Some Results about Cohomology of Integral Specht Modules

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Conjecture (Szczepański, 2003)

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i.e., an α with $\mathrm{res}\downarrow^G_U(\alpha) \neq 0$ for all subgroups $1 \neq U \leq G$.

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$\rightsquigarrow$ Conjecture

For any $n \geq 5$ there exist distinct partitions $\lambda_1, \ldots, \lambda_m \vdash n$ such that $H^2(S_n, \bigoplus_{j=1}^m S_{\mathbb{Z}}^{\lambda_j})$ contains a special element.

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Definition

Let $p, \iota \in \mathbb{Z}_{>0}$, where p is prime and $p\iota \leq n$.
Then α is called (p, ι)-special if $\text{res}\downarrow_{\langle g \rangle}^{S_n}(\alpha) \neq 0$ for all products $g \in S_n$ of ι disjoint p-cycles.

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$$H^2(S_n, \bigoplus_{j=1}^{m} S_{\mathbb{Z}}^{\lambda_j}) \cong \bigoplus_{j=1}^{m} H^2(S_n, S_{\mathbb{Z}}^{\lambda_j})$$

Plan: For every $p\iota \leq n$ find a (p, ι)-special partition $\lambda_{(p,\iota)}$.
Then $H^2(S_n, \bigoplus_{p,\iota} S_{\mathbb{Z}}^{\lambda_{(p,\iota)}})$ contains a special element.

Necessary condition for "*$\lambda \vdash n$ is a (p, ι)-special partition*":

$$p \text{ divides } |H^2(S_n, S_{\mathbb{Z}}^{\lambda})|.$$

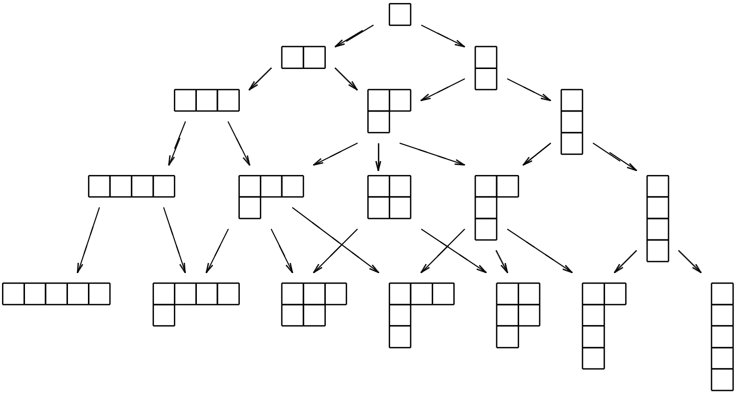
Partitions with this property can be described by certain subgraphs of the Young graph.

Young graph: $\mathcal{Y} := \{\lambda \vdash n \mid n \in \mathbb{N}\}$,

$\lambda \vdash n$, $\mu \vdash n+1$:

$\lambda \to \mu \iff \mu \in \lambda+ \iff \lambda \in \mu-$

Then: μ is a successor of λ, λ is a predecessor of μ.



p prime, $i \in \mathbb{Z}_{>0}$

$$\mathcal{C}_p^i := \bigcup_{n \in \mathbb{Z}_{\geq 0}} \{\lambda \vdash n \mid p \text{ divides } |H^i(S_n, S_{\mathbb{Z}}^{\lambda})|\} \quad \subseteq \mathcal{Y}$$

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Theorem (W., 2006)

Let $\lambda \in \mathcal{C}_p^i$.

1. *There exists a successor of λ in $\mathcal{C}_p^i$.*
2. *If $\lambda \vdash n$ with $p \nmid n$, then there exists a predecessor of λ in $\mathcal{C}_p^i$.*

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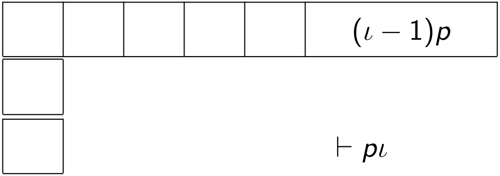
- Branching rules
- Eckmann-Shapiro Lemma (for 1.)
- $\mathrm{tr}\uparrow_{S_{n-1}}^{S_n} \circ \mathrm{res}\downarrow_{S_{n-1}}^{S_n} = n \cdot \mathrm{id}$ (for 2.)

Example for paths for odd p in $\mathcal{C}_p^2$ (proved):



$\vdash p\iota$

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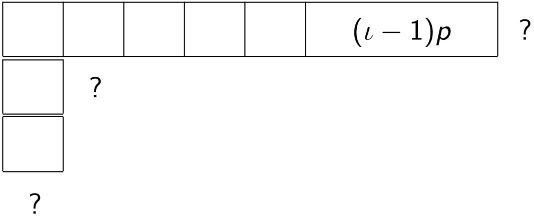

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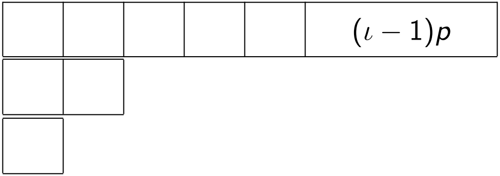

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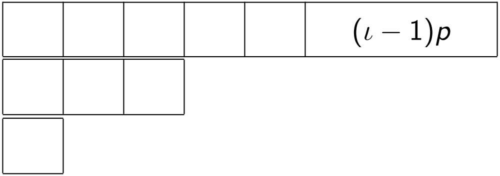

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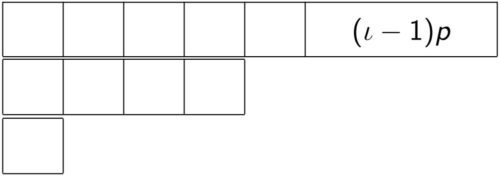

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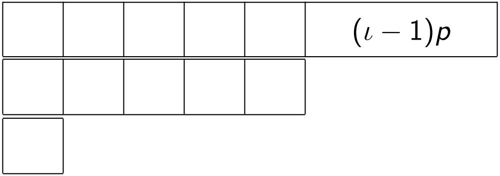

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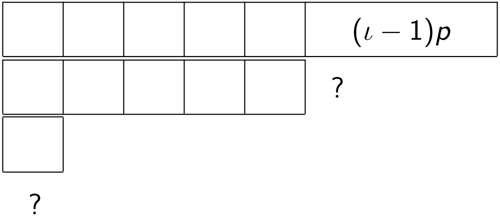

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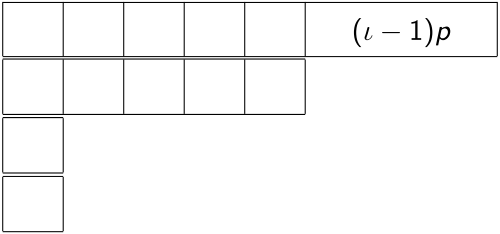

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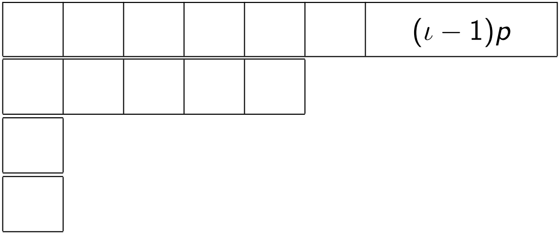

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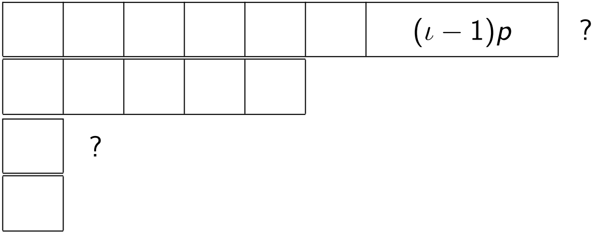

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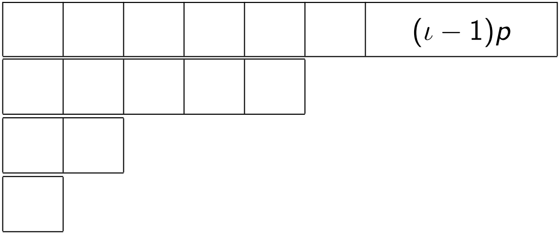

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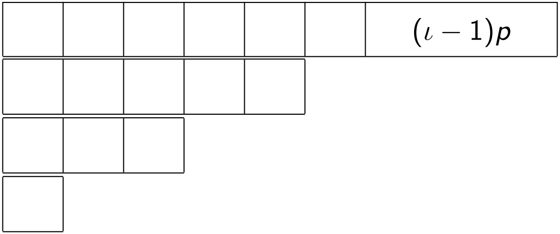

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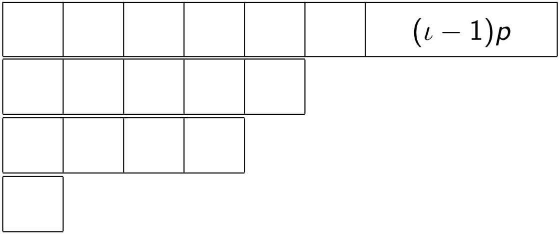

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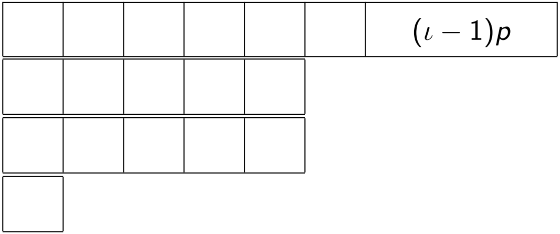

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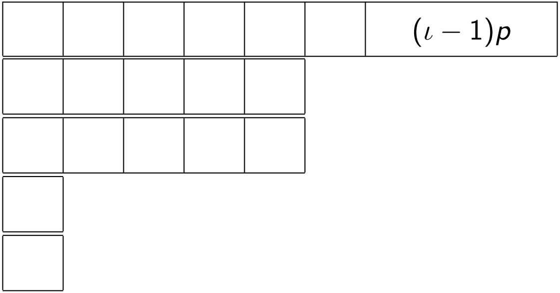

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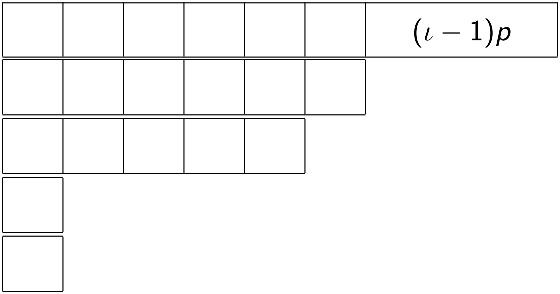

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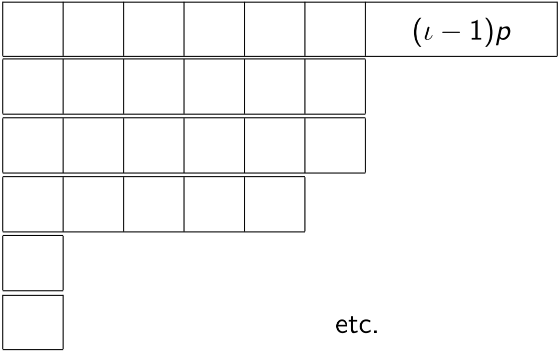

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Proposition

For odd p, $\lambda := (p\iota - 2, 1^2)$ is (p, ι)-special.

- $p > 2$ prime
- $n \in \mathbb{Z}_{>0}$ with $5 \leq n = \iota p$
- $b := (1, 2, \ldots, n)$
- $g := b^{\iota}$
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Proposition

λ is (p, ι)-special, i.e., $\operatorname{res}\downarrow^{S_n}_{\langle g \rangle} H^2(S_n, S^{\lambda}_{\mathbb{Z}_p}) \neq 0.$

Lemma (1)

$$H^2(\langle b\rangle, S_{\mathbb{Z}}^{\lambda}\downarrow_{\langle b\rangle}^{S_n}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & \textit{for } n \textit{ odd,} \\ \mathbb{Z}/\frac{n}{2}\mathbb{Z} & \textit{for } n \textit{ even.} \end{cases}$$

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Lemma (2)

$\text{res}\downarrow^{S_n}_{\langle b\rangle}: H^2(S_n, S^\lambda_{\mathbb{Z}}) \to H^2(\langle b\rangle, S^\lambda_{\mathbb{Z}}\downarrow^{S_n}_{\langle b\rangle})$ *is surjective.*

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For a finite group G acting on $\mathbb{Z}^k$:

$$H^2(G, \mathbb{Z}^k) \quad \cong \quad H^1(G, \mathbb{Q}^k/\mathbb{Z}^k)$$

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- Then $H^1(G, \mathbb{Q}^k/\mathbb{Z}^k)$
  $\cong \{v \in \mathbb{Q}^{gk} \mid Z_G v \in \mathbb{Z}^{rk}\} / \left(\ker Z_G + \mathbb{Z}^{rk}\right)$.

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$\mathrm{tr}\uparrow^{\langle b\rangle}_{\langle g\rangle}\colon H^2(\langle g\rangle, S^\lambda_{\mathbb{Z}} \downarrow^{S_n}_{\langle g\rangle}) \to H^2(\langle b\rangle, S^\lambda_{\mathbb{Z}} \downarrow^{S_n}_{\langle b\rangle})$ the transfer map

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because $H^2(\langle b\rangle, S^\lambda_{\mathbb{Z}} \downarrow^{S_n}_{\langle b\rangle})$ has an element of order n or $\frac{n}{2}$, resp.

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$$\Rightarrow \mathrm{res}\downarrow^{\langle b\rangle}_{\langle g\rangle} \neq 0$$

$$H^2(S_n, S_{\mathbb{Z}}^{\lambda})$$

$$\Big\downarrow \operatorname{res}\downarrow^{S_n}_{\langle b\rangle}$$

$$H^2(\langle b\rangle, S_{\mathbb{Z}}^{\lambda}\downarrow^{S_n}_{\langle b\rangle})$$

$$\Big\downarrow \operatorname{res}\downarrow^{\langle b\rangle}_{\langle g\rangle}$$

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$$
\begin{array}{ccc}
H^2(S_n, S_{\mathbb{Z}}^{\lambda}) & & \\
\Big\downarrow \mathrm{res}\downarrow^{S_n}_{\langle b\rangle} & & \text{surjective} \\
H^2(\langle b\rangle, S_{\mathbb{Z}}^{\lambda}\downarrow^{S_n}_{\langle b\rangle}) & & \neq 0 \\
\Big\downarrow \mathrm{res}\downarrow^{\langle b\rangle}_{\langle g\rangle} & & \neq 0 \\
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\end{array}
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$$
\begin{array}{cc}
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H^2(\langle g\rangle, S_{\mathbb{Z}}^{\lambda}\downarrow^{S_n}_{\langle g\rangle}) &
\end{array}
$$

$$\Rightarrow \operatorname{res}\downarrow^{S_n}_{\langle g\rangle} \neq 0$$

Conjecture

Let $n \geq 2\iota$.

$\iota = 1$: (n) *is* $(2,1)$*-special (proved).*

$\iota \geq 2$: $(2\iota - 2, 2, 1^{n-2\iota})$ *is* $(2,\iota)$*-special.*

Conjecture

For any $n \in \mathbb{Z}_{>0}$ there exist distinct partitions $\lambda_1, \ldots, \lambda_m \vdash n$, such that $\bigoplus_{j=1}^m S_{\mathbb{Z}}^{\lambda_j}$ is a faithful $\mathbb{Z}S_n$-lattice and $H^2(S_n, \bigoplus_{j=1}^m S_{\mathbb{Z}}^{\lambda_j})$ contains a special element.

The only nonfaithful Specht modules are $S_{\mathbb{Z}}^{(n)}$ (for $n \geq 2$), $S_{\mathbb{Z}}^{(1^n)}$ (for $n \geq 3$) and $S_{\mathbb{Z}}^{(2,2)}$.

- n=1: (1)
- n=2: (2) for a special element, (1^2) for faithfulness
- n=3: $(3), (1^3)$ for s. e., $(2,1)$ for f.
- n=4: $(4), (2,2), (1^4)$ for s. e., $(3,1)$ or $(2,1^2)$ for f.
- n≥5: Every direct sum of Specht modules, that leads to a special element, is faithful.

$$S_n \quad \cong \quad \langle\, a, b \mid a^2, b^n, (ab)^{n-1}, [a, b^j]^2 \;\; \forall\, 2 \leq j \leq \tfrac{n}{2} \,\rangle.$$

$$a \mathrel{\hat{=}} (1,2), \quad b \mathrel{\hat{=}} (1,\ldots,n)$$

corresponding matrices:

$a \mapsto A,\ b \mapsto B$

$$\left(\begin{array}{c|c} 1+A & 0 \\ \hline 0 & \sum_{i=0}^{n-1} B^i \\ \hline \sum_{i=0}^{n-2} (AB)^i & \sum_{i=0}^{n-2} (AB)^i A \\ \hline (1+AB^jAB^{n-j})(1+AB^j) & (1+AB^jAB^{n-j})A(\sum_{i=0}^{j-1} B^i + B^jA \sum_{i=0}^{n-j-1} B^i) \\ 2 \leq j \leq \frac{n}{2} & 2 \leq j \leq \frac{n}{2} \end{array}\right)$$

V a faithful $\mathbb{Z}G$-lattice
$\alpha \in H^2(G, V)$ corresponding to extension $0 \to V \to \Gamma \xrightarrow{\pi} G \to 1$

$$\Gamma \text{ torsion free } \iff \text{res}\downarrow^G_U (\alpha) \neq 0 \text{ for all } 1 \neq U \leq G$$

1. $\text{res}\downarrow^G_U$ corresponds to $0 \to V \to \pi^{-1}(U) \to U \to 1$.
   If $\text{res}\downarrow^G_U = 0$, the extension splits.
   $\rightsquigarrow \pi^{-1}(U)$ (and hence Γ) contains elements of finite order.
2. $1 \neq \hat{U} \leq \Gamma$ finite.
   $\text{res}\downarrow^G_{\pi(\hat{U})}$ corresponds to $0 \to V \to V\hat{U} \to \pi(\hat{U}) \to 1$.
   The extension splits, hence $\text{res}\downarrow^G_{\pi(\hat{U})} = 0$.