On a formula for all sets of constant width in 3d

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Abstract

In a recent paper [12] a constructive formula was given for all two-dimensional sets of constant width. Based on that result we derive here a formula for the parametrization of the boundary of bodies of constant width in 3 dimensions, depending on one function defined on $\mathbb{S}^2$ and a large enough constant. Moreover, it is proven that all bodies of constant width in 3d have such a parametrization. The last result needs a tool that we describe as 'shadow domain' and is explained in an appendix.

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1 Introduction and two dimensions

We start by recalling the definition of those sets. A closed convex set $G \subset \mathbb{R}^n$ is called a set of constant width in $\mathbb{R}^n$, if its directional width is constant. The width in direction $\omega \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n; |x| = 1\}$ is given by

$$d_G(\omega) = \max\{\langle \omega, x\rangle ; x \in G\} - \min\{\langle \omega, x\rangle ; x \in G\}.$$

Thus, if $d_G(\omega) = d_G$, a constant, then G is a set of constant width. In 3 dimensions G is called a body of constant width.

The interest in the subject started with Leonhard Euler, who around 1774 considered 2d curves of constant width, which he called 'curva orbiformis'. He not only studied such sets for 2 dimensions but also gave a formula describing such curves. See §10 of [5]. In 3 dimensions a ball is obviously the classical example of a body of constant width but the famous Meissner bodies also have this property. See [16, 17] or [13]. Quite simple examples can also be constructed by taking a symmetric 2d set of constant width and rotating that around its axis of symmetry.

Let us mention that famous mathematicians such as Minkowski [19] and Hilbert [10] were intrigued by the subject. The first interest of most scholars focused on deriving properties of such domains. A wonderful survey on sets of constant width (up to 1983) was provided by Chakerian and Groemer in [3], and a more recent updated and thorough treatment can be found in the book by Martini, Montejano and Oliveros [15]. Let us recall that the 3d question, motivated by Blaschke's 2d result [2], as to which body of constant fixed width has the smallest volume or, equivalently, the smallest surface area, is still open.

In the last century Hammer and Sobczyk described a construction for 2 dimensions in [7, 8, 9], based on a characterization of what they called 'outwardly simple line families'. More recently a direct concise formula was given in [12] to describe all those sets in two dimensions. In the present paper we use that result to find a formula that describes all bodies of constant width in 3d.

Let us start by recalling the 2d formula, which we need for the 3d construction:

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Theorem 1 ([12, Theorem 3.2]) *Let $\boldsymbol{x}_0 \in \mathbb{R}^2$, $r \in \mathbb{R}$ and $a \in L^\infty(\mathbb{R})$ satisfy*

$$r \geq \|a\|_\infty , \tag{1}$$

$$a(\varphi + \pi) = -a(\varphi) \text{ for all } \varphi, \tag{2}$$

$$\int_0^\pi a(s) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

Define the closed curve $\boldsymbol{x} : [0, 2\pi] \to \mathbb{R}^2$ by

$$\boldsymbol{x}(\varphi) = \boldsymbol{x}_0 + \int_0^\varphi (r - a(s)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds. \tag{4}$$

Then $\boldsymbol{x}$ describes the boundary of a set of constant width $2r$.

Remark 1.1 *If $r > \|a\|_\infty$, then $\tilde{\boldsymbol{x}}$ is a Jordan curve, meaning a continuous one-to-one mapping when considered from $\mathbb{S}^1$ to ∂G. For $r > a(\varphi)$ the outside normal at $\boldsymbol{x}(\varphi)$ is $\binom{-\sin\varphi}{\cos\varphi}$. If $r = a(\varphi)$ for $\varphi \in [\varphi_1, \varphi_2]$, then one finds for $\varphi \in [\varphi_1, \varphi_2]$ that $\boldsymbol{x}(\varphi) = \boldsymbol{x}(\varphi_1)$ and a family of outside 'normals':*

$$\left\{ \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} ; \varphi \in [\varphi_1, \varphi_2] \right\}.$$

Not only does the formula in (4) describe the boundary of a 2d domain of constant width, one can even show that all those sets are described this way:

Theorem 2 ([12, Theorem 4.1]) *If $G \subset \mathbb{R}^2$ is a closed convex set of constant width $2r$, then there exists $\boldsymbol{x}_0$ and a as in Theorem 1, such that $\partial G = \boldsymbol{x}([0, 2\pi])$ with $\boldsymbol{x}$ as in (4).*

We should mention that there have been previous attempts to provide a description of all 3d bodies of constant width. In [14] Lachand-Robert and Oudet present a geometric construction that generates 3d bodies of constant width from 2d sets of constant width. This construction, however, does not capture all 3d bodies of constant width because a counterexample is provided in the paper [4] by Danzer. In [20] Montejano and Roldan-Pensado generalize the construction of Meissner bodies to generate so-called Meissner polyhedra. This construction does not generate all 3d bodies either, because the rotated Reuleaux triangle is a counterexample. Furthermore Bayen, Lachand-Robert and Oudet give a different description of (all) 3d sets of constant width in Theorem 2 of [1]. Compared to [1] our paper provides an alternative construction, based on the method from [12].

Most of the time we will use a column notation for vectors in $\mathbb{R}^n$ ($n \in \{2, 3\}$). For the standard inner product of $u, v \in \mathbb{R}^n$ we use $\langle u, v \rangle$. The notation $u \cdot v$ is used for componentwise multiplication, which includes but can be more general than the inner product.

2 A formula in three dimensions

Aside from our results from [12] for two dimensions we will use a result by Hadwiger in [6], which can be roughly described as: convex bodies in $\mathbb{R}^n$ are uniquely determined by the projections in $\mathbb{R}^{n-1}$ perpendicular to one fixed direction. The result holds for $n \geq 4$ and, whenever the one fixed direction is regular, also for $n = 3$. This last addendum is due to [11]. Regular means here, that the planes perpendicular to that fixed direction which touch the convex domain, do that in precisely one point. Since sets of constant width are necessarily strictly convex, this is obviously the case for sets of constant width and any choice of the fixed direction.

Let us define for $\omega \in \mathbb{S}^2$ the orthogonal projection P_ω on the plane $E_\omega \subset \mathbb{R}^3$ through 0 perpendicular to ω. To exploit the result of Hadwiger we will use for a fixed $u \in \mathbb{S}^2$ all projections in the directions $\omega \in \mathbb{S}^2$ with $\langle \omega, u \rangle = 0$. See Figure 1. So we have

$$P_\omega x = \langle u, x \rangle u + \langle u \times \omega, x \rangle (u \times \omega), \tag{5}$$

and when identifying the projections on E_ω with coordinates in $\mathbb{R}^2$ through

$$\hat{P}_\omega x = \begin{pmatrix} \langle u, x \rangle \\ \langle u \times \omega, x \rangle \end{pmatrix}. \tag{6}$$

We may now explain the result by Hadwiger in [6] in more detail. He proved that for two convex bodies G_1 and G_2 in $\mathbb{R}^3$ the following holds.

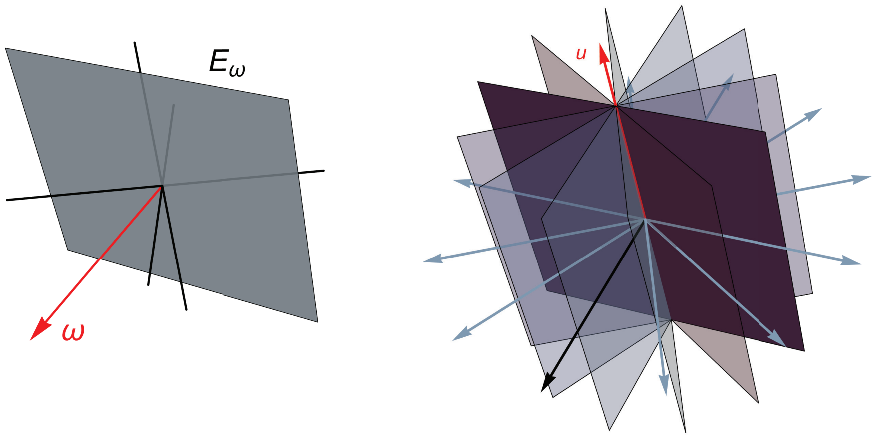

Figure 1: The plane E_ω for one ω and 'all' planes E_ω with ω such that $\langle \omega, u\rangle = 0$. Those E_ω contain u as a common direction.

- *If $P_u G_1 \simeq P_u G_2$ and $P_\omega G_1 \simeq P_\omega G_2$ for all $\omega \in \mathbb{S}^2$ with $\langle \omega, u\rangle = 0$, then $G_1 \simeq G_2$.*

Here $A \simeq B$ means that A equals B after a translation. In other words, there is a fixed $v \in \mathbb{R}^3$ such that $A = v + B$. Groemer showed in [11] that one could drop the condition $P_u G_1 \simeq P_u G_2$, whenever u is a regular direction for G_1. Here regular means that $\max \{\langle u, x\rangle ; x \in G_1\}$ is attained for a unique $x \in G_1$. Since domains G of constant width are precisely those domains for which

$$G^* := \tfrac{1}{2} G + \tfrac{1}{2}(-G) := \left\{\tfrac{1}{2}x - \tfrac{1}{2}y; x, y \in G\right\}$$

is a ball, which has only regular directions, one finds that $\left(\tilde{P}_\omega G\right)^*$ is a disc for all $\omega \in \mathbb{S}^2$ with $\langle \omega, u\rangle = 0$, if and only if G^* is a ball. Necessarily those discs and the ball have the same radius. This implies that a convex closed set $G \subset \mathbb{R}^3$ is a body of constant width if and only if there is a direction $u \in \mathbb{S}^2$, such that for some fixed $\rho > 0$ one finds

$$\left(\tilde{P}_\omega G\right)^* \simeq D_\rho := \left\{y \in \mathbb{R}^2; |y| \leq \rho\right\} \text{ for all } \omega \in \mathbb{S}^2 \text{ with } \langle \omega, u\rangle = 0.$$

This means that all those $P_\omega G$ should be two-dimensional convex sets of constant width ρ. So by taking $u = (1, 0, 0)$ we find that the boundary of $P_\omega G$ is described by (4) with some a depending on ω. This leads us to the result in Theorem 6 that will be formulated using an admittedly unusual parametrization of $\mathbb{S}^2$, which we introduce next:

Definition 3 *We parametrize $\mathbb{S}^2 = \boldsymbol{V}(S)$ by taking $S := [0, 2\pi] \times \left[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right]$ and*

$$\boldsymbol{V}(\varphi, \theta) := \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{pmatrix}. \tag{7}$$

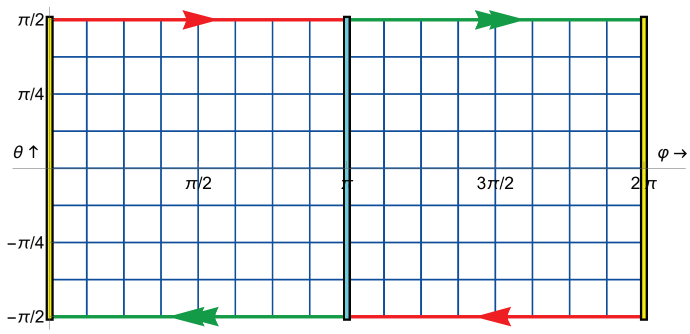

Figure 2: The parametrization v and its (non)uniqueness

Lemma 4 *Concerning (non)uniqueness of the parametrization $\boldsymbol{V}: S \to \mathbb{S}^2$ from (7) the following holds with*

$$S_\circ := ((0,\pi)\cup(\pi,2\pi)) \times \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right). \tag{8}$$

- *The restriction $\boldsymbol{V}: S_\circ \to \mathbb{S}^2$ is one-to-one.*
- *On $S \setminus S_\circ$ one has for all $\theta \in \left[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right]$:*

$$\boldsymbol{V}(0,\theta) = \boldsymbol{V}(2\pi,\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \boldsymbol{V}(\pi,\theta) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \tag{9}$$

and for all $\varphi \in [0,\pi]$:

$$\begin{cases} \boldsymbol{V}\left(\varphi, \tfrac{1}{2}\pi\right) = \boldsymbol{V}\left(2\pi-\varphi, -\tfrac{1}{2}\pi\right), \\ \boldsymbol{V}\left(\varphi, -\tfrac{1}{2}\pi\right) = \boldsymbol{V}\left(2\pi-\varphi, \tfrac{1}{2}\pi\right). \end{cases} \tag{10}$$

Remark 4.1 *The factor $a(\cdot,\cdot)$ will determine through $\kappa = (r - a(\varphi,\theta))^{-1}$ the curvature for fixed θ along*

$$\varphi \mapsto \boldsymbol{V}(\varphi,\theta), \tag{11}$$

the twodimensional projection of the body on the plane $E_{(0,-\sin\theta,\cos\theta)}$. Necessary for a correct parametrization is $|a(\varphi,\theta)| \le r$ and this gives a supremum bound for a. For $\varphi \notin \{0,\pi,2\pi\}$ the 'curvature' in the other direction is determined through h but in a more intricate way that uses $\partial_\theta a(\varphi,\theta)$. The parametrization has two special points, namely the east- and westpole $(\pm 1,0,0)$. For $\varphi \in \{0,\pi,2\pi\}$ the curvature in $(\pm 1,0,0)$ along the curve parametrized by (11) for fixed θ is determined by $a(0,\theta)$ and although $a(0,\theta)$ refers to just one point for all $\theta \in \left[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right]$ we maintain the dependence on θ.

So we have to identify $\left(\varphi, \tfrac{1}{2}\pi\right)$ with $\left(2\pi-\varphi, -\tfrac{1}{2}\pi\right)$ and $(0,\theta)$ with $(2\pi,\theta)$, but not take a constant at these two points. This leads to the following non-standard C^k-differentiability on $\mathbb{S}^2$.

Definition 5 *If we say $u \in C^k_{\text{per}}(S)$, then we mean that $\tilde{u} \in C^k(\mathbb{R}^2)$, where $\tilde{u}$ is the periodic extension of u, defined as follows:*

1. *for $(\varphi,\theta) \in S$ and $\tilde{\varphi} \in \mathbb{R}$ set*

$$u_1(\tilde{\varphi},\theta) := u(\varphi,\theta) \quad \text{if} \quad \tilde{\varphi} - \varphi \in 2\pi\mathbb{Z},$$

2. *for $(\tilde{\varphi},\theta) \in \mathbb{R} \times \left[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right]$ and $\tilde{\theta} \in \mathbb{R}$ set*

$$\begin{aligned} \tilde{u}(\tilde{\varphi},\tilde{\theta}) &:= u_1(\tilde{\varphi},\theta) \quad &&\text{if} \quad \tilde{\theta} - \theta \in 2\pi\mathbb{Z}, \\ \tilde{u}(\tilde{\varphi},\tilde{\theta}) &:= u_1(2\pi - \varphi, -\theta) \quad &&\text{if} \quad \tilde{\theta} - \theta + \pi \in 2\pi\mathbb{Z}. \end{aligned}$$

The parametrization is illustrated in Fig. 2 and Fig. 3. The relations in (9) for $\varphi \in \{0,\pi,2\pi\}$ are depicted in Fig. 2 by the yellow and light blue lines that coincide with the yellow and light blue dots in Fig. 3; the red and green curves in both figures relate to (10).

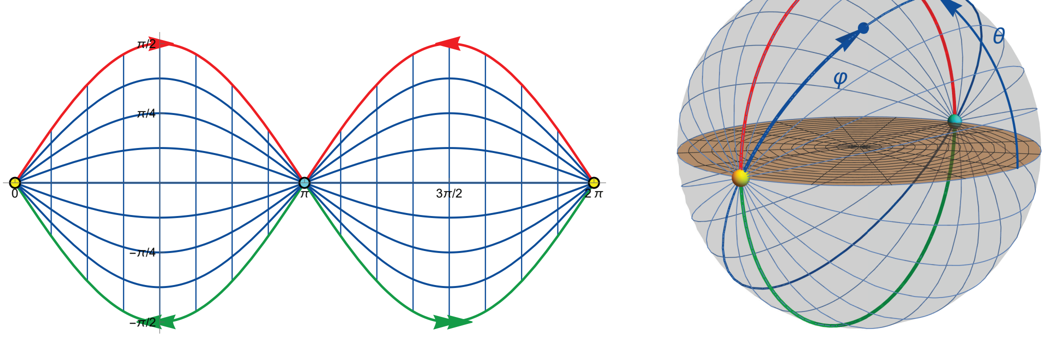

Figure 3: This parametrization of $\mathbb{S}^2$ yields geodesics through the east- and westpole.

Theorem 6 (Constructing bodies of constant width) *Suppose $a \in C^2_{\text{per}}(S)$ satisfies*

$$a(\varphi+\pi,\theta) = -a(\varphi,\theta) \qquad \textit{for all } (\varphi,\theta)\in S, \tag{12}$$

$$\int_0^\pi a(s,\theta)\begin{pmatrix}-\sin s\\ \cos s\end{pmatrix} ds = \begin{pmatrix}0\\0\end{pmatrix} \quad \textit{for all } \theta\in\left[-\tfrac12\pi,\tfrac12\pi\right]. \tag{13}$$

1. *Let $h \in C(S)$ be defined for $(\varphi,\theta)\in S_\circ$ by:*

$$h(\varphi,\theta) = -\frac{\int_0^\varphi \sin(\varphi - s)\ \partial_\theta a(s,\theta)\ ds}{\sin\varphi}. \tag{14}$$

   Then there exist $r_0(a) \geq \|a\|_\infty$, such that for all

$$r \geq r_0(a) \tag{15}$$

   and $\boldsymbol{X}_0 \in \mathbb{R}^3$, the surface $\boldsymbol{X}(S)$, defined for $(\varphi,\theta)\in S$ by

$$\boldsymbol{X}(\varphi,\theta) = \boldsymbol{X}_0 + \int_0^\varphi (r - a(s,\theta))\begin{pmatrix}-\sin s\\ \cos s\cos\theta\\ \cos s\sin\theta\end{pmatrix} ds + h(\varphi,\theta)\begin{pmatrix}0\\ -\sin\theta\\ \cos\theta\end{pmatrix} \tag{16}$$

   describes the boundary of a convex body of constant width.

2. *Moreover, with a as above, the function h in (14) is the unique possibility in order that $\boldsymbol{X}$ in (16) describes the boundary of a body of constant width.*

Remark 6.1 *Our construction will be illustrated by an example in Section 3. Although $a \in C^2_{\text{per}}(S)$ will imply that*

$$(\varphi,\theta)\mapsto h(\varphi,\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \in C^1_{\text{per}}(S)$$

and hence $\boldsymbol{X}\in C^1_{\text{per}}(S)$, it does not mean that the parametrization $\mathbb{S}^2 \to \partial G$ is a diffeomorphism. This will only be the case for $r > r_0(a)$ and in general not for $r = r_0(a)$. In the next section the example is such that $r = r_0(a) = 1$ and $\boldsymbol{X}$ will not be a diffeomorphism near $(1,0,0)$.

We have stated that $a \in C^2_{\text{per}}(S)$, which is sufficient for describing a 3d set of constant width, but certainly more than necessary for h and $\boldsymbol{X}$ to be well-defined. Necessary will be $a(\cdot,\theta)\in L^\infty(0,2\pi)$ and $\partial_\theta a(\cdot,\theta)\in L^1(0,2\pi)$. However, without more regularity the parametrization will not be differentiable and, if correct at all, will display a nonsmooth surface. For the 2d case a necessary and sufficient restriction appears, namely $r \geq r_0(a) := \|a\|_\infty$. There is obviously no h in 2d and hence no further restriction concerning regularity. To have a differentiable parametrization in 3d a bound appears that contains $\partial_\theta h$.

Lemma 7 *With $a \in C^2_{\text{per}}(S)$ defined according to Remark 4.1, and such that (12) and (13) hold, one finds that h defined in (14) satisfies for all $\theta\in\left[-\frac12\pi,\frac12\pi\right]$*

$$h(\varphi,\theta) = h(\varphi+\pi,\theta) \quad \textit{for } \varphi\in[0,\pi], \tag{17}$$

$$h(\varphi,\theta) = 0 \quad \textit{for } \varphi\in\{0,\pi,2\pi\} \tag{18}$$

and

$$h\left(\varphi,\tfrac12\pi\right) = -h\left(2\pi-\varphi,-\tfrac12\pi\right) \quad \textit{for all } \varphi\in[0,2\pi]. \tag{19}$$

Moreover

$$h(\varphi,\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \in C^1_{\text{per}}(S). \tag{20}$$

Remark 7.1 *Note that h in Theorem 6 is defined only for $(\varphi,\theta)\in S_\circ$. The lemma states that h can be extended uniquely, first to a continuous function on S which then satisfies the above results.*

Remark 7.2 *The formula in (16) for $\boldsymbol{X}$ is defined on S and should be such that the points in $S\setminus S_\circ$, that correspond to the same element in $\mathbb{S}^2$, give the same result. That is, $\boldsymbol{X}$ needs to satisfy (9) and (10). One may check that this indeed holds true by the conditions on a in (12) and (13) and the results for h in Lemma 7.*

Proof of Lemma 7. A straightforward computation shows, using (12) and (13), that (17) holds:

$$
\begin{aligned}
h(\varphi+\pi,\theta) &= -\frac{\int_0^{\varphi+\pi}\sin(\varphi+\pi-s)\,\partial_\theta a(s,\theta)\,ds}{\sin(\varphi+\pi)} = -\frac{\int_0^{\varphi+\pi}\sin(\varphi-s)\,\partial_\theta a(s,\theta)\,ds}{\sin\varphi}\\
&= -\frac{\binom{\cos\varphi}{\sin\varphi}\cdot\partial_\theta\int_0^{\pi}\binom{-\sin s}{\cos s}a(s,\theta)\,ds+\int_\pi^{\varphi+\pi}\sin(\varphi-s)\,\partial_\theta a(s,\theta)\,ds}{\sin\varphi}\\
&= -\frac{0+\int_0^{\varphi}\sin(\varphi-s-\pi)\,\partial_\theta a(s+\pi,\theta)\,ds}{\sin\varphi} = h(\varphi,\theta)\,.
\end{aligned}
$$

To prove (18) note that the definition of h from a implies that h is $C^1(S)$ plus a possible singularity at $\varphi\in\{0,\pi,2\pi\}$. We find by (13) with $k\in\{1,2\}$ that

$$
\int_0^{\varphi}\sin(\varphi-s)\;a(s,\theta)\;ds=\int_{k\pi}^{\varphi}\sin(\varphi-s)\;a(s,\theta)\;ds.
$$

By taking $k\in\{0,1,2\}$ such that $|\varphi-k\pi|\leq\frac{1}{2}\pi$, one finds

$$
\begin{aligned}
\left|\int_0^{\varphi}\sin(\varphi-s)\;\partial_\theta a(s,\theta)\;ds\right| &\leq \|\partial_\theta a\|_\infty\left|\int_{k\pi}^{\varphi}|\sin(\varphi-s)|\,ds\right|\\
&= 2\,\|\partial_\theta a\|_\infty\left(\sin\left(\tfrac{k\pi-\varphi}{2}\right)\right)^2\leq\|\partial_\theta a\|_\infty\,|\sin\varphi|^2\,.
\end{aligned}
$$

We used that $\left|\sqrt{2}\sin(\varphi/2)\right|\leq|\sin\varphi|$ for $\varphi\in[-\pi/2,\pi/2]$. Thus it follows that

$$
|h(\varphi,\theta)|\leq\|\partial_\theta a\|_\infty\,|\sin\varphi| \tag{21}
$$

and $h(\varphi,\theta)=0$ for $\varphi\in\{0,\pi,2\pi\}$ is the continuous extension, which implies $h\in C^1(S)$ with the appropriate extension in the φ-direction.

For (19) one should notice that $a\in C^1_{\text{per}}(S)$, defined as in Definition 5, means

$$
a(\varphi,\theta)=a(2\pi-\varphi,\theta-\pi)
$$

and implies

$$
\partial_\theta a\left(\varphi,\tfrac{1}{2}\pi\right)=\partial_\theta a\left(2\pi-\varphi,-\tfrac{1}{2}\pi\right), \tag{22}
$$

and hence, by using consecutively (12), (13), a substitution and (22), that

$$
\begin{aligned}
h\left(2\pi-\varphi,-\tfrac{1}{2}\pi\right) &= -\frac{\int_0^{2\pi-\varphi}\sin(2\pi-\varphi-s)\,\partial_\theta a\left(s,-\frac{1}{2}\pi\right)ds}{\sin(2\pi-\varphi)}\\
= \frac{\left(\binom{-\cos\varphi}{\sin\varphi}\cdot\partial_\theta\int_0^{2\pi}\binom{-\sin s}{\cos s}a(s,\theta)\,ds\right)_{[\theta=-\frac{1}{2}\pi]}}{\sin(2\pi-\varphi)} &- \frac{\int_{2\pi}^{2\pi-\varphi}\sin(2\pi-\varphi-s)\,\partial_\theta a\left(s,-\frac{1}{2}\pi\right)ds}{\sin(-\varphi)}\\
= -\frac{\int_{2\pi}^{2\pi-\varphi}\sin(2\pi-\varphi-s)\,\partial_\theta a\left(s,-\frac{1}{2}\pi\right)ds}{\sin(-\varphi)} &= \frac{\int_0^{\varphi}\sin(s-\varphi)\,\partial_\theta a\left(2\pi-s,-\frac{1}{2}\pi\right)ds}{\sin(-\varphi)}\\
&= \frac{\int_0^{\varphi}\sin(\varphi-s)\,\partial_\theta a\left(s,\frac{1}{2}\pi\right)ds}{\sin(\varphi)} = -h\left(\varphi,\tfrac{1}{2}\pi\right).
\end{aligned}
$$

Checking (20) is straightforward. The only delicate point is the last condition in Definition 5, but this follows from (19). In particular the last term in (16) is thus in $C^1_{\text{per}}(S)$. ■

The following theorem states that (16) describes not only some but all sets of constant width.

Theorem 8 (All bodies of constant width are represented by (16))

1. *Each body of constant width is described by (16) for some $a\in L^\infty(S)$ satisfying (12) and (13), with some $r\geq\|a\|_{L^\infty(S)}$ and with h such that (17), (18), (19) and*

$$
h(\varphi,\theta)=\frac{\lim_{\varepsilon\to0}\int_0^{\varphi}\frac{a(s,\theta)-a(s,\theta+\varepsilon)}{\varepsilon}\sin(\varphi-s)\,ds}{\sin\varphi}. \tag{23}
$$

2. *Concerning regularity we have*

$$
h(\varphi,\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\in C^{0,1}_{\text{per}}(S)
$$

and moreover,

   i. *if $a,\partial_\theta a\in C^0_{\text{per}}(S)$, then also (14) holds true;*
   ii. *if $a,\partial_\theta a\in C^1_{\text{per}}(S)$, then also (20) holds true.*

3 An example

The formulas are rather technical and in order to illustrate that the formula does deliver a body of constant width, we give an actual construction in a case that is computable. The example yields a body of constant width connecting two triangular 2d-domains of constant width based on the 2d-formula. In addition to $\boldsymbol{x}_0 = 0$ and $r = 1$ we use in Fig. 4:

- for the figure on the left: $a(s) = a_1(s) := -\cos(3s)$;
- for the figure in the middle: $a(s) = a_2(s) := \sin(3s)$.

The figure on the right of Fig. 4 combines these two curves in a 3d-setting in orthogonal planes with the red line as common intersection. In order to find a smooth perturbation from the horizontal to the vertical curve by curves whose projections will be 2d-curves of constant width 1, we use the following:

$$a(\varphi, \theta) := (\cos\theta)^2 \, a_1(\varphi) + (\sin\theta)^2 \, a_2(\varphi). \tag{24}$$

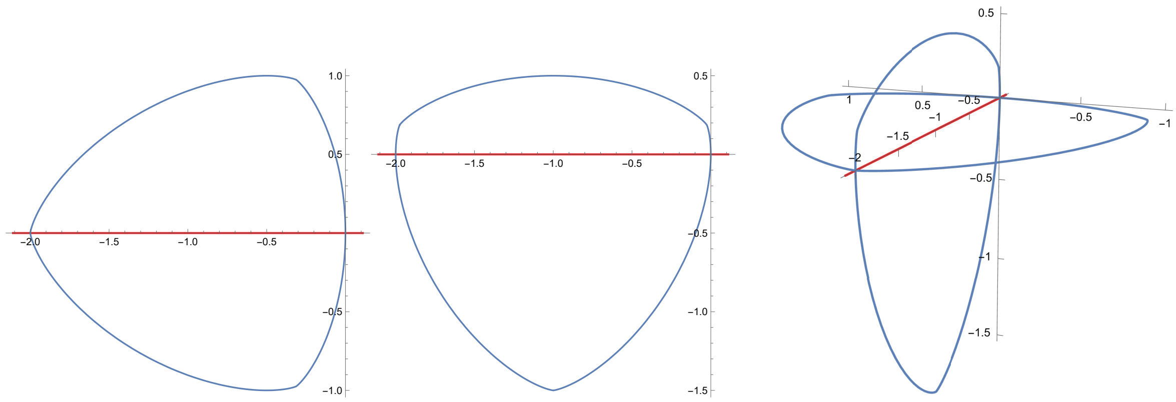

Figure 4: The 2d sets with a_1 and a_2, showing the common axis, and the combination in 3d by joining the red axes with the first curve horizontally and the second vertically

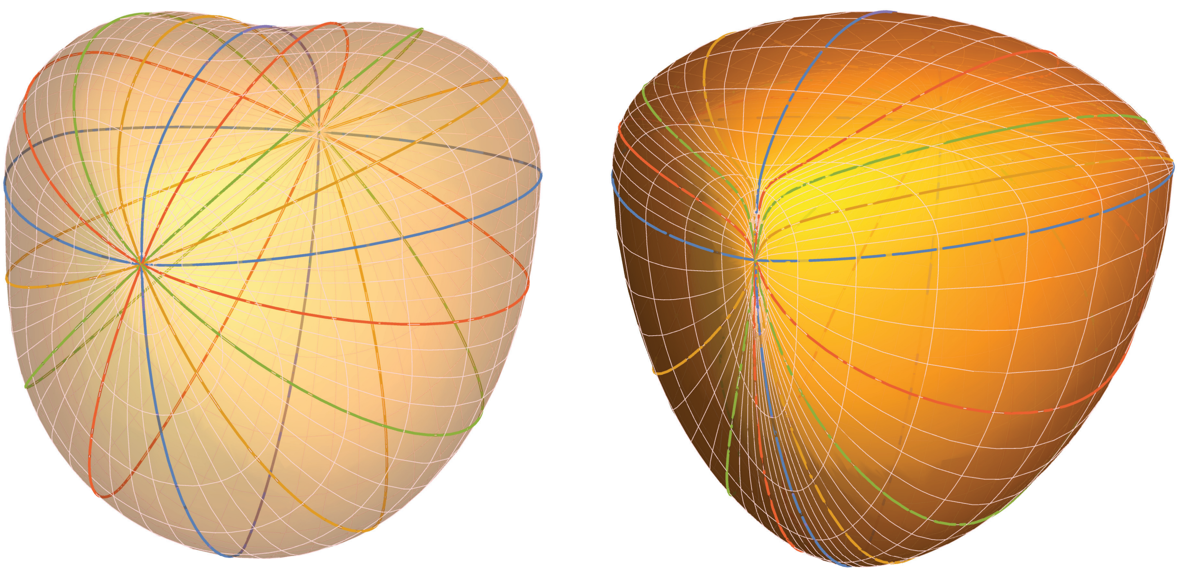

Figure 5: The intermediate construction without h and the final result. By the special choice of the θ-dependence in (24) the curves from Fig. 4 remain unmodified. They appear in blue. The surface on the right is not everywhere smooth.

The a in (24) is used to produce the sketch on the left in Fig. 5 using the formula in (16) without the h-term. Each intersection with a plane containing the horizontal (red) line $\{\lambda(1,0,0); \lambda \in \mathbb{R}\}$ will produce a 2d set of constant width although the body itself will not be a 3d set of constant width. Only after the modification which includes the additional h-term in (16) does one indeed find a 3d set of constant width.

The 2d-curves from Figure 4 appear in the horizontal and a vertical plane in blue. The special choice of $(\cos\theta)^2$ and $(\sin\theta)^2$ in (24) implies that $\partial_\theta a(\varphi, 0) = 0 = \partial_\theta a(\varphi, \pm\pi/2)$ for all φ. Therefore $h(\varphi, 0) = 0 = h(\varphi, \pm\pi/2)$ for all φ or, in other words, those two curves remain unmodified.

So in the left part of Fig. 5 one finds rotated 2d-domains of constant width 'rotating' from the the 2d figure with a_1 to the 2d figure with a_2. Since that rotated form is not even convex it cannot be a 3d-body of constant width. On the right one finds the same projections as on the left but the curves are now moved in a perpendicular direction with factor h. The function h in (14) is the one and only such that the figure on the right is a 3d-body of constant width.

4 Proofs of the two theorems

Let us start by introducing three vectors for a more concise notation:

$$\boldsymbol{\Xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{\Theta} = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } \boldsymbol{\Psi} = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{25}$$

These three directions constitute a θ-dependent orthonormal basis in $\mathbb{R}^3$ that turns out to be convenient for our parametrization. Also note that

$$\boldsymbol{V}(\varphi, \theta) = \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{pmatrix} = \cos\varphi\ \boldsymbol{\Xi} + \sin\varphi\ \boldsymbol{\Theta} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} \tag{26}$$

and that the identities

$$\partial_\theta \boldsymbol{\Theta} = \boldsymbol{\Psi} \text{ and } \partial_\theta \boldsymbol{\Psi} = -\boldsymbol{\Theta} \tag{27}$$

hold true.

The notation with $\cdot$ in (26) might seem artificial. However, it coincides with the usual definition of 'scalar' product and is most convenient for a concise formulation of the following proofs.

Proof of Theorem 6. We will have to show that $\boldsymbol{X}$ in (16) is a regular parametrization and secondly, that the resulting surface will yield a body of constant width. For both aspects we need to consider $\partial_\varphi \boldsymbol{X}(\varphi, \theta)$ and $\partial_\theta \boldsymbol{X}(\varphi, \theta)$.

► *Computation of $\partial_\varphi X$ and $\partial_\varphi X$.* We will check first that $\boldsymbol{X}$ in (16) is a regular parametrization for r large enough, that is

$$\tilde{\boldsymbol{X}} : \mathbb{S}^2 \to \partial G \text{ defined by } \tilde{\boldsymbol{X}}(\omega) := \boldsymbol{X}(\varphi, \theta) \tag{28}$$

is C^1, one-to-one and onto and even a diffeomorphism. With the notation from (25) we can rewrite (16) in

$$\boldsymbol{X}(\varphi, \theta) = \boldsymbol{X}_0 + \int_0^\varphi (r - a(s, \theta)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} + h(\varphi, \theta)\ \boldsymbol{\Psi}. \tag{29}$$

One computes that

$$\partial_\varphi \boldsymbol{X}(\varphi, \theta) = (r - a(\varphi, \theta)) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} + \partial_\varphi h(\varphi, \theta)\ \boldsymbol{\Psi}$$

and

$$\begin{aligned} \partial_\theta \boldsymbol{X}(\varphi, \theta) = -\int_0^\varphi \partial_\theta a(s, \theta) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} - h(\varphi, \theta)\ \boldsymbol{\Theta}\ + \\ \left(\int_0^\varphi (r - a(s, \theta)) \cos s\ ds + \partial_\theta h(\varphi, \theta) \right) \boldsymbol{\Psi}. \end{aligned}$$

► *Invariant normal direction.* Before continuing with showing that the parametrization is appropriate, notice that one directly finds whenever this is the case, then the outward normal at $\boldsymbol{X}(\varphi, \theta)$ satisfies:

$$\boldsymbol{V}\big(\tilde{\boldsymbol{X}}(\omega)\big) = \omega \text{ for all } \omega \in \mathbb{S}^2. \tag{30}$$

Indeed, with $\omega = \cos\varphi\ \boldsymbol{\Xi} + \sin\varphi\ \boldsymbol{\Theta}$ we find

$$\omega \cdot \partial_\varphi \boldsymbol{X}\,(\varphi,\theta) = 0 \tag{31}$$

and by the definition of h:

$$\omega \cdot \partial_\theta \boldsymbol{X}\,(\varphi,\theta) = \int_0^\varphi \partial_\theta a(s,\theta) \sin(s-\varphi)\, ds - h(\varphi,\theta)\sin\varphi = 0. \tag{32}$$

Note that for the inverse of $\omega \mapsto \boldsymbol{X}\,(\varphi,\theta)$ to be the Gauss map of the body of constant width we need (32) to be zero and hence this is the only possible definition of h.

► *Well defined parametrization.* For $a \in C^2(S)$ Lemma 7 implies that h is well-defined and lies in $C^1(S)$. So with (12) and (13) also the expression in (29) lies in $C^1(S)$.

In order to have a regular parametrization it is sufficient that:

- $\partial_\varphi \boldsymbol{X} \times \partial_\theta \boldsymbol{X}$ is nontrivial on $\{(\varphi,\theta)\in S; \varphi \notin \{0,\pi,2\pi\}\}$, and
- $\partial_\varphi \boldsymbol{X}\,(\varphi,0) \times \partial_\varphi \boldsymbol{X}\left(\varphi,\tfrac12\pi\right)$ is nontrivial for $\varphi \in \{0,\pi,2\pi\}$.

Let us start with the second case for $\varphi = 0$, with $\varphi \in \{\pi, 2\pi\}$ similarly:

$$\partial_\varphi \boldsymbol{X}(\varphi,0) \times \partial_\varphi \boldsymbol{X}(\varphi,\tfrac12\pi) = \begin{pmatrix} 0 \\ r-a(0,0) \\ \partial_\varphi h\,(0,0) \end{pmatrix} \times \begin{pmatrix} 0 \\ -\partial_\varphi h\left(0,\tfrac12\pi\right) \\ r-a(0,\tfrac12\pi) \end{pmatrix} =: \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix},$$

where

$$T = (r-a(0,0))\left(r-a(0,\tfrac12\pi)\right) + \partial_\varphi h\,(0,0)\,\partial_\varphi h\left(0,\tfrac12\pi\right). \tag{33}$$

Since

$$\partial_\varphi h(\varphi,\theta) = \frac{\int_0^\varphi \sin s\ \partial_\theta a(s,\theta) ds}{(\sin\varphi)^2} \tag{34}$$

one obtains as in (21)

$$|\partial_\varphi h(\varphi,\theta)| \le \|\partial_\theta a\|_{L^\infty(S)}.$$

A sufficient condition for $T > 0$ is

$$r > \|a\|_{L^\infty(S)} + \|\partial_\theta a\|_{L^\infty(S)}.$$

For $\varphi \notin \{0,\pi,2\pi\}$, using (31) and (32), which state that ω is perpendicular to $\partial_\varphi \boldsymbol{X}$ and $\partial_\theta \boldsymbol{X}$, a simple way of checking that $\partial_\varphi \boldsymbol{X} \times \partial_\theta \boldsymbol{X}$ is nontrivial, is to show $\omega \cdot (\partial_\varphi \boldsymbol{X} \times \partial_\theta \boldsymbol{X}) \neq 0$. With the orthonormal basis $\{\boldsymbol{\Xi}, \boldsymbol{\Theta}, \boldsymbol{\Psi}\}$ we obtain

$$\omega \cdot \big(\partial_\varphi \boldsymbol{X}(\varphi,\theta) \times \partial_\theta \boldsymbol{X}(\varphi,\theta)\big)$$

$$= \det \begin{pmatrix} \cos\varphi & -(r-a(\varphi,\theta))\sin\varphi & \int_0^\varphi \partial_\theta a(s,\theta)\sin s\ ds \\ \sin\varphi & (r-a(\varphi,\theta))\cos\varphi & -h(\varphi,\theta) - \int_0^\varphi \partial_\theta a(s,\theta)\cos s\ ds \\ 0 & \partial_\varphi h(\varphi,\theta) & \partial_\theta h(\varphi,\theta) + \int_0^\varphi (r-a(s,\theta))\cos s\ ds \end{pmatrix}$$

$$\begin{aligned} &= (r-a(\varphi,\theta))\left(\partial_\theta h(\varphi,\theta) + \int_0^\varphi (r-a(s,\theta))\cos s\ ds\right) + \\ &\quad \partial_\varphi h(\varphi,\theta)\left(\cos\varphi\ h(\varphi,\theta) + \cos\varphi \int_0^\varphi \partial_\theta a(s,\theta)\cos s\ ds + \int_0^\varphi \partial_\theta a(s,\theta)\sin\varphi\sin s ds\right) = \end{aligned} \tag{35}$$

Using the expression for h from (14) we obtain

$$\begin{aligned} (35) &= (r-a(\varphi,\theta))\left(\partial_\theta h(\varphi,\theta) + \int_0^\varphi (r-a(s,\theta))\cos s\ ds\right) + \\ &\quad \partial_\varphi h(\varphi,\theta)\left(\cos\varphi \frac{-\int_0^\varphi \sin(\varphi-s)\ \partial_\theta a(s,\theta)\ ds}{\sin\varphi} + \int_0^\varphi \partial_\theta a(s,\theta) \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} ds\right) \end{aligned}$$

$$= (r - a(\varphi,\theta)) \left(\partial_\theta h(\varphi,\theta) + \int_0^\varphi (r - a(s,\theta)) \cos s \; ds \right) +$$
$$\partial_\varphi h(\varphi,\theta) \left(\frac{\cos\varphi}{\sin\varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} + \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \right) \cdot \int_0^\varphi \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \partial_\theta a(s,\theta) \; ds$$

$$= (r - a(\varphi,\theta)) \left(\partial_\theta h(\varphi,\theta) + \int_0^\varphi (r - a(s,\theta)) \cos s \; ds \right) +$$
$$\partial_\varphi h(\varphi,\theta) \left(\left(\frac{(\cos\varphi)^2}{\sin\varphi} + \frac{(\sin\varphi)^2}{\sin\varphi} \right) \int_0^\varphi \sin s \; \partial_\theta a(s,\theta) \; ds \right)$$

$$= (r - a(\varphi,\theta)) \left(\partial_\theta h(\varphi,\theta) + \int_0^\varphi (r - a(s,\theta)) \cos s \; ds \right) + \partial_\varphi h(\varphi,\theta) \frac{\int_0^\varphi \sin s \; \partial_\theta a(s,\theta) ds}{\sin\varphi} = \tag{36}$$

In (34) we computed $\partial_\varphi h(\varphi,\theta)$ and since

$$\partial_\theta h(\varphi,\theta) = \frac{-\int_0^\varphi \sin(\varphi - s) \, \partial_\theta^2 a(s,\theta) ds}{\sin\varphi}$$

we may continue by

$$(36) = (r - a(\varphi,\theta)) \left(\int_0^\varphi (r - a(s,\theta)) \cos s \; ds - \frac{\int_0^\varphi \sin(\varphi - s) \, \partial_\theta^2 a(s,\theta) ds}{\sin\varphi} \right) +$$
$$\frac{\left(\int_0^\varphi \sin s \; \partial_\theta a(s,\theta) ds\right)^2}{(\sin\varphi)^3}. \tag{37}$$

With the bounds based on (12) and (13) we get:

$$\left| \int_0^\varphi \sin s \; \partial_\theta a(s,\theta) ds \right| \le |\sin\varphi|^2 \left\| \partial_\theta a \right\|_{L^\infty(S)},$$
$$\left| \int_0^\varphi \sin(\varphi - s) \, \partial_\theta^2 a(s,\theta) ds \right| \le |\sin\varphi|^2 \left\| \partial_\theta^2 a \right\|_{L^\infty(S)}.$$

So we may estimate

$$\left| (37) - (r - a(\varphi,\theta)) \int_0^\varphi (r - a(s,\theta)) \cos s \; ds \right| \le \left(2r \left\| \partial_\theta^2 a \right\|_{L^\infty(S)} + \left\| \partial_\theta a \right\|_{L^\infty(S)}^2 \right) |\sin\varphi|$$

which shows that whenever $\varphi \in (0,\pi)$ for r large enough the expression in (37) is positive. Whenever $\varphi \in (\pi, 2\pi)$ for r large enough the expression in (37) is negative. Note that the orientation of $d\varphi \wedge d\theta$ changes on $\mathbb{S}^2$ for $\varphi = \pi$, which explains the sign change. Moreover, for each $(\varphi,\theta) \in S$ with $\varphi \in [0,\pi]$ and $r \ge \|a\|_{L^\infty(S)}$ the function $r \mapsto (37)$ is increasing, meaning that once $(37) > 0$ is satisfied for $r = r_0$ it is so for $r > r_0$. The same holds for (33). So there exists a minimal $r_0(a)$ such that the parametrization is well-defined for all $r > r_0(a)$. For $r = r_0(a)$ the parametrization no longer is C^1 or one-to-one. However, since for all $r > r_0(a)$ one finds a body of constant width and all functions involved are continuous, also the limit by taking $t \downarrow r_0(a)$ gives a body of constant width.

► *Homotopy to the sphere.* The parametrization is well-defined for all $r > r_0(a)$ and to consider the explicit dependence on r we use for the expression in (16) in this paragraph

$$\boldsymbol{X}_e(r,\varphi,\theta) := \boldsymbol{X}(\varphi,\theta).$$

With $\tilde{\boldsymbol{X}}_e$ as in (28) we define

$$(0,1] \times \mathbb{S}^2 \ni (\rho,\omega) \mapsto \tilde{\boldsymbol{Y}}(\rho,\omega) := \frac{1}{r} \tilde{\boldsymbol{X}}_e\left(\rho^{-1} r, \omega\right) \in \mathbb{R}^3.$$

One finds that

$$\tilde{\boldsymbol{Y}}(1,\omega) = \frac{1}{r} \tilde{\boldsymbol{X}}_e(r,\omega) \text{ and } \tilde{\boldsymbol{Y}}(0,\omega) := \lim_{\rho \downarrow 0} \tilde{\boldsymbol{Y}}(\rho,\omega) = \omega$$

with all $\tilde{\boldsymbol{Y}}(\rho,\cdot)$ being regular parametrizations satisfying

$$\nu\left(\tilde{\boldsymbol{Y}}(\rho,\omega)\right) = \omega \text{ for all } \omega \in \mathbb{S}^2. \tag{38}$$

So $\mathbb{R}^3 \setminus \tilde{\boldsymbol{X}}_e\left(1, \mathbb{S}^2\right)$ has precisely two connected components. The bounded one we call A.

► *Convexity.* Since the extreme value of $\tilde{\boldsymbol{X}}(\mathbb{S}^2)$ in the direction ω has normal ω, and since $\nu_{\tilde{\boldsymbol{X}}(\omega)} = \omega$, that extreme point is $\tilde{\boldsymbol{X}}(\omega)$. So for each ω it holds that $\tilde{\boldsymbol{X}}(\mathbb{S}^2)$, except for $\tilde{\boldsymbol{X}}(\omega)$ itself, is on one side of that tangen plane. Hence $\bar{A}$ lies on one side of all the tangent planes for $\partial A = \tilde{\boldsymbol{X}}(\mathbb{S}^2)$, which implies that $\bar{A}$ is convex. See also the proof of Hadamard's Theorem [18, page 194].

► *Body of constant width.* According to the results proved above it is sufficient to show that

$$\tilde{\boldsymbol{X}}(\omega) - \tilde{\boldsymbol{X}}(-\omega) = 2r\omega \text{ for all } \omega \in \mathbb{S}^2.$$

For the parametrization $\boldsymbol{X}$ with $\boldsymbol{V}$ as in (7) this coincides with

$$\boldsymbol{X}(\varphi,\theta) - \boldsymbol{X}(\varphi+\pi,\theta) = 2r\ \boldsymbol{V}(\varphi,\theta) \text{ for all } (\varphi,\theta) \in S.$$

Using (29) we find with (12), (13) and (17) that

$$\begin{aligned}
&\boldsymbol{X}(\varphi+\pi,\theta) - \boldsymbol{X}(\varphi,\theta) \\
&= \int_{\varphi}^{\varphi+\pi} (r - a(s,\theta)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} + (h(\varphi+\pi,\theta) - h(\varphi,\theta))\ \boldsymbol{\Psi} \\
&= r \begin{pmatrix} \cos(\varphi+\pi) - \cos\varphi \\ \sin(\varphi+\pi) - \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} = -2r\ \boldsymbol{V}(\varphi,\theta),
\end{aligned}$$

as desired. ■

Proof of Theorem 8, the derivation of the formula with some h. Suppose that G is a body of constant width. Define $\boldsymbol{X}_0 \in \mathbb{R}^3$ as the point on ∂G with the largest x_1-coordinate. Taking $u = (1,0,0)^T$ and $\omega = (0, -\sin\theta, \cos\theta)^T$ the result of Hadwiger, extended by the remark of Groemer that bodies of constant width have only regular boundary points, states that is is sufficient that the projections $P_\omega G$ of G on the planes

$$E_\omega := c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

with $\theta \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ are curves of constant width $d_{\tilde{P}_\omega G} = 2r$. Thus all those sets can be described by the formula in Recipe 1 with for each θ some function a as in Recipe 1 depending on θ as a parameter. The value of r is the same for all projections and does not depend on θ. In other words, a fixed r exists and for each θ a mapping $\varphi \mapsto a(\varphi,\theta) \in L^\infty(0,2\pi)$ such that for the corresponding $\boldsymbol{x}$ as in Theorem 1 we have

$$\partial\tilde{P}_\omega G = \boldsymbol{x}([0,2\pi],\theta)$$

with some $\boldsymbol{x}(0,\theta) \in \mathbb{R}^2$ and $\sup\{|a(\varphi;\theta)|; 0 \le \varphi \le \pi\} \le r$ for all $\theta \in [-\pi/2, \pi/2]$. Moreover, the mapping $\varphi \mapsto a(\varphi,\theta)$ satisfies (2) and (3). Hence (12), (13) and $r_0(a) \ge \|a\|_{L^\infty(S)}$ are necessary conditions.

Since $G \subset P_\omega G + [\omega]$ with $[\omega] = \{\lambda\omega; \lambda \in \mathbb{R}\}$, it follows that for each

$$X_* \in \partial G \cap (\partial P_\omega G + [\omega])$$

there is $(\varphi,\theta) \in S$ and $h(\varphi,\theta) \in \mathbb{R}$ with

$$h(\varphi,\theta) = h(\varphi+\pi,\theta) \text{ for all } (\varphi,\theta) \in S \tag{39}$$

such that

$$X_* = \boldsymbol{x}(\varphi,\theta) \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} + h(\varphi,\theta)\ \boldsymbol{\Psi}. \tag{40}$$

Here (39) follows from the fact that the line through the points of farthest distance is perpendicular to the plane through $(0,0,0)$, $(1,0,0)$ and $(0,\cos\theta,\sin\theta)$. Since for $\varphi \in \{0,\pi,2\pi\}$ the X_* in (40) does not depend on θ, one finds for all $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, that

$$h(0,\theta) = h(\pi,\theta) = h(2\pi,\theta) = 0$$

and

$$\boldsymbol{x}(0,0) = \boldsymbol{x}(0,\theta) = \boldsymbol{x}(2\pi,\theta) = \boldsymbol{x}(\pi,\theta) + 2r\boldsymbol{\Xi}.$$

The first vector on the right in (40) inherits the conditions of the two-dimensional formula and so $\varphi \mapsto \boldsymbol{x}(\varphi,\theta)$ is for each $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ as in (4), and the formula in (40) gives a parametrization $\boldsymbol{X} : S \to \mathbb{R}^3$ of ∂G given by

$$\boldsymbol{X}(\varphi,\theta) = \boldsymbol{X}_0 + \int_0^\varphi (r - a(s,\theta)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix} + h(\varphi,\theta)\ \boldsymbol{\Psi}. \tag{41}$$

Since the two vector functions in $\boldsymbol{X}(\varphi,\theta)$ have independent directions, each one should be continuous on S and with $\boldsymbol{\Theta}$ and $\boldsymbol{\Psi}$ reversing sign in $\theta = \pm\pi/2$, hence

$$\begin{aligned} (\varphi,\theta) &\mapsto \int_0^\varphi (r - a(s,\theta)) \sin s \, ds \in C^0_{\text{per}}(S), \\ (\varphi,\theta) &\mapsto \int_0^\varphi (r - a(s,\theta)) \cos s \, ds \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \in C^0_{\text{per}}(S), \\ (\varphi,\theta) &\mapsto h(\varphi,\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \in C^0_{\text{per}}(S). \end{aligned}$$

For later use we will also define

$$\boldsymbol{X}_{oh}(\varphi,\theta) := \boldsymbol{X}_0 + \int_0^\varphi (r - a(s,\theta)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta}(\theta) \end{pmatrix}. \tag{42}$$

We now have shown the formula in (16) but without a more specific formula for h. In order to find (23) we need some additional tools we explain next. ■

The first step to address formula for h of Theorem 8 is to find a formulation of how Lipschitz-continuity on $\mathbb{S}^2$ translates to our parametrization that uses S. This translation into our (or any) spherical coordinates $(\varphi,\theta) \in S$ is however not so obvious. The formulation is found in Lemma 10 of Appendix A.

It is not clear how to show directly that the parametrization $\omega \mapsto \tilde{\boldsymbol{X}}(\omega)$ is Lipschitz. Instead of a direct approach we make a detour through $\omega \mapsto \tilde{\boldsymbol{X}}_{oh}(\omega)$ from (42) that turns out to be something that parametrizes what we will call a shadow domain. See Appendix B. Without loss of generality we may assume that the domain satisfies the assumptions in Definition 13. There one also finds the definition of $Sh_{\boldsymbol{\Xi}}(\Omega)$.

Lemma 9 *Suppose that $\boldsymbol{X}$ from (41) parametrizes the body of constant width. Let $\boldsymbol{X}_{oh}$ be as in (42), $\boldsymbol{V}$ as in (7) and $\boldsymbol{\Psi}$ as in (25). Then the function*

$$\omega \mapsto \tilde{\boldsymbol{X}}_{oh}(\omega) : \mathbb{S}^2 \to \mathbb{R}^3$$

defined by $\tilde{\boldsymbol{X}}_{oh}(\boldsymbol{V}(\varphi,\theta)) := \boldsymbol{X}_{oh}(\varphi,\theta)$ for $(\varphi,\theta) \in S$ is Lipschitz-continuous and satisfies

1. *$P_{\boldsymbol{\Psi}(\theta)}\big(\boldsymbol{X}(\varphi,\theta)\big) = \boldsymbol{X}_{oh}(\varphi,\theta)$ and*
2. *if $\Omega \subset \mathbb{R}^3$ is the bounded domain with $\partial\Omega = \tilde{\boldsymbol{X}}(\mathbb{S}^2)$ then the 3d-shadow satisfies*

$$\partial Sh_{\boldsymbol{\Xi}}(\Omega) = \tilde{\boldsymbol{X}}_{oh}(\mathbb{S}^2).$$

Remark 9.1 *The sketch on the left of Figure 5 shows such a domain with boundary $\tilde{\boldsymbol{X}}_{oh}(\mathbb{S}^2)$.*

Proof. From our construction one finds that the function $\tilde{\boldsymbol{X}}_{oh} : \mathbb{S}^2 \to \mathbb{R}^3$ parametrizes the collection of boundaries of '2d-shadows' in the directions $\boldsymbol{\Psi}(\theta)$ for $\theta \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ and gives a bounded two-dimensional manifold in $\mathbb{R}^3$.

Each 2d-shadow $P_{\boldsymbol{\Psi}(\theta)}(\Omega)$ for $\partial\Omega = \boldsymbol{X}(\theta,\mathbb{S}^1)P_{\boldsymbol{\Psi}(\theta)}\big(\boldsymbol{X}(\theta,\mathbb{S}^1)\big)\ \boldsymbol{X}_{oh}(\theta,\varphi)$ is a two-dimensional set of constant width. The 3d-domain bounded by these curves is in general not a body of constant width and not even convex. In fact, if we rotate a body G of constant width around the axis through $\boldsymbol{e}_1$ and $-\boldsymbol{e}_1$, assuming G has width 2 and lies between these points, we may use that each projection is a curve of constant width. Moreover, its boundary lies between the extreme cases of two-dimensional curves of constant width. These extreme cases are the Reuleaux triangle pointing upwards and the one pointing downwards. See Fig. 6.

Writing $\mathcal{P}(\varphi;\theta)$ for the boundary of the projection of G in the direction $(0,\cos\theta,\sin\theta)$ we find for both points on the same side, that

$$\begin{aligned} |\mathcal{P}(\varphi;\theta) - \mathcal{P}(\varphi;\theta_0)| &\le \sqrt{4 - (1+|x|)^2}\, |\theta - \theta_0| \\ &\le 2\sqrt{1-x^2}\, |\theta - \theta_0| \le 2\,|\sin\varphi|\, |\theta - \theta_0|, \end{aligned}$$

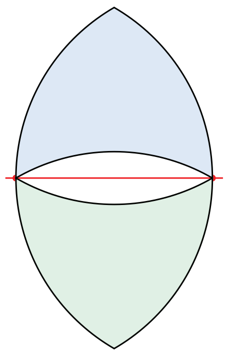

Figure 6: The axis in red with the shaded parts showing the possible areas for the projections

where $x = \cos\varphi$. Moreover, since all projections have boundaries that are curves of equal width 2 and the Lipschitz constant with respect to the φ-variable is uniform with respect to θ, there exists $L \in \mathbb{R}^+$, independent of θ, such that

$$|\mathcal{P}(\varphi;\theta_0) - \mathcal{P}(\varphi_0;\theta_0)| \leq L\,|\varphi - \varphi_0|\,.$$

Combining the last two estimates shows the Lipschitz-condition for $\mathcal{P}$ with φ and φ_0 on the same side. If φ and φ_0 are such that points are on opposite sides, a similar argument gives the estimate of the expression in (51). Hence $\tilde{\mathcal{P}} : \mathbb{S}^2 \to \mathbb{R}^3$ is Lipschitz-continuous according to Lemma 10. ∎

Continued proof of Theorem 8, a formula for h and regularity. If $\mathbb{S}^2 \ni \omega \mapsto R(\omega)\omega$ for a positive R describes the boundary of a convex domain, then this function is Lipschitz continuous. Such a result does not hold if one just has that $\omega \mapsto \tilde{\boldsymbol{X}}(\omega)$ describes the boundary of a convex domain. To show the Lipschitz-continuity of this function, we use the $3d$-shadow domain from Definition 13 in Appendix B. It is proven in Lemma 9 that the function $\tilde{\boldsymbol{X}}_{oh}$, defined by

$$S \ni (\varphi,\theta) \mapsto \boldsymbol{X}_{oh}(\varphi,\theta) = P_{\boldsymbol{\Psi}(\theta)}\big(\boldsymbol{X}(\varphi,\theta)\big), \tag{43}$$

is Lipschitz-continuous on $\mathbb{S}^2$. It remains to show that this tranfers to $\tilde{\boldsymbol{X}}$. Note that for $\boldsymbol{\Theta}$ and $\boldsymbol{\Psi}$ as functions of θ:

$$\boldsymbol{\Theta}(\theta+\varepsilon) = \cos\varepsilon\ \boldsymbol{\Theta}(\theta) + \sin\varepsilon\ \boldsymbol{\Psi}(\theta) \ \text{ and } \ \boldsymbol{\Psi}(\theta+\varepsilon) = \cos\varepsilon\ \boldsymbol{\Psi}(\theta) - \sin\varepsilon\ \boldsymbol{\Theta}(\theta). \tag{44}$$

When there is no misunderstanding we skip the θ-dependence of $\boldsymbol{\Theta}$ and $\boldsymbol{\Psi}$ and use only $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\theta)$ and $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\theta)$. Thus one computes

$$\begin{aligned}
&\boldsymbol{X}_{oh}(\varphi,\theta+t) - \boldsymbol{X}_{oh}(\varphi,\theta) = \\
&\qquad \int_0^{\varphi} (r - a(s,\theta+t)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \cos t\ \boldsymbol{\Theta} + \sin t\ \boldsymbol{\Psi} \end{pmatrix} \\
&\qquad\qquad - \int_0^{\varphi} (r - a(s,\theta)) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} ds \cdot \begin{pmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Theta} \end{pmatrix}.
\end{aligned}$$

Hence for $t \neq 0$ and small enough such that $\theta + t \in [-\pi/2, \pi/2]$, it holds by the Lipschitz-continuity that

$$\frac{\boldsymbol{X}_{oh}(\varphi,\theta+t) - \boldsymbol{X}_{oh}(\varphi,\theta)}{t\sin\varphi} \cdot \boldsymbol{\Psi} = \frac{\sin t}{t\sin\varphi} \int_0^{\varphi} (r - a(s,\theta+t)) \cos s\ ds, \tag{45}$$

is bounded and moreover positive for $\varphi \notin \{0, \pi, 2\pi\}$. The contribution by h in (41) is in the $\pm\boldsymbol{\Psi}$-direction but, contrary to $\boldsymbol{X}_{oh}$, has no a-priori fixed direction although the directions in φ and $\varphi+\pi$ are the same since $h(\varphi,\theta) = h(\varphi+\pi,\theta)$. Since the paramerization $\boldsymbol{X}$ as a function of θ has to be pointing in the direction of $\sin\varphi$ in order to be well-defined, we obtain

$$\tfrac{\boldsymbol{X}_{oh}(\varphi+\pi,\theta+t) - \boldsymbol{X}_{oh}(\varphi+\pi,\theta)}{t} \cdot \boldsymbol{\Psi} \leq \tfrac{h(\varphi,\theta+t) - h(\varphi,\theta)}{t} \leq \tfrac{\boldsymbol{X}_{oh}(\varphi,\theta+t) - \boldsymbol{X}_{oh}(\varphi,\theta)}{t} \cdot \boldsymbol{\Psi}.$$

This estimate implies, since $|\sin t| \leq |t|$, that

$$\left|\tfrac{h(\varphi,\theta+t) - h(\varphi,\theta)}{t}\right| \leq 2r\,|\sin\varphi|\,.$$

So $\boldsymbol{X}=\boldsymbol{X}_{oh}+h\boldsymbol{\Psi}$ is Lipschitz-continuous and we find

$$\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi,\theta)=\int_0^{\varphi}(r-a(s,\theta+\varepsilon))\begin{pmatrix}-\sin s\\ \cos s\end{pmatrix}ds\cdot\begin{pmatrix}\boldsymbol{\Xi}\\ \cos\varepsilon\ \boldsymbol{\Theta}+\sin\varepsilon\ \boldsymbol{\Psi}\end{pmatrix}+$$
$$+h(\varphi,\theta+\varepsilon)(\cos\varepsilon\ \boldsymbol{\Psi}-\sin\varepsilon\ \boldsymbol{\Theta})-\int_0^{\varphi}(r-a(s,\theta))\begin{pmatrix}-\sin s\\ \cos s\end{pmatrix}ds\cdot\begin{pmatrix}\boldsymbol{\Xi}\\ \boldsymbol{\Theta}\end{pmatrix}-h(\varphi,\theta)\ \boldsymbol{\Psi}$$

$$=\int_0^{\varphi}(a(s,\theta)-a(s,\theta+\varepsilon))\begin{pmatrix}-\sin s\\ \cos s\end{pmatrix}ds\cdot\begin{pmatrix}\boldsymbol{\Xi}\\ \boldsymbol{\Theta}\end{pmatrix}+(h(\varphi,\theta+\varepsilon)-h(\varphi,\theta))\ \boldsymbol{\Psi}+$$
$$-2\sin(\varepsilon/2)\left(h(\varphi,\theta+\varepsilon)\begin{pmatrix}\cos(\varepsilon/2)\\ \sin(\varepsilon/2)\end{pmatrix}+\int_0^{\varphi}(r-a(s,\theta+\varepsilon))\cos s ds\begin{pmatrix}\sin(\varepsilon/2)\\ -\cos(\varepsilon/2)\end{pmatrix}\right)\cdot\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\Psi}\end{pmatrix}.$$

Since $\boldsymbol{X}(\varphi,\theta)$ describes the surface of a body of constant width $2r$ and

$$\boldsymbol{X}(\varphi,\theta)-\boldsymbol{X}(\varphi+\pi,\theta)=2r\boldsymbol{V}(\varphi,\theta)$$

we need that for all ε

$$|\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi+\pi,\theta)|\leq|\boldsymbol{X}(\varphi,\theta)-\boldsymbol{X}(\varphi+\pi,\theta)|. \tag{46}$$

Note that

$$(\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi+\pi,\theta))\cdot\boldsymbol{V}(\varphi,\theta)=(\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi,\theta))\cdot\boldsymbol{V}(\varphi,\theta)+2r$$

and thus we need

$$(\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi,\theta))\cdot\boldsymbol{V}(\varphi,\theta)\leq 0. \tag{47}$$

Since $\boldsymbol{V}(\varphi,\theta)=\cos\varphi\ \boldsymbol{\Xi}+\sin\varphi\ \boldsymbol{\Theta}$ we find, using the Lipschitz-continuity of $\theta\mapsto\boldsymbol{X}(\varphi,\theta)$, that

$$(\boldsymbol{X}(\varphi,\theta+\varepsilon)-\boldsymbol{X}(\varphi,\theta))\cdot\boldsymbol{V}(\varphi,\theta)=\int_0^{\varphi}(a(s,\theta)-a(s,\theta+\varepsilon))\sin(\varphi-s)\,ds$$
$$-\sin\varphi\left(\sin\varepsilon\ h(\varphi,\theta+\varepsilon)+2(\sin(\varepsilon/2))^2\int_0^{\varphi}(r-a(s,\theta+\varepsilon))\cos s\ ds\right)$$
$$=\varepsilon\left(\int_0^{\varphi}\frac{a(s,\theta)-a(s,\theta+\varepsilon)}{\varepsilon}\sin(\varphi-s)\,ds-\sin\varphi\ h(\varphi,\theta)\right)+\mathcal{O}\left(\varepsilon^2\right).$$

For (47) to hold it follows that for all ε small:

$$\int_0^{\varphi}\frac{a(s,\theta)-a(s,\theta+\varepsilon)}{\varepsilon}\sin(\varphi-s)\,ds-\sin\varphi\ h(\varphi,\theta)=\mathcal{O}(\varepsilon).$$

And with h being Lipschitz-continuous itself, we find

$$h(\varphi,\theta)=\frac{\lim_{\varepsilon\to 0}\int_0^{\varphi}\frac{a(s,\theta)-a(s,\theta+\varepsilon)}{\varepsilon}\sin(\varphi-s)\,ds}{\sin\varphi}. \tag{48}$$

It remains to show the regularity properties stated in the second item of the theorem. These follow rather immediately. Whenever $a,\partial_\theta a\in C^0_{\mathrm{per}}(S)$ one finds from (48) that

$$h(\varphi,\theta)=-\frac{\int_0^{\varphi}a_\theta(s,\theta)\sin(\varphi-s)\,ds}{\sin\varphi}$$

as in (14). With h satisfying (14) one finds for $a,\partial_\theta a\in C^1_{\mathrm{per}}(S)$ that also (20) is satisfied. ■

A Lipschitz-continuity on the sphere

By definition a function $\tilde{\boldsymbol{Z}}:\mathbb{S}^2\to\mathbb{R}^3$ is Lipschitz-continuous, if there exists $\tilde{L}>0$ such that

$$\left|\tilde{\boldsymbol{Z}}(\omega)-\tilde{\boldsymbol{Z}}(\omega_0)\right|\leq\tilde{L}\,|\omega-\omega_0|\ \text{ for all }\omega,\omega_0\in\mathbb{S}^2. \tag{49}$$

With the parametrisation $\omega=\boldsymbol{V}(\varphi,\theta)$ as in (2) and $(\varphi,\theta)\in S$ we want to reformulate the Lipschitz-continuity in (49) for $\boldsymbol{Z}:S\to\mathbb{R}^3$ defined by $\boldsymbol{Z}(\varphi,\theta)=\tilde{\boldsymbol{Z}}(\omega)$. The formulation using (φ,θ) instead of ω is somewhat elaborate.

Lemma 10 *Setting* $\omega = \boldsymbol{V}(\varphi,\theta)$ *and* $\omega_0 = \boldsymbol{V}(\varphi_0,\theta_0)$*, one finds that:*

- *for* $\varphi,\varphi_0 \in [0,\pi]$ *or* $\varphi,\varphi_0 \in [\pi,2\pi]$*:*
$$|\omega-\omega_0| \le |\varphi-\varphi_0| + |\theta-\theta_0| \min\left(|\sin\varphi|, |\sin\varphi_0|\right) \le \pi\,|\omega-\omega_0|\,; \tag{50}$$
- *for* $\varphi \in [0,\pi]$ *and* $\varphi_0 \in [\pi,2\pi]$*, or vice versa:*
$$|\omega-\omega_0| \le |2\pi-\varphi-\varphi_0| + (\pi-|\theta-\theta_0|)\min\left(|\sin\varphi|, |\sin\varphi_0|\right) \le \pi\,|\omega-\omega_0|\,. \tag{51}$$

Proof. Assuming $\varphi,\varphi_0 \in [0,\pi]$ or $\varphi,\varphi_0 \in [\pi,2\pi]$ one considers as an intermediate point $\omega_* = \boldsymbol{V}(\varphi_0,\theta)$ and uses the following estimates:

- The triangle inequality in $\mathbb{R}^3$: $|\omega-\omega_0| \le |\omega-\omega_*| + |\omega_*-\omega_0|$.
- Comparing the length via the circle with fixed φ_0 on the sphere through the points ω and ω_* with the straight line in $\mathbb{R}^3$ through those points gives:
$$|\omega-\omega_*| \le |\varphi-\varphi_0| \le \tfrac{\pi}{2}\,|\omega-\omega_*|\,.$$
- A direct computation shows that
$$|\omega_*-\omega_0| = 2\,|\sin\varphi_0|\,\left|\sin\left(\tfrac12(\theta-\theta_0)\right)\right|$$
and since $\theta-\theta_0 \in [-\pi,\pi]$ one finds
$$\tfrac{2}{\pi}\,|\theta-\theta_0| \le 2\left|\sin\left(\tfrac12(\theta-\theta_0)\right)\right| \le |\theta-\theta_0|\,,$$
implying
$$|\omega_*-\omega_0| \le |\theta-\theta_0|\,|\sin\varphi_0| \le \tfrac{\pi}{2}\,|\omega_*-\omega_0|\,.$$
- Both ω_* and ω_0 lie on the circle on the unit sphere with fixed φ_0. Since ω_* is the point on that circle that is closest to ω, one obtains
$$|\omega-\omega_*| \le |\omega-\omega_0|\,.$$
A similar argument now for the circle on the unit sphere with fixed θ_0 shows
$$|\omega_*-\omega_0| \le |\omega-\omega_0|\,. \tag{52}$$

Combining these inequalities gives the estimates in (50). By symmetry we may replace $|\sin\varphi_0|$ by $\min\left(|\sin\varphi|, |\sin\varphi_0|\right)$.

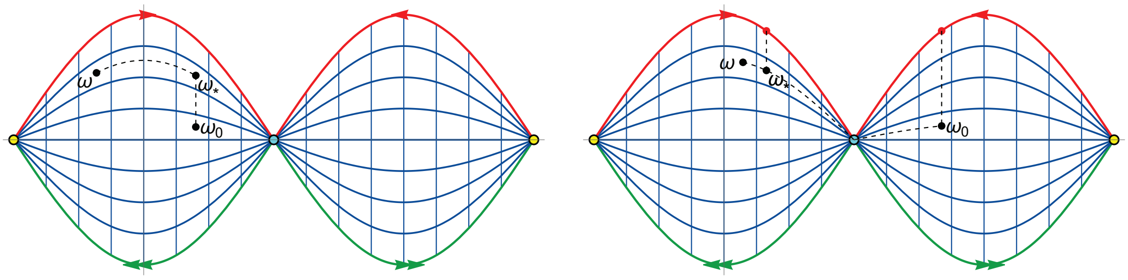

Figure 7: For (50) see left and for (51) see right.

For the second case we assume $\varphi \in [0,\pi]$, $\varphi_0 \in [\pi,2\pi]$ as in Fig. 7 on the right. We consider the shortest path from ω to ω_0 through $\omega_* = V(2\pi-\varphi_0,\theta)$ and the top or bottom boundary of S. Obviously $|\omega-\omega_0| \le |\omega-\omega_*| + |\omega_*-\omega_0|$ still holds. As before one finds
$$|\omega-\omega_*| \le |2\pi-\varphi_0-\varphi| \le \tfrac{\pi}{2}\,|\omega-\omega_*|$$
and since
$$|\omega_*-\omega_0| = 2\,|\sin\varphi_0|\sin\left(\frac{\pi-|\theta-\theta_0|}{2}\right)$$
with $0 \le \pi-|\theta-\theta_0| \le \pi$ one obtains
$$|\omega_*-\omega_0| \le (\pi-|\theta-\theta_0|)\,|\sin\varphi_0| \le \tfrac{\pi}{2}\,|\omega_*-\omega_0|\,.$$
Also as before we have
$$|\omega-\omega_*| \le |\omega-\omega_0|$$
but the last inequality (52) holds if $|\sin\varphi_0| \le |\sin\varphi|$. So in (51) the minimum term is necessary if one wants to keep the same constant. See Fig. 7. ∎

B Shadow domains

Definition 11 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected domain with $0 \in \Omega$. We define $R_\Omega : \mathbb{R} \to \mathbb{R}^+$ by*

$$R_\Omega(\psi) := \sup\left\{ x\cos\psi + y\sin\psi;\ \begin{pmatrix} x \\ y \end{pmatrix} \in \Omega \right\} \tag{53}$$

and define the shadow domain of Ω by

$$Sh(\Omega) := \left\{ \begin{pmatrix} r\cos\psi \\ r\sin\psi \end{pmatrix}; 0 \le r < R_\Omega(\psi) \text{ and } \psi \in [0, 2\pi] \right\}.$$

Remark 11.1 *The intersection of $Sh(\Omega)$ with the line*

$$\ell(\psi) := \left\{ t\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}; t \in \mathbb{R} \right\}$$

gives precisely the shadow of Ω with the light at infinity in the direction $\left(\begin{smallmatrix} -\sin\psi \\ \cos\psi \end{smallmatrix}\right)$. See Fig. 8 in the case of a triangle.

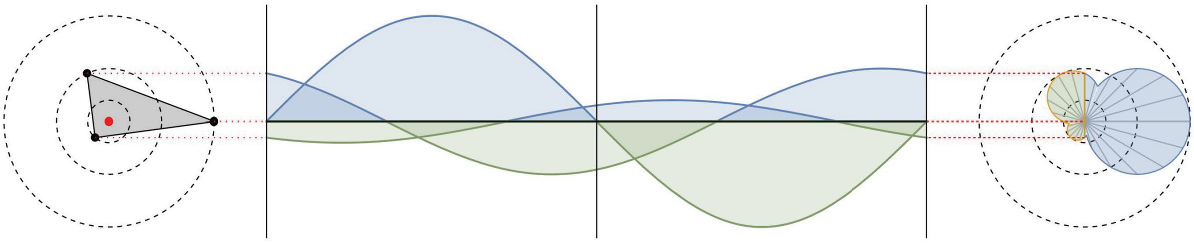

Figure 8: On the left a triangle, $\psi \mapsto R_\Omega(\psi)$ in the middle as the maximum of the three functions and on the right the shadow domain of the triangle

Lemma 12 *Let Ω be as in Definition 11. The function R_Ω in (53) is Lipschitz-continuous with Lipschitz-constant at most*

$$L = \sup\{\|x\|;\ x \in \Omega\}. \tag{54}$$

Proof. Let $co(\Omega)$ denote the convex hull of Ω. It holds that $R_{co(\Omega)}(\psi) = R_\Omega(\psi)$. Note that taking the convex hull also does not change L. Hence we may assume without loss of generality that Ω is convex. The boundary of a bounded convex domain in $\mathbb{R}^2$ with $0 \in \Omega$ can be parametrized in polar coordinates with $r(t) > 0$ as follows:

$$\partial\Omega = \left\{ r(t)\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}; t \in [0, 2\pi] \right\}.$$

For such a parametrization one finds

$$\begin{aligned} R_\Omega(\psi) &= \sup\{r(t)\cos t\cos\psi + r(t)\sin t\sin\psi; t \in [0, 2\pi]\} \\ &= \sup\{r(t)\cos(\psi - t); t \in [0, 2\pi]\}. \end{aligned} \tag{55}$$

The function $\psi \mapsto r(t)\cos(\psi - t)$ is Lipschitz-continuous with constant $\|r\|_\infty = L$ as in (54). A function defined as the supremum of Lipschitz-functions with a uniform constant is Lipschitz-continuous with that same constant. ∎

Remark 12.1 *Notice that (55) leads to*

$$R_\Omega(\psi) = \sup\left\{ r(\psi - s)\cos(s); |s| < \tfrac{1}{2}\pi \right\},$$

which again explains, why we call $Sh(\Omega)$ the shadow domain.

Next we extend this shadow in 2 dimensions to $3d$-shadows of a bounded convex domain $\Omega \subset \mathbb{R}^3$. With the basis $\{\boldsymbol{\Xi}, \boldsymbol{\Theta}(\theta), \boldsymbol{\Psi}(\theta)\}$ as in (25) we define $P_{\boldsymbol{\Psi}(\theta)} : \mathbb{R}^3 \to \mathbb{R}^3$, consistent with (5), by

$$P_{\boldsymbol{\Psi}(\theta)}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \langle \boldsymbol{\Xi}, x \rangle \boldsymbol{\Xi} + \langle \boldsymbol{\Theta}(\theta), x \rangle \boldsymbol{\Theta}(\theta) = \begin{pmatrix} x_1 \\ (\cos\theta\ x_2 + \sin\theta\ x_3)\cos\theta \\ (\cos\theta\ x_2 + \sin\theta\ x_3)\sin\theta \end{pmatrix}.$$

Definition 13 *Supposing that $\Omega \subset \mathbb{R}^3$ is convex, bounded and such that*

- *the domain lies in the Ξ-direction between -1 and 1:*

$$-1 = \inf \{\langle \Xi, x \rangle ; x \in \Omega\} \text{ and } \sup \{\langle \Xi, x \rangle ; x \in \Omega\} = 1,$$

- *with Ξ, $-\Xi \in \partial\Omega$,*

then we define the 3d-shadow domain in the directions perpendicular to the Ξ-axis by

$$Sh_{\Xi}(\Omega) := \bigcup \left\{ P_{\Psi(\theta)}(\Omega); |\theta| \leq \tfrac{1}{2}\pi \right\}.$$

One may notice that this $3d$-shadow domain is related to the $2d$-shadows for fixed x_1 through the formula

$$Sh_{\Xi}(\Omega) = \bigcup_{|x_1|<1} \left(\begin{array}{c} x_1 \\ Sh\left(\hat{P}_{\Xi}\left(\Omega \cap x_1\Xi\right)\right) \end{array} \right), \tag{56}$$

with $\hat{P}_{\Xi}$ as in (6).

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