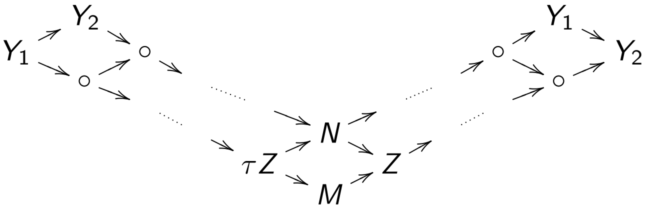

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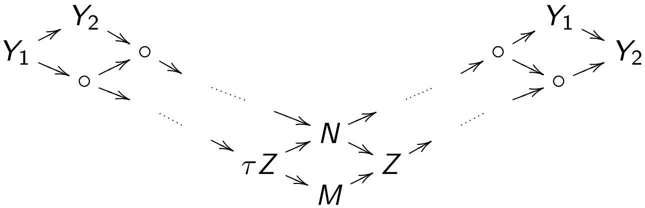

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Applying LEMMA again

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Theorem

Let A be an algebra and $\mathcal{C}$ a generalized standard almost cyclic coherent component of Γ_A. The following conditions are equivalent:

1. *$\mathcal{C}$ contains neither a Möbius configuration nor a coil configuration.*
2. *The partial orders $\leq_{\text{deg}}$ and $\leq_{\text{ext}}$ coincide on $\text{add}(\mathcal{C})$.*

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Theorem (M – Skowroński)

Let $\mathcal{C}$ be a connected component of Γ_A. Then $\mathcal{C}$ is coherent and almost cyclic if and only if $\mathcal{C}$ is a generalized multicoil.

Separating family of components

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Let A be an algebra. A family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ of components of Γ_A is said to be **separating** in $\operatorname{mod} A$ if the modules in $\operatorname{ind} A$ split into three disjoint classes $\mathcal{P}_A$, $\mathcal{C}_A = \mathcal{C}$ and $\mathcal{Q}_A$ such that:

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Note that then $\mathcal{P}_A$ and $\mathcal{Q}_A$ are uniquely determined by $\mathcal{C}_A$.

Results

Following Drozd an algebra A is said to be **tame** if, for each dimension d, there exists a finite number of $k[x]-A$-bimodules M_i which are finitely generated and free as left $k[x]$-modules, and all but finite number of isomorphism classes of indecomposable A-modules of dimensional d are of the form $k[x]/(x-\lambda)\otimes_{k[x]} M_i$ for some i and some $\lambda \in k$.

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Corollary

Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A. Then the orders $\leq_{\deg}$ and $\leq_{\text{ext}}$ coincide on mod A *if and only if A is tame and $\mathcal{C}_A$ contains neither a Möbius configuration nor a coil configuration.*

Main references

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