Hereditary Categories which are Fractionally Calabi-Yau

Adam-Christiaan van Roosmalen

Max-Planck-Institut für Mathematik

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Outline

1. Definitions and main result
   - Definitions
   - Derived categories and derived equivalences
   - (Fractionally) Calabi-Yau
2. Hereditary categories which are (fractionally) Calabi-Yau
   - Related to representations of Dynkin quivers
   - Related to tubes
   - Related to elliptic curves
   - Related to weighted projective lines

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Conventions and general definitions

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- We assume k is an algebraically closed field.
- All categories are k-linear.
- $\mathcal{A}$ will always be an abelian hereditary category.

Definition

- A category $\mathcal{A}$ is *hereditary* if $\operatorname{Ext}^i(X, Y) = 0$, for all $i > 1$, and for all $X, Y \in \operatorname{Ob} \mathcal{A}$.
- A category is *Ext-finite* if $\dim_k \operatorname{Ext}^i(X, Y) < \infty$ for all $i \in \mathbb{N}$ and all $X, Y \in \operatorname{Ob} \mathcal{A}$.

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Serre duality

Definition (Serre duality)

An Ext-finite abelian category $\mathcal{A}$ has Serre duality if there is an autoequivalence $F : D^b\mathcal{A} \to D^b\mathcal{A}$ admitting natural isomorphisms

$$\mathrm{Hom}_{D^b\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{D^b\mathcal{A}}(Y, FX)^*$$

where $(-)^*$ is the vector space dual.

Remarks

- $\mathcal{A}$ has Serre duality if and only if $D^b\mathcal{A}$ has Auslander-Reiten triangles. We have $F \cong \tau[1]$.
- if $\mathcal{A}$ is hereditary, then $\mathcal{A}$ has Serre duality if and only if
  - $\mathcal{A}$ has almost split sequences, and
  - the Nakayama functor $N : \mathcal{P} \to \mathcal{I}$ is an equivalence.

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Derived categories

Main idea

Instead of objects, we consider complexes. Two complexes which are resolutions of the same object are identified.

Construction

In the category of complexes over $\mathcal{A}$, all quasi-isomorphisms (=induce isomorphisms on the homologies) are formally inverted.

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Although derived categories are generally not abelian, they have the structure of a triangulated category.

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The hereditary case

Let $\mathcal{A}$ be an Ext-finite abelian hereditary category.

We may describe $D^b\mathcal{A}$ by

Objects $\operatorname{Ob} D^b\mathcal{A} = \operatorname{add}(\bigcup_{n\in\mathbb{Z}} \mathcal{A}[n])$

Morphisms induced by

$$\operatorname{Hom}_{D^b\mathcal{A}}(X[n], Y[m]) = \operatorname{Ext}^{m-n}_{\mathcal{A}}(X, Y)$$

where $X, Y \in \mathcal{A}[0]$.

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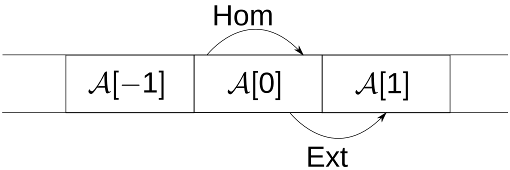

Derived equivalences (hereditary case)

Theorem ([1, 2])

Let $\mathcal{A}$ be a hereditary category, and let $\mathcal{H}$ be a strictly full subcategory of $D^b\mathcal{A}$ such that

$$\operatorname{Ob} D^b\mathcal{A} = \operatorname{add}\Big(\bigcup_{n\in\mathbb{Z}} \mathcal{H}[n]\Big).$$

If $\operatorname{Hom}_{D^b\mathcal{A}}(\mathcal{H}[m], \mathcal{H}[n]) = 0$ for $n > m$, then $\mathcal{H}$ is hereditary and $D^b\mathcal{A} \cong D^b\mathcal{H}$ as triangulated categories.

C. F. Berg, A.-C. van Roosmalen, *Projective components in hereditary categories with Serre duality*, in preperation.

C. M. Ringel, *Hereditary Triangulated Categories*, accepted.

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Notation

$$D^{\geq i} = \operatorname{add}(\bigcup_{n\leq i} \mathcal{H}[n]) \qquad D^{\leq i} = \operatorname{add}(\bigcup_{n\geq i} \mathcal{H}[n])$$

Representations of $x \rightrightarrows y$

$$R_{1,0}: \; k \underset{(0)}{\overset{(1)}{\rightrightarrows}} k$$

Indecomposable representations

Preprojective $\dim V(x) + 1 = \dim V(y)$

Preinjective $\dim V(x) - 1 = \dim V(y)$

Regular $\dim V(x) = \dim V(y)$

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Representations of $x \rightrightarrows y$

$$R_{0,1} : \; k \overset{(0)}{\underset{(1)}{\rightrightarrows}} k$$

Indecomposable representations

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Representations of $x \rightrightarrows y$

$$R_{a,b} : \ k \overset{(a)}{\underset{(b)}{\rightrightarrows}} k$$

Indecomposable representations

Preprojective	$\dim V(x) + 1 = \dim V(y)$
Preinjective	$\dim V(x) - 1 = \dim V(y)$
Regular	$\dim V(x) = \dim V(y)$

These representations are isomorphic if and only if $(a_1 : b_1) = (a_2 : b_2)$.

Auslander-Reiten quiver of rep $x \rightrightarrows y$

Preprojective

$$\begin{tikzcd}[column sep=small]
 & P_x \arrow[dr, shift left] \arrow[dr, shift right] & & \bullet \arrow[dr, shift left] \arrow[dr, shift right] & & \bullet \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots\cdots \\
P_y \arrow[ur, shift left] \arrow[ur, shift right] & & \bullet \arrow[ur, shift left] \arrow[ur, shift right] & & \bullet \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots\cdots &
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Preinjective

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\cdots\cdots \arrow[dr, shift left] \arrow[dr, shift right] & & \bullet \arrow[dr, shift left] \arrow[dr, shift right] & & \bullet \arrow[dr, shift left] \arrow[dr, shift right] & & I_x \\
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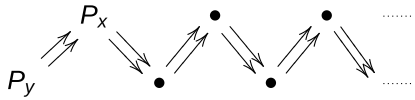
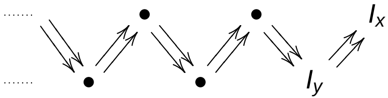
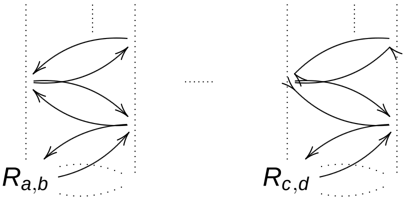
Preprojective



Preinjective

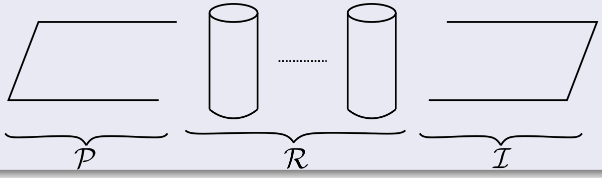


Regular



Relation between rep $x \rightrightarrows y$ and coh $\mathbb{P}^1$

rep $x \rightrightarrows y$



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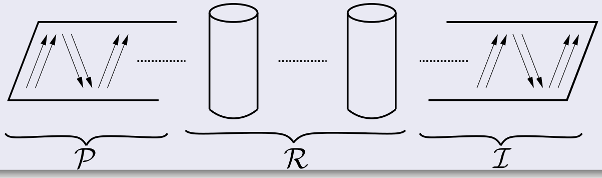

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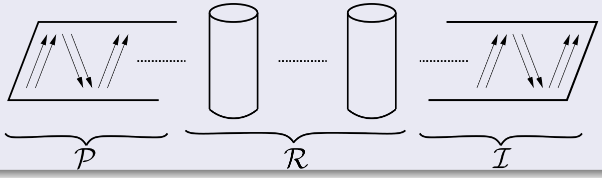

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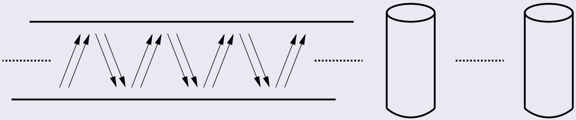

These categories are not equivalent,
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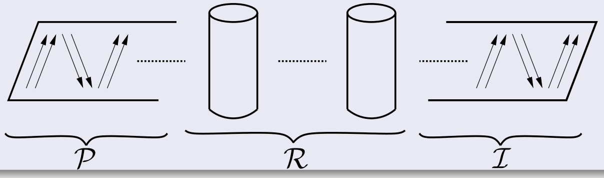

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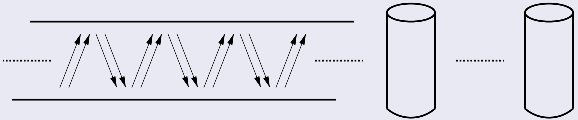

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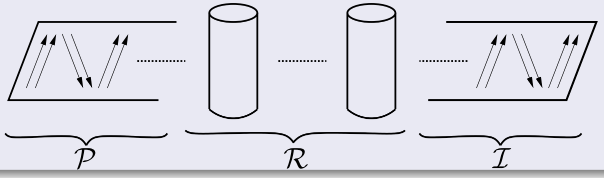

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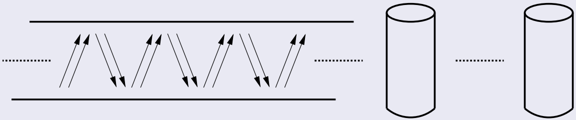

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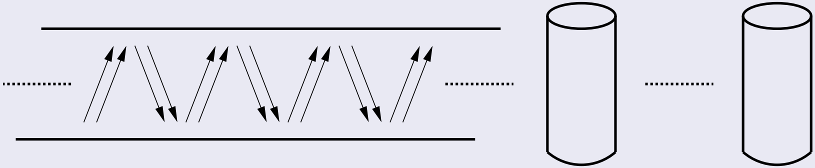

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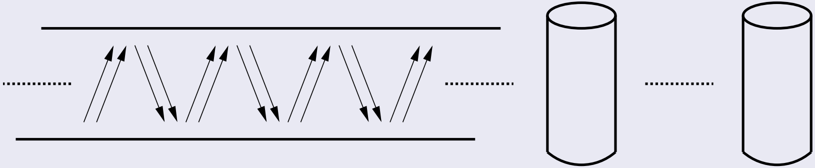

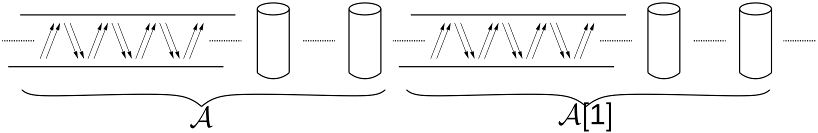

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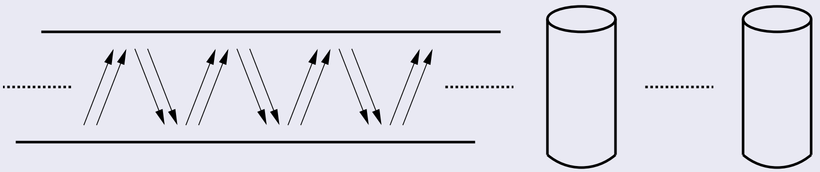

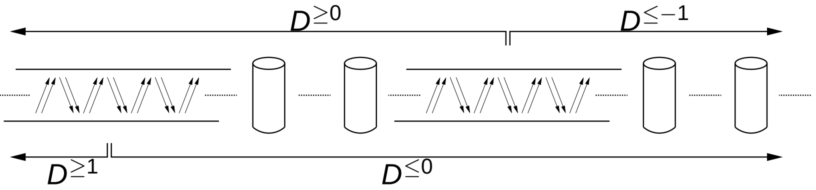

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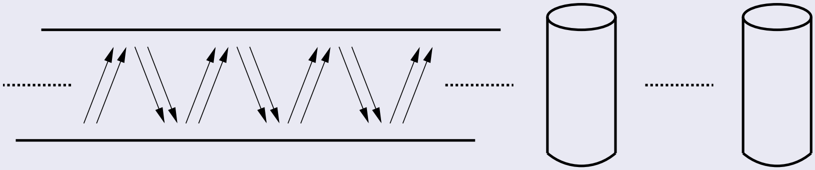

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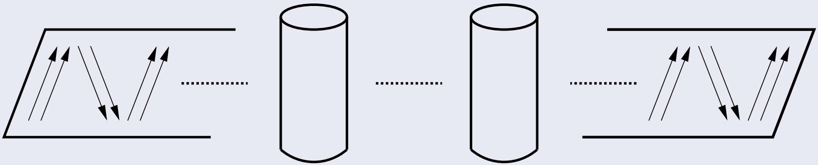

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Calabi-Yau

Definition

An Ext-finite abelian category $\mathcal{A}$ is n-Calabi-Yau if $[n] : D^b\mathcal{A} \to D^b\mathcal{A}$ is a Serre functor.

Remark

Since $F \cong \tau[1]$, an abelian category is 1-Calabi-Yau if $\tau \cong id$.

Theorem

An abelian n-Calabi-Yau category has global dimension n.

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Remark

Since $F \cong \tau[1]$, a category is fractionally Calabi-Yau if $\tau^n \cong [m-n]$.

Remark

$\mathcal{A}$ is fractionally Calabi-Yau of dimension 1 if and only if $\tau^n \cong id$.

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A quick example

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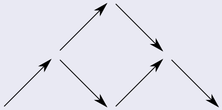

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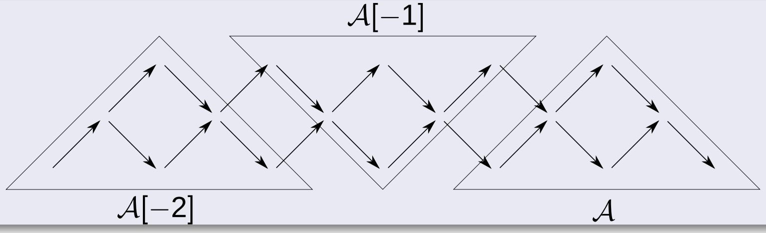

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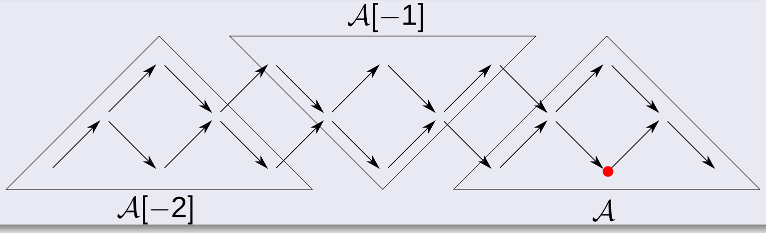

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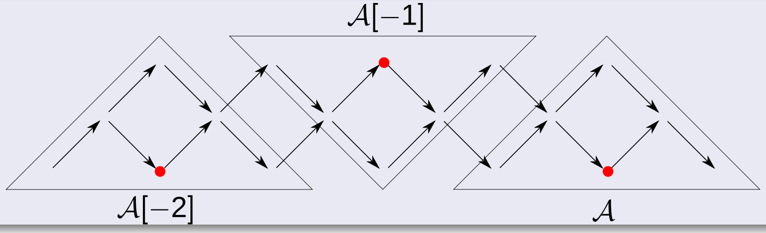

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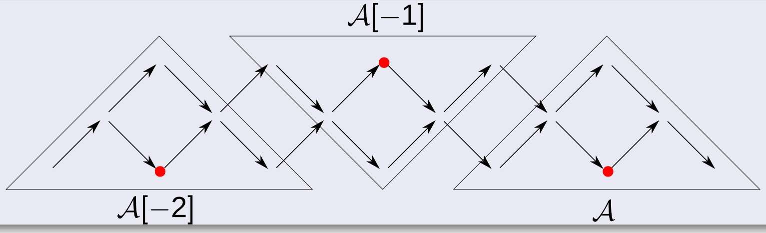

Thus $\tau^4 = [-2]$, or equivalently $F^4 = [2]$.

This category is fractionally Calabi-Yau of dimension $\frac{2}{4} = \frac{1}{2}$.

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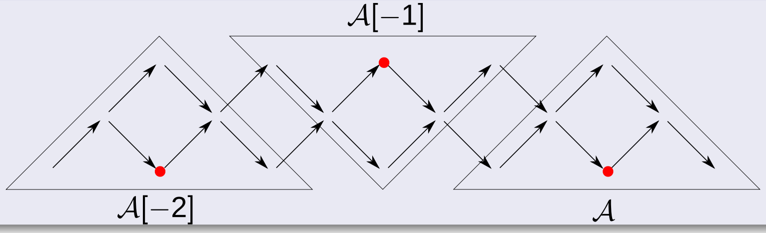

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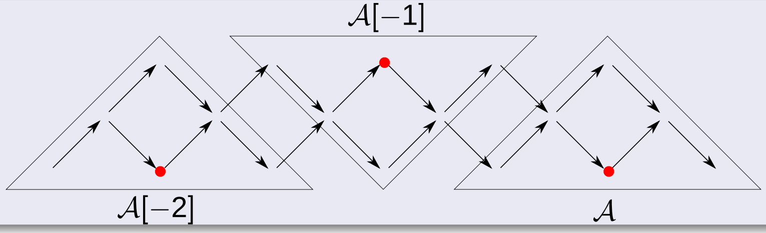

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This category is fractionally Calabi-Yau of dimension $\frac{2}{4} = \frac{1}{2}$.

Outline

1. Definitions and main result
   - Definitions
   - Derived categories and derived equivalences
   - (Fractionally) Calabi-Yau

2. Hereditary categories which are (fractionally) Calabi-Yau
   - Related to representations of Dynkin quivers
   - Related to tubes
   - Related to elliptic curves
   - Related to weighted projective lines

Nilpotent representations of $\bullet \circlearrowright f$

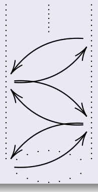
Indecomposable objects :

1-dimensional $f \mapsto \begin{pmatrix} 0 \end{pmatrix}$

2-dimensional $f \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

3-dimensional $f \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Auslander-Reiten Quiver



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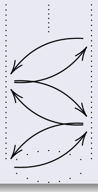

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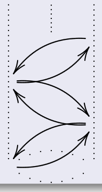

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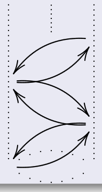

Nilpotent representations of $\bullet \circlearrowright f$

Representation

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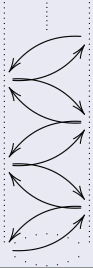

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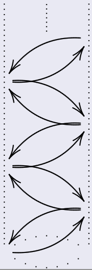

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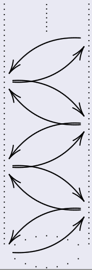

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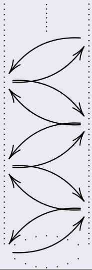

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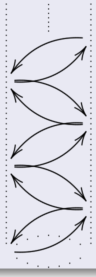

This category is 1-Calabi-Yau.

Nilpotent representations of $\bullet \rightleftarrows \bullet$

Representation

Auslander-Reiten quiver

This category is not 1-Calabi-Yau
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Tubes in hereditary categories

Definition

A tube in an abelian hereditary category $\mathcal{A}$ with Serre duality is the essential image of an embedding $i : \operatorname{Nilp} \tilde{A}_n \to \mathcal{A}$ where i commutes with τ.

Theorem

Let $\mathcal{A}$ be a hereditary category with Serre duality.

- *Every τ-periodic element lies in a tube, and*
- *a tube $\mathcal{T}$ is directing in the sense that, if there is a path*

$$X_0 \to \cdots \to X_n$$

in $D^b \mathcal{A}$ with $X_0, X_n \in \mathcal{T}$, then $X_i \in \mathcal{T}$, for all i.

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Definition of elliptic curve

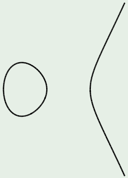
Definition

An *elliptic curve* is a smooth curve in $\mathbb{P}^2_k$ of genus 1.

Equation (char $k \neq 2$)

$y^2 = x(x-1)(x-\lambda)$ with $\lambda \in k$.

Example



Example

Description of coh $\mathbb{E}$

- coh $\mathbb{E}$ is 1-Calabi-Yau.
- Every Auslander-Reiten component is a homogeneous tube.
- Every tube $\mathcal{T}$ has a slope, $\mu(\mathcal{T})$, which lies in $\mathbb{Q} \cup \{\infty\}$.
- The tubes with infinite slope are in 1-1-correspondence with the points of V.
- For two different tubes, $\mathcal{T}_1, \mathcal{T}_2$, we have

$$\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2) \neq 0 \Leftrightarrow \mu(\mathcal{T}_1) < \mu(\mathcal{T}_2).$$

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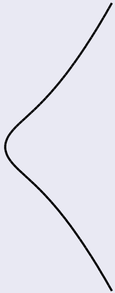
Auslander-Reiten Quiver of coh $\mathbb{E}$

Auslander-Reiten Quiver



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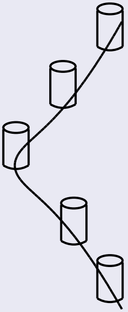

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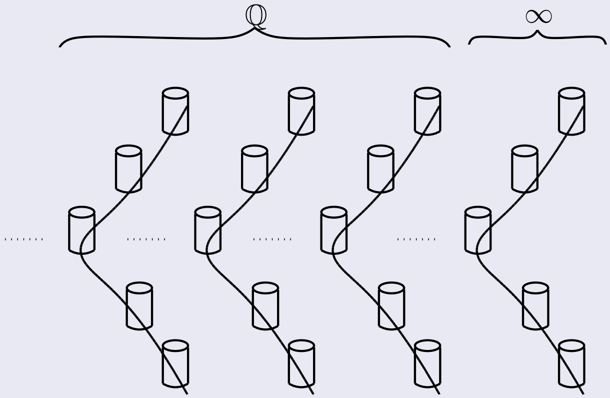


Categories derived equivalent with coh $\mathbb{E}$

Theorem ([1])

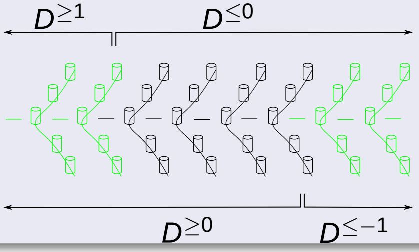
Let $\mathcal{A} = \operatorname{coh} \mathbb{E}$. All bounded t-structures on $D^b \mathcal{A}$ whose heart is derived equivalent to $\mathcal{A}$ may be given by

- $\theta \in \mathbb{Q} \cup \{\infty\}$, *$n \in \mathbb{Z}$, and a set S of tubes in $\mathcal{A}_\theta[n]$*
- $\theta \in \mathbb{R} \setminus \mathbb{Q}$, *$n \in \mathbb{Z}$.*

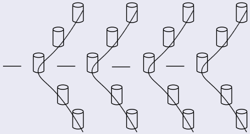
A.L. Gorodentsev, S.A. Kuleshov, A.N. Rudakov *t-stabilities and t-structures on triangulated categories*, Izv. Math. **68** (2004), no. 4, 749-781.

Hereditary categories derived equivalent with coh $\mathbb{X}$

t-structure ($\theta \in \mathbb{Q} \cup \{\infty\}$)



Heart



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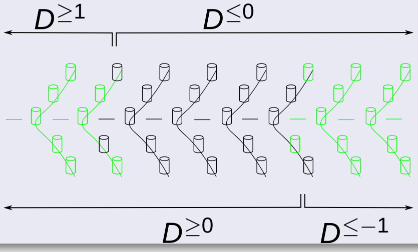

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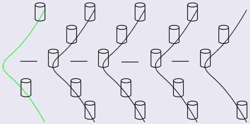

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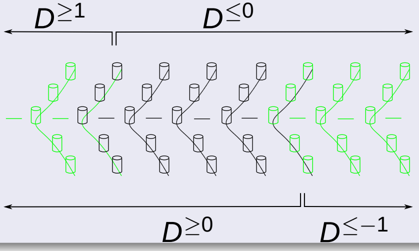

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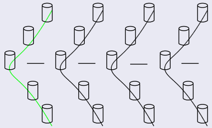

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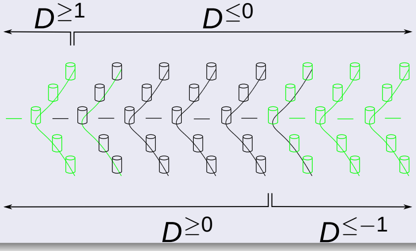

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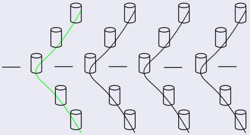

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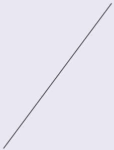
Beilinson’s equivalence and weighted projective lines

Projective line $\mathbb{X}$



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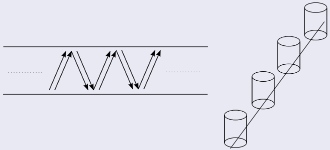
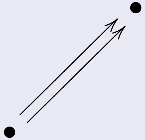


The quiver Q



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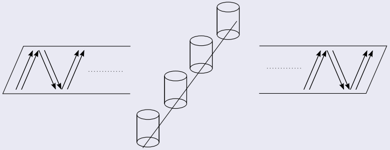

Beilinson's equivalence and weighted projective lines

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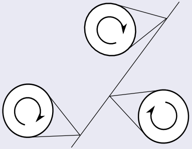

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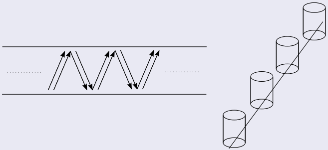

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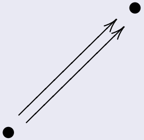

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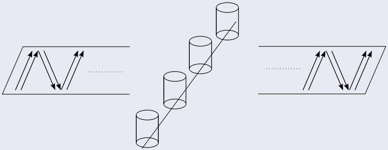

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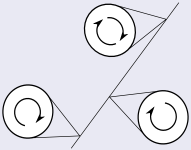

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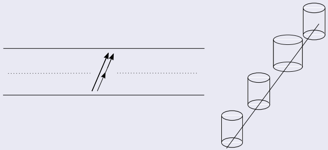

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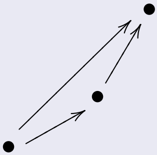

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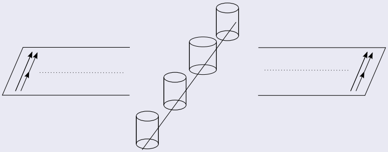

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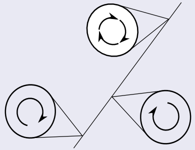

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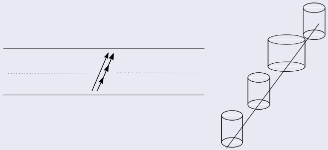

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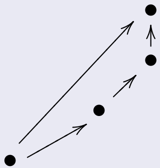

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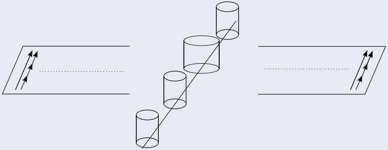

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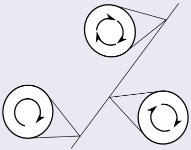

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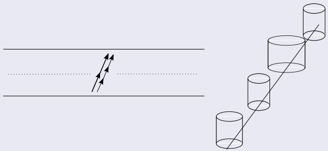
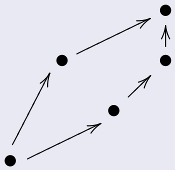
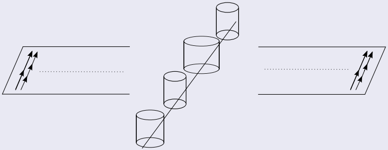


The quiver Q



rep Q



Canonical algebras

Definition

A canonical algebra is the path algebra of a quiver of the form

$$\begin{tikzcd}
& L_1^{(1)} \arrow[r, "f_1"] & L_2^{(1)} \arrow[r, "f_1"] & \cdots \arrow[r, "f_1"] & L_{p_1-1}^{(1)} \arrow[ddr, "f_1"] & \\
& L_1^{(2)} \arrow[r, "f_2"'] & L_2^{(2)} \arrow[r, "f_2"'] & \cdots \arrow[r, "f_2"'] & L_{p_2-1}^{(2)} \arrow[dr, "f_2"'] & \\
L \arrow[uur, "f_1"] \arrow[ur, "f_2"'] \arrow[dr, "f_t"'] & \vdots & & & \vdots & L_1 \\
& L_1^{(t)} \arrow[r, "f_t"] & L_2^{(t)} \arrow[r, "f_t"] & \cdots \arrow[r, "f_t"] & L_{p_t-1}^{(t)} \arrow[ur, "f_t"'] &
\end{tikzcd}$$

with relations $f_i^{p_i} = f_2^{p_2} - \lambda_i f_1^{p_1}$, for all $2 \leq i \leq t$, where $\lambda_i \neq \lambda_j$ for $i \neq j$.

Weighted projective lines

Definition ([1, 2])

A connected Ext-finite abelian hereditary noetherian category with a tilting complex and no nonzero projectives is said to be a category of coherent sheaves coh $\mathbb{X}$ over a weighted projective line $\mathbb{X}$.

W. Geigle, H. Lenzing *A class of weighted projectives lines arising in the representation theory of finite dimensional algebras*, Lect. Notes Math. **1273** (1987), 265–297.

H. Lenzing *Hereditary Noetherian categories with a tilting complex*, Proc. Amer. Math. Soc. **125** (1997), 1893–1901.

Quivers of ‘tubular’ canonical algebras

$$\begin{tikzcd}
& \cdot \arrow[ddr] & \\
& \cdot \arrow[dr] & \\
\cdot \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \cdot \\
& \cdot \arrow[ur] & \\
& \cdot \arrow[uur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \cdot \arrow[drr] & & \\
\cdot \arrow[urr] \arrow[r] \arrow[dr] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \cdot \arrow[r] & \cdot \arrow[dr] & \\
\cdot \arrow[ur] \arrow[r] \arrow[dr] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
& \cdot \arrow[r] & \cdot \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& & & \cdot \arrow[drr] & & \\
\cdot \arrow[urrr] \arrow[rr] \arrow[dr] & & \cdot \arrow[r] & \cdot \arrow[rr] & & \cdot \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[ur] &
\end{tikzcd}$$

Properties of coh $\mathbb{X}$ where $\mathbb{X}$ is tubular

- coh $\mathbb{X}$ is fractionally Calabi-Yau of dimension 1.
- Every Auslander-Reiten component is a tube.
- Every tube $\mathcal{T}$ has a slope, $\mu(\mathcal{T})$, which lies in $\mathbb{Q} \cup \{\infty\}$.
- The tubes with infinite slope are in 1-1-correspondence with the points of $\mathbb{X}$.
- For two different tubes, $\mathcal{T}_1, \mathcal{T}_2$, we have

$$\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2) \neq 0 \Leftrightarrow \mu(\mathcal{T}_1) < \mu(\mathcal{T}_2).$$

- Denote by $\mathcal{A}_\mu$ the additive category given by all tubes of slope μ.
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Auslander-Reiten Quiver of coh $\mathbb{X}$

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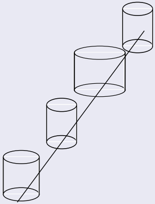

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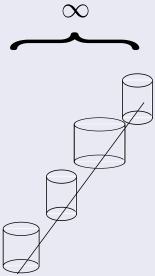

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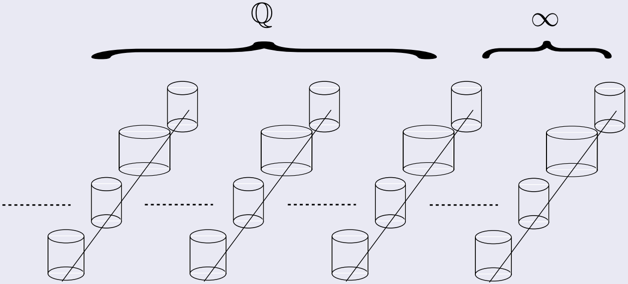


Hereditary categories derived equivalent with coh $\mathbb{X}$

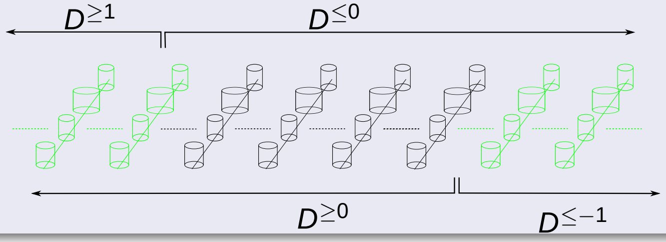
Theorem

Let $\mathcal{A} = \operatorname{coh}\mathbb{X}$. All bounded t-structures on $D^b\mathcal{A}$ whose heart is hereditary and derived equivalent to $\mathcal{A}$ may be given by

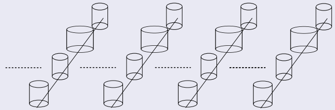
- $\theta \in \mathbb{Q} \cup \{\infty\}$, $n \in \mathbb{Z}$, *and a set* S *of tubes in* $\mathcal{A}_\theta[n]$
- $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $n \in \mathbb{Z}$.

Hereditary categories derived equivalent with coh $\mathbb{X}$

t-structure ($\theta \in \mathbb{Q} \cup \{\infty\}$)



Heart



Hereditary categories derived equivalent with coh $\mathbb{X}$

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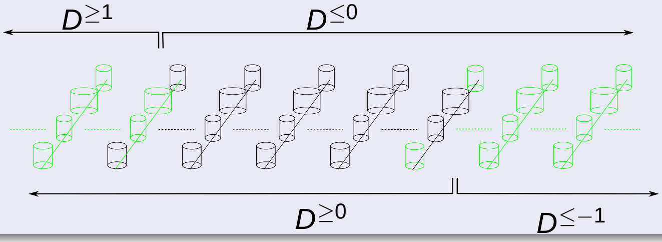
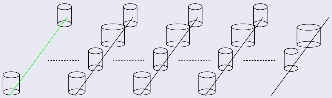


Heart



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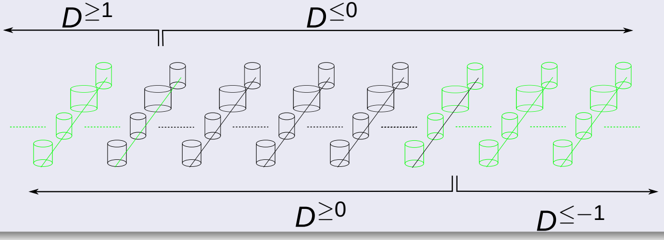
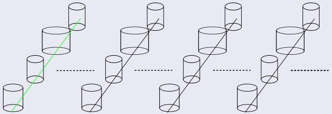
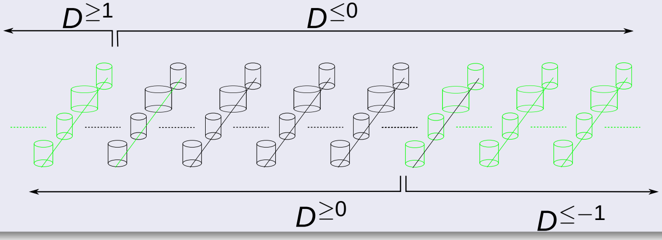


Heart

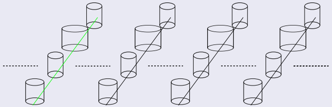


Hereditary categories derived equivalent with coh $\mathbb{X}$

t-structure $(\theta \in \mathbb{R} \setminus \mathbb{Q})$



Heart



Theorem

Let $\mathcal{A}$ be a connected abelian hereditary category which is fractionally Calabi-Yau, then $\mathcal{A}$ is derived equivalent to either

1. *the category of nilpotent representations of the one-loop quiver, or*
2. *the category of coherent sheaves on an elliptic curve, or*
3. *the category of nilpotent representations of $\tilde{A}_n$ with $n \geq 1$, or*
4. *the category of coherent sheaves over a weighted projective line of tubular type, or*
5. *the category of finite presented modules $\operatorname{mod} Q$ over a Dynkin quiver Q.*