Fusion systems and blocks of finite groups

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Outline

- Fusion system
- Broué's Conjecture
- Generalization of perfect isometry

Fusion system

Definition

Let p be a prime number and P a finite p-group. A fusion system $\mathcal{F}$ over P is a category whose
objects : the subgroups of P
morphisms : $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ for $Q_1,\ Q_2 \leq P$ satisfy the following;
(i) Elements in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ are injective homomorphisms and all the homomorphisms from Q_1 to Q_2 given by the conjugation of elements of P lie in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$.
(ii) Every element f in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ can be written as a composition of the isomorphism $f : Q_1 \to f(Q_1)$ and the inclusion $f(Q_1) \subseteq Q_2$, and the both lie in $\mathcal{F}$.

Fusion system

Definition

Let $\mathcal{F}$ be a fusion system over P.
(i) A subgroup Q of P is said to be fully centralized in $\mathcal{F}$, if $|C_P(Q)| \geq |C_P(Q_1)|$ for all those $Q_1 \leq P$ that are $\mathcal{F}$-conjugate to Q.
(ii) A subgroup Q of P is said to be fully normalized in $\mathcal{F}$, if $|N_P(Q)| \geq |N_P(Q_1)|$ for all those $Q_1 \leq P$ that are $\mathcal{F}$-conjugate to Q.

Fusion system

Definition (Puig)

Let $\mathcal{F}$ be a fusion system over P.
We say that $\mathcal{F}$ is a saturated fusion system if the following are satisfied.

(a) Every fully normalized subgroup Q of P is fully centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Hom}_{\mathcal{F}}(Q, Q)$.

(b) For $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ with $\varphi(Q)$ is fully centralized, let $N = \{g \in N_P(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q))\}$, where c_g is the conjugation by g. Then, there is $\varphi' \in \mathrm{Hom}_{\mathcal{F}}(N, P)$ such that the restriction of φ' to Q is equal to φ.

Fusion system

Proposition (Broto-Levi-Oliver)

Let G be a finite group, and let P be a Sylow p-subgroup of G. Then the fusion system $\mathcal{F}_P(G)$ over P is saturated.

Here we let $\mathcal{F}_P(G)$ be the fusion system over P defined by setting $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q_1, Q_2) = \mathrm{Hom}_G(Q_1, Q_2)$ for all $Q_1, Q_2 \leq P$.

Broué's Conjecture

$(K, \mathcal{O}, k)$: a splitting p-modular system

$\mathcal{O}$: a complete discrete valuation ring
K : the field of fractions, char(K)$= 0$
k : the residue class field of $\mathcal{O}$, char(k)$= p$

G, H : finite groups

B, B' : the principal p-blocks of G and H

$$\mathrm{Irr}(B) := \{ \chi \in \mathrm{Irr}(G) \mid \chi \in B \}$$

Definition

$\exists$ bijection $f : \mathbf{Irr}(B) \to \mathbf{Irr}(B')$, $\exists$ map $\epsilon : \mathbf{Irr}(B) \to \{\pm 1\}$

$$\mu(g,h) := \sum_{\chi \in \mathbf{Irr}(B)} \epsilon(\chi)\chi(g) f(\chi)(h) \qquad (g \in G,\ h \in H)$$

If μ satisfies the following, then we say that μ is perfect isometry.

(i) If $\mu(g,h) \neq 0$, then either both g and h are p-regular or both are p-singular.

(ii) $\mu(g,h)/|C_G(g)|$ and $\mu(g,h)/|C_H(h)|$ lie in $\mathcal{O}$.

Broué's Conjecture

Conjecture (Broué's Abelian Defect Group Conjecture)

Let B be the principal p-block of G with abelian Sylow p-subgroup P and let b be the principal p-block of $N_G(P)$. Then B and b are derived equivalent and are perfectly isometric.

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Conjecture (Reformulation of Broué's Conjecture)

G, H : finite groups with common Sylow p-subgroup P
B, B' : the principal p-blocks of G and H
If P is abelian and if $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, then B and B' are derived equivalent and are perfectly isometric.

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Conjecture (Reformulation of Broué's Conjecture)

G, H *: finite groups with common Sylow p-subgroup P*
B, B' *: the principal p-blocks of G and H*
If P is abelian and if $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, then B and B' are derived equivalent and are perfectly isometric.

Remark

(Burnside) If a Sylow p-subgroup P of G is abelian, then G and $N_G(P)$ give rise to same fusion system over P.

Generalization of perfect isometry

Let P be a p-group, and let Q be a normal subgroup of P.

$$X(P;Q) := \{\, \theta \in \mathbb{Z}\mathrm{Irr}(P) \mid \theta(g) = 0 \ \ \forall g \in P \setminus Q \,\}$$

$$V(P;Q) := \{\, \sum_{\varphi \in \mathrm{Irr}(Q)} a_\varphi \varphi \uparrow^P \ \mid a_\varphi \in \mathbb{Z} \,\}$$

Then, $X(P;Q)$ and $V(P;Q)$ are $\mathbb{Z}$-submodules of $\mathbb{Z}\mathrm{Irr}(P)$. Moreover, we have $V(P;Q) \subseteq X(P;Q)$. Furthermore$V(P;Q)$ and $X(P;Q)$ have the same $\mathbb{Z}$-rank.

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Proposition (Robinson)

With the notation above, there exists a non-negative integer c such that $p^c X(P;Q) \subseteq V(P;Q)$.

Generalization of perfect isometry

Definition

Let P be a p-group and Q a normal subgroup of P. We denote by $c(P;Q)$ the non-negative integer c smallest among those c which satisfy $p^c X(P;Q) \subseteq V(P;Q)$.

Generalization of perfect isometry

Definition

$\exists$ bijection $f : \mathbf{Irr}(B) \to \mathbf{Irr}(B')$, $\exists$ map $\epsilon : \mathbf{Irr}(B) \to \{\pm 1\}$

$$\mu(g,h) := \sum_{\chi \in \mathbf{Irr}(B)} \epsilon(\chi)\chi(g)f(\chi)(h) \qquad (g \in G,\ h \in H)$$

If μ satisfies the following, then we say that μ is Q-perfect isometry.

(i) If $\mu(g,h) \neq 0$, then $(g_p, h_p) \in_{G\times H} (Q \times Q)\Delta(P)$.

(ii) $p^{c(P;Q)}\mu(g,h)/|C_G(g)|$ and $p^{c(P;Q)}\mu(g,h)/|C_H(h)|$ lie in $\mathcal{O}$.

Here $\Delta(P) = \{(u, u^{-1}) \mid u \in P\}$.

Generalization of perfect isometry

Conjecture

G, H : finite groups with common Sylow p-subgroup P
B, B' : the principal p-blocks of G and H
If $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, then B and B' are $[P, P]$-perfectly isometric.

Generalization of perfect isometry

Theorem (N-Uno)

G, H : finite groups with common Sylow p-subgroup $P \cong p_+^{1+2}$
B, B' : the principal p-blocks of G and H
Assume $\mathcal{F}_P(G) = \mathcal{F}_P(H)$.
Then B and B' are $[P, P]$-perfectly isometric.

Generalization of perfect isometry

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Assume $\mathcal{F}_P(G) = \mathcal{F}_P(H)$.
Then B and B' are $[P, P]$-perfectly isometric.

Remark

(Ruiz-Viruel) For $P \cong p_+^{1+2}$, saturated fusion systems over P are classified.

Generalization of perfect isometry

Theorem (N)

Let B be the principal p-block of finite simple G with T.I. Sylow p-subgroup P and let b be the principal p-block of $N_G(P)$. Then B and b are $[P, P]$-perfectly isometric.

We say $H \leq G$ is a trivial intersection (T. I.) set in G if $H \cap H^x = 1$ for $\forall x \in G \setminus N_G(H)$.