On a formula for sets of constant width in 2d

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May 28, 2018

Abstract

A formula for smooth orbiforms originating from Euler can be adjusted to describe all sets of constant width in 2d. Moreover, the formula allows short proofs of some laborious approximation results for sets of constant width.

1 Introduction

Around 1774 Leonhard Euler considered curves, which he called 'curva orbiformis'. A formula describing such curves can be found in §10 of [9]. In a series of papers [10], [11], [12] in the 1950's Hammer and Sobczyk recalled the formula to construct all 'curves of constant breadth' in the plane. They started from a characterization of 'outwardly simple line families' $F = \bigcup_\varphi F_\varphi$, with

$$F_\varphi = \left\{ \boldsymbol{y} \in \mathbb{R}^2;\ \boldsymbol{y} \cdot \binom{\cos\varphi}{\sin\varphi} = q(\varphi) \right\} \text{ for all } \binom{\cos\varphi}{\sin\varphi} \in \mathbb{S}^1.$$

Since the orientation of a line should play no role, the Lipschitz-continuous function q satisfies

$$q(\varphi + \pi) = -q(\varphi) \text{ for all } \varphi. \tag{1}$$

They use q to define a curve $\boldsymbol{x} : [0, 2\pi] \to \mathbb{R}^2$ of 'constant breadth' by

$$\boldsymbol{x}(\varphi) = \left(c - \int_0^\varphi q(s)\, ds \right) \binom{\cos\varphi}{\sin\varphi} - q(\varphi) \binom{-\sin\varphi}{\cos\varphi}. \tag{2}$$

Indeed, for c large enough in (2), that is,

$$c \geq \sup_\varphi \left\{ Dq(\varphi) + \int_0^\varphi q(s)\, ds \right\}, \tag{3}$$

with $Dq(\varphi) = \limsup_{h\to 0} \frac{q(\varphi+h)-q(\varphi)}{h}$, the function $\boldsymbol{x}$ describes a 'curve of constant breadth' and, moreover, each curve of constant breadth can be obtained this way. The detour through 'outwardly simple line families' and the result spread over three papers does not make it simple for the reader.

Over the years mathematicians have been intrigued by sets of constant width and we would like to mention some of the papers of a more general nature: [18, 19], [3], [7], [8], [16] and [15]. Also several books dealt with the subject [14], [13], [6]. Relatively elementary sets of constant width in two dimensions that can be constructed geometricallly, are the so called Reuleaux polygons. These sets are defined by having boundaries consisting of finitely many circular arcs. See the two examples in Fig. 1. The obvious question is, whether one can approximate any set of constant width by a set of constant width that is a Reuleaux polygon. In 1915 Blaschke [4] proved such a result in two dimensions. In 1951 Jaglom and Boltjanski published in Russian the result that all arcs in the approximating Reuleaux polygon can be choosen to have the same curvature. Translations appeared in 1956 in German [14] and in 1961 in English. Their proof is

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purely geometrical and fills several pages. Also Meissner, who was interested in the applications of sets of constant width, focused on geometry in [18, 19] around 1909. The approximations of Tanno in [20] and Wegner in [21] went in the other direction, showing that any set of constant width in two dimensions can be approximated by such sets having a smooth and even analytical boundary.

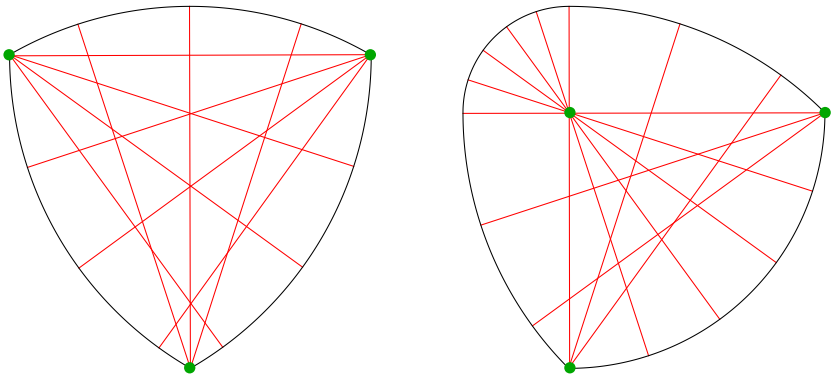

Figure 1: Two sets of constant width with a boundary of circular arcs. On the left the Reuleaux triangle. The segments connect boundary points with opposite normal directions for which the constant width is attained. The points denote the centers of rotation.

In 2 dimensions there are both the analytical approach originating from Euler as well as the geometrical approach; for example from Meissner. We want to unify these approaches and we will do so by modifying the formula in (2) to one that displays some geometrical quantities. The modified formula is not entirely new. The rudimentary idea seems to be given by Barbier [2] in 1860, and Blaschke [4] sort of mentions it in footnote ** about an alternative way to describe the area-minimisation problem for sets of constant width, but he does not give a derivation or reference.

In Section 3 we will recall the modified formula. Section 4 is devoted to the converse, namely that any set of constant width is described by this formula. Moreover, the formula will be more convenient to give short proofs of the approximations we have just mentioned. This can be found in Section 5.

Let us start by recalling what is today called a set of constant width in $\mathbb{R}^n$.

Definition 1 *A closed, bounded and convex set $G \subset \mathbb{R}^n$ is called 'a set of constant width' $L \in \mathbb{R}^+$ if the following holds for each $x \in \partial G$:*

1. *for all $y \in G$ one has $\|x - y\| \leq L$;*
2. *there exists $y \in \partial G$ with $\|x - y\| = L$.*

An equivalent way of defining a 'set of constant width' G is to say that the width of G in direction $\omega \in \mathbb{S}^{n-1}$, given by

$$d_G(\omega) = \max\{x \cdot \omega; x \in G\} - \min\{x \cdot \omega; x \in G\},$$

does not depend on ω. Here $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \text{ with } |\omega| = 1\}$. Again this can be reformulated by defining for a closed bounded set G the support function $p_G : \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$p_G(\omega) = \max\{x \cdot \omega; x \in G\}. \tag{4}$$

Note that $d_G(\omega) = p_G(\omega) + p_G(-\omega)$.

Meissner [18] already knew that a closed, bounded and convex set G is a set of constant width L if and only if

$$p_G(\omega) + p_G(-\omega) = L. \tag{5}$$

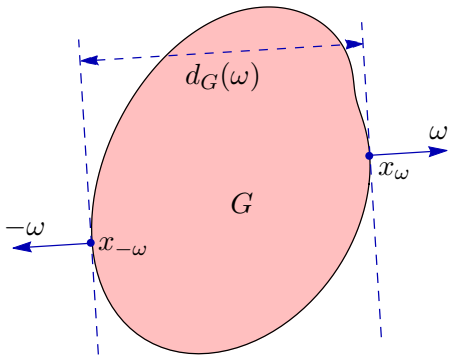

Figure 2: The relation between the directional width and the support function

Since being convex and of constant width implies being strictly convex, a set of constant width G has for each $\omega \in \mathbb{S}^{n-1}$ a unique $x_\omega \in \partial G$ such that

$$p_G(\omega) = x_\omega \cdot \omega \tag{6}$$

and ω is an outer normal direction at x_ω. Note that $x \in \partial G$ maybe reached by several $\omega \in \mathbb{S}^{n-1}$: the set

$$\left\{\omega \in \mathbb{S}^{n-1}; x = x_\omega\right\}$$

is connected, but not necessarily single valued.

2 The construction by Euler

This corresponding formula, recalling (2),

$$\boldsymbol{x}(\varphi) = \left(c - \int_0^\varphi q(s)\, ds\right) \binom{\cos\varphi}{\sin\varphi} - q(\varphi) \binom{-\sin\varphi}{\cos\varphi},$$

indeed yields the following result:

Theorem 2 (Euler, Hammer-Sobczyk) *For any function $q \in C^{0,1}(\mathbb{R})$ satisfying (1) and for any constant c satisfying (3) the function $\boldsymbol{x}$ defined in (2) describes the boundary of a set of constant width in $\mathbb{R}^2$.*

Proof. Before giving the proof we should point out that the width $2r$ of this set can be expressed in terms of q by (15) and (18) below. Notice that (1) implies that q is 2π-periodic and that

$$\int_\varphi^{\varphi+2\pi} q(s)\, ds = 0 \text{ for all } \varphi.$$

So $\boldsymbol{x}$ as in (2) is 2π-periodic and moreover, for $q \in C^1(\mathbb{R})$ one finds that

$$\boldsymbol{x}'(\varphi) = \left(c - q'(\varphi) - \int_0^\varphi q(s)\, ds\right) \binom{-\sin\varphi}{\cos\varphi}. \tag{7}$$

For $q \in C^{0,1}(\mathbb{R})$ we want to use an approximation argument that preserves the conditions in (1) and (3). To do so we consider the function $g \in L^\infty(\mathbb{R})$, defined by

$$g(\varphi) = q'(\varphi) + \int_0^\varphi q(s)\, ds, \tag{8}$$

which is differentiable almost everywhere The function q we recover from $q(0)$ and g through

$$q(\varphi) = q(0)\cos\varphi + \int_0^\varphi \cos(\varphi - s)\, g(s)\, ds. \tag{9}$$

Note that the 2π-periodicity of g shows the 2π-periodicity of q. Moreover, the formula in (9) gives a solution to the integro-differential equation (8). Indeed,

$$q'(\varphi) = -q(0)\sin(\varphi) - \int_0^\varphi \sin(\varphi - s)\, g(s)\, ds + g(\varphi)$$

and

$$\begin{aligned}\int_0^\varphi q(s)\, ds &= q(0)\sin(\varphi) + \int_0^\varphi \int_0^t \cos(t-s)\, g(s)\, ds dt \\ &= q(0)\sin(\varphi) + \int_0^\varphi \int_s^\varphi \cos(t-s)\, dt\ g(s)\, ds \\ &= q(0)\sin(\varphi) + \int_0^\varphi \sin(\varphi - s)\, g(s)\, ds. \end{aligned} \tag{10}$$

Using a mollifier to approximate g by g_n we find $\|g_n - g\|_{L^p} \to 0$ for all p and also $\|g_n\|_\infty \leq \|g\|_\infty$. Through (9) one finds from $\|g_n - g\|_{L^p} \to 0$ that $\|q_n - q\|_\infty \to 0$. For c satisfying (3) it follows from $\|g_n\|_\infty \leq \|g\|_\infty$ that

$$c - q_n'(\varphi) - \int_0^\varphi q_n(s)\, ds \geq 0. \tag{11}$$

Thus, the smooth function q_n inherits the properties of q in (1) and satisfies (3) with the same constant c as for q. Hence we may proceed by assuming that q is smooth. With the property in (11) and the formula in (7) one finds that $\varphi \mapsto \boldsymbol{x}(\varphi)$ rotates counterclockwise. Moreover, the outside normal $\nu(\varphi)$ at $\boldsymbol{x}(\varphi)$, when $|\boldsymbol{x}'(\varphi)| \neq 0$, satisfies $\nu(\varphi) = \binom{\cos\varphi}{\sin\varphi}$. The curvature satisfies

$$\kappa(\varphi) = \frac{x_1'(\varphi)\, x_2''(\varphi) - x_1''(\varphi)\, x_2'(\varphi)}{\left(x_1'(\varphi)^2 + x_2'(\varphi)^2\right)^{3/2}} = \frac{1}{\left|c - q'(\varphi) - \int_0^\varphi q(s)\, ds\right|} \in (0, \infty].$$

Being 2π-periodic, rotating to the left and $\kappa > 0$ imply that $\boldsymbol{x}([0, 2\pi])$ is the boundary of a well-defined strictly convex domain.

Moreover, (2) implies with (1)

$$\begin{aligned}\boldsymbol{x}(\varphi+\pi) - \boldsymbol{x}(\varphi) = &\left(-c + \int_0^{\varphi+\pi} q(s)\, ds - c + \int_0^\varphi q(s)\, ds\right)\tbinom{\cos\varphi}{\sin\varphi} + \\ &+ \ (q(\varphi+\pi) + q(\varphi))\tbinom{-\sin\varphi}{\cos\varphi} = \left(-2c + \int_0^\pi q(s)\, ds\right)\tbinom{\cos\varphi}{\sin\varphi}\end{aligned}$$

and, with p_G as in (6),

$$p_G\tbinom{\cos(\varphi+\pi)}{\sin(\varphi+\pi)} + p_G\tbinom{\cos\varphi}{\sin\varphi} = \left(-\boldsymbol{x}(\varphi+\pi) + \boldsymbol{x}(\varphi)\right)\cdot\tbinom{\cos\varphi}{\sin\varphi} = 2c - \int_0^\pi q(s)\, ds. \tag{12}$$

A convex domain for which the expression (12) is constant is a set of constant width. ∎

3 A modified construction

We will not use (2) directly. Our related formulation was inspired by a video on the educational website [22]. After 3 minutes and 11 seconds this video shows an informative recipe for constructing sets of constant width consisting of circular arcs by rotating sticks of length $2r$ over a variable center located on the stick with the center at a signed distance $a(\varphi)$ from its center. The video uses a piecewise constant function a. It is not mentioned how the values were chosen. But obviously one cannot take arbitrary values if the curve has to be closed. As mentioned before, the formula was known to Blaschke [3]. Even Barbier seems to explain the construction as early as 1860, but only in words [2, Footnote on Page 279].

The video motivated us to ask a) under what condition on a does this construction work, b) if one can generalise the construction to functions $a \in L^\infty$, and c) if all sets of constant width can be constructed in this way. The first two questions are treated in this section and the last one in the next section.

Recipe 3 *For $a \in L^\infty(\mathbb{R})$ with*

$$a(\varphi+\pi) = -a(\varphi) \text{ for all } \varphi, \tag{13}$$

$$\int_0^\pi a(s) \binom{-\sin s}{\cos s}\, ds = \binom{0}{0}, \tag{14}$$

$$r \geq \|a\|_\infty \tag{15}$$

and $\tilde{\boldsymbol{x}}_0 \in \mathbb{R}^2$ set

$$\tilde{\boldsymbol{x}}(\varphi) = \tilde{\boldsymbol{x}}_0 + \int_0^\varphi (r - a(s)) \binom{-\sin s}{\cos s} ds. \tag{16}$$

Blaschke must have known the formulation since only then does the remark in [4, Footnote**) on page 505f] showing conditions (14) and (15) make sense. He, however, gives no reference, nor does he seem to make any further use of it.

For q smooth one may translate (2) into (16), with the assumptions in Recipe 3 satisfied, through

$$\tilde{\boldsymbol{x}}_0 = \binom{c}{-q(0)}, \quad r = c - \tfrac{1}{2}\int_0^\pi q(s)\, ds \quad \text{and} \tag{17}$$

$$a(\varphi) = \tfrac{1}{2}\int_0^\pi q(s)\, ds - q'(\varphi) - \int_0^\varphi q(s)\, ds. \tag{18}$$

So one finds that g from (8) is related to a by $a + g = \frac{1}{2}\int_0^\pi q(s)\, ds$. However, instead of transferring the formula in (16) back to (2) we prefer to give a direct proof that (16) describes the boundary of a set of constant width.

Theorem 4 *With a, r and $\tilde{\boldsymbol{x}}_0$ as in Recipe 3 the function $\tilde{\boldsymbol{x}} : [0, 2\pi] \to \mathbb{R}^2$ in (16) describes a closed curve through $\tilde{\boldsymbol{x}}_0$ and $\tilde{\boldsymbol{x}}_0 - \binom{2r}{0}$, which is the boundary of a set of constant width $2r$.*

Examples relating a and $\tilde{\boldsymbol{x}}_0$ for $r = \|a\|_\infty$ can be found in Fig. 3.

Proof. Setting

$$u(\varphi) := \int_0^\varphi (r - a(s)) \sin(\varphi - s)\, ds = r(1 - \cos\varphi) - \int_0^\varphi a(s) \sin(\varphi - s)\, ds \tag{19}$$

one finds

$$u'(\varphi) = \int_0^\varphi (r - a(s)) \cos(\varphi - s)\, ds = r\sin\varphi - \int_0^\varphi a(s)\cos(\varphi - s)\, ds,$$
$$\text{and} \quad u''(\varphi) = r - a(\varphi) - u(\varphi) \quad \text{a.e.} \tag{20}$$

Since a satisfies (13), one finds that u is periodic with period 2π. With some straightforward calculations it follows that $\tilde{\boldsymbol{x}}$ from (16) can be rewritten to

$$\tilde{\boldsymbol{x}}(\varphi) = \tilde{\boldsymbol{x}}_0 + u(\varphi) \binom{\cos\varphi}{\sin\varphi} + u'(\varphi) \binom{-\sin\varphi}{\cos\varphi}, \tag{21}$$

showing that $\tilde{\boldsymbol{x}}$ is periodic. The derivative of $\tilde{\boldsymbol{x}}$ is defined almost everywhere and then satisfies

$$\tilde{\boldsymbol{x}}'(\varphi) = (r - a(\varphi)) \binom{-\sin\varphi}{\cos\varphi} \quad \text{and} \quad \kappa(\varphi) = \frac{1}{r - a(\varphi)}. \tag{22}$$

Since $r \geq \|a\|_\infty$, it follows from the sign of the integrand in (19) that $u(\varphi) \geq 0$ for $\varphi \in [-\pi, \pi]$ and hence on $\mathbb{R}$ and with (22) that $\tilde{\boldsymbol{x}}$ rotates counterclockwise. Since the curvature satisfies

$$\kappa(\varphi) \in \left[\frac{1}{r + \|a\|}, \infty\right],$$

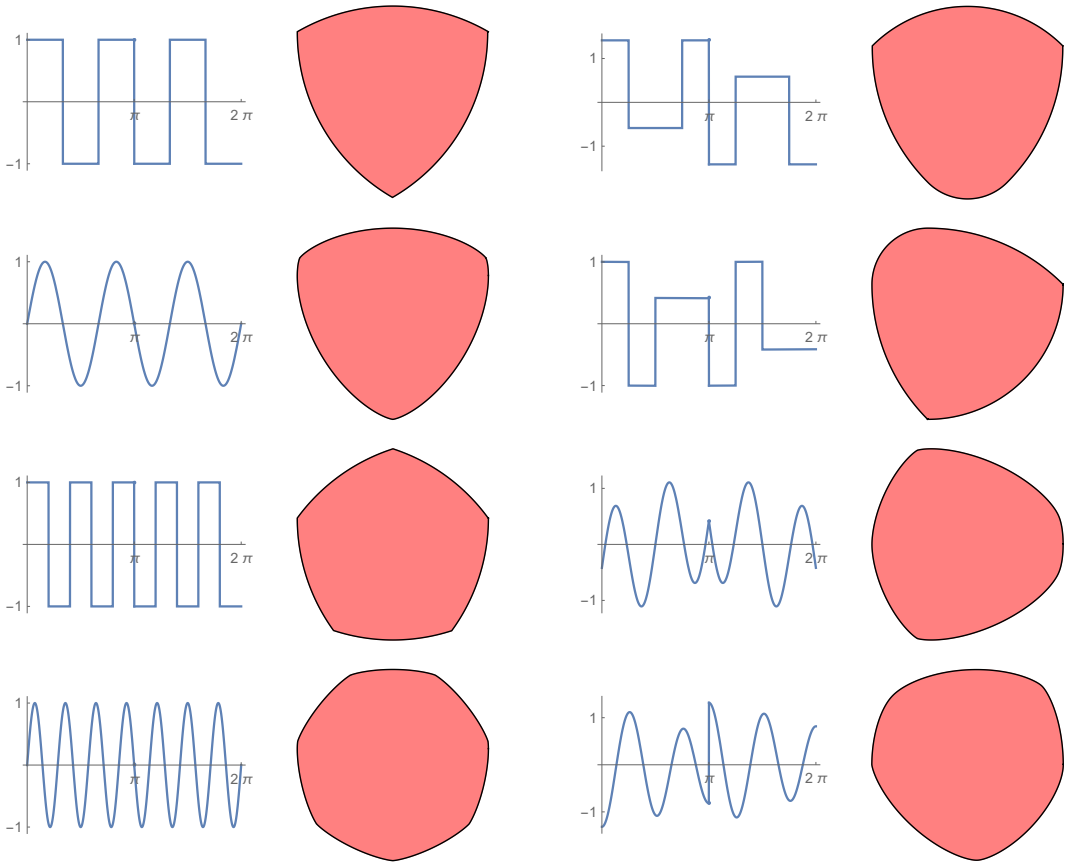

Figure 3: Graphs of the functions a in Recipe 3 with the corresponding sets of constant width for $r = \|a\|_\infty$.

the set $\boldsymbol{x}([0, 2\pi])$ is the boundary of a well-defined strictly convex domain. From (21), (19) and (13), (14) we also find that

$$p\binom{\cos(\varphi+\pi)}{\sin(\varphi+\pi)} + p\binom{\cos\varphi}{\sin\varphi} = \left(-\tilde{\boldsymbol{x}}(\varphi+\pi) + \tilde{\boldsymbol{x}}(\varphi)\right) \cdot \binom{\cos\varphi}{\sin\varphi} =$$
$$= u(\varphi+\pi) + u(\varphi) = r(1-\cos(\varphi+\pi)) + r(1-\cos\varphi) = 2r.$$

By (5) this implies that the enclosed domain is a set of constant width. The conditions $u(0) = u'(0) = 0$ imply $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$. Since $u(\pi) = 2r$ and $u'(\pi) = 0$ imply $\tilde{\boldsymbol{x}}(\pi) = \tilde{\boldsymbol{x}}_0 - \binom{2r}{0}$, the claim follows. ∎

As mentioned above one can see the construction in (16) of Recipe 3 as the rotation of a stick of length $2r$ over a variable center located on the stick with the center of rotation at a distance $a(\varphi)$ from its center. The inequality in (15) takes care that this center of rotation indeed lies inside the stick. In fact this rotation center reaches r as the smallest distance to one of the ends when $r = \|a\|_\infty$. One also finds that the maximal curvature of the set of constant width is $(r - \|a\|_\infty)^{-1}$ for $r > \|a\|_\infty$ and infinity for $r = \|a\|_\infty$. The minimal curvature is $(r + \|a\|_\infty)^{-1}$. The conditions in (13) and (14) make sure that ends meet.

If one wraps a set G of constant width L in a layer of width ε, i.e. if

$$G_\varepsilon := \{x + y \ ; \ x \in G, \ y \in B_\varepsilon(0)\}$$

is the Minkowski sum of G with a ball of radius ε, then G_ε has constant width $L + 2\varepsilon$. So one may modify r and mollify corners by adding any positive constant ε to find that $r + \varepsilon$ also gives a set of constant width.

In the case that a is piecewise constant, one finds a set of constant width with a boundary consisting of finitely many circular arcs. This is nicely illustrated in [22].

4 From the set of constant width to the function a

In this section we will show the converse of the result in Theorem 4.

Theorem 5 *Let $G \subset \mathbb{R}^2$ be a set of constant width. Then there exist a, r and $\tilde{\boldsymbol{x}}$ as in Recipe 3 such that $\tilde{\boldsymbol{x}}([0, 2\pi]) = \partial G$. More precisely:*

1. *For $\tilde{\boldsymbol{x}}_0$ the point on ∂G with maximal x_1-coordinate and $\tilde{\boldsymbol{x}}_*$ the point on ∂G with minimal x_1-coordinate, one has $r = \frac{1}{2}\|\tilde{\boldsymbol{x}}_0 - \tilde{\boldsymbol{x}}_*\|$.*
2. *For $\kappa_{\min} \in \mathbb{R}^+$ the minimal curvature of ∂G and $\kappa_{\max} \in (0, \infty]$ the maximal curvature, one has $\|a\|_\infty = \frac{1}{2}\left(\kappa_{\min}^{-1} - \kappa_{\max}^{-1}\right)$ and $r \geq \frac{1}{2}\left(\kappa_{\min}^{-1} + \kappa_{\max}^{-1}\right)$.*

Remark 5.1 *The Reuleaux triangle in Fig. 1 has $r = 1$ and*

$$a(t) := \begin{cases} -1 & \text{for } t \in \left(0, \frac{1}{3}\pi\right), \\ 1 & \text{for } t \in \left(\frac{1}{3}\pi, \frac{2}{3}\pi\right), \\ -1 & \text{for } t \in \left(\frac{2}{3}\pi, \pi\right). \end{cases} \tag{23}$$

One finds $\kappa_{\min}^{-1} = 2$ and $\kappa_{\max}^{-1} = 0$.

Proof. Fix $\tilde{\boldsymbol{x}}_0$, $\tilde{\boldsymbol{x}}_*$ and r as stated in the theorem. Since G is a set of constant width, for each $\omega = \binom{\cos\varphi}{\sin\varphi} \in \mathbb{S}^1$ there is a unique point $x_\omega \in \partial G$ such that

$$x \cdot \omega < x_\omega \cdot \omega \text{ for all } x \in \partial G \setminus \{x_\omega\}.$$

Instead of G we consider $G_\varepsilon = \bigcup \left\{\overline{B_\varepsilon(x)}; x \in G\right\}$, which is again a set of constant width $2r+2\varepsilon$. Moreover, if x_ω^ε denotes the point with

$$x \cdot \omega < x_\omega^\varepsilon \cdot \omega \text{ for all } x \in \partial G_\varepsilon \setminus \{x_\omega^\varepsilon\},$$

we find $x_\omega^\varepsilon = x_\omega + \varepsilon\omega$. Since G_ε is convex, Alexandrov's Theorem implies that the curvature of ∂G_ε is defined almost everywhere. Moreover, since each $x_\omega^\varepsilon \in \partial G_\varepsilon$ lies on the boundary of $B_\varepsilon(x_\omega)$ and some $B_{2r+\varepsilon}(x_{-\omega})$ with

$$\overline{B_\varepsilon(x_\omega)} \subset G_\varepsilon \subset \overline{B_{2r+\varepsilon}(x_{-\omega})}, \tag{24}$$

the curvature $\kappa(x_\omega^\varepsilon)$ of ∂G_ε at x_ω^ε lies in the interval $\left[(2r+\varepsilon)^{-1}, \varepsilon^{-1}\right]$. See Fig 4.

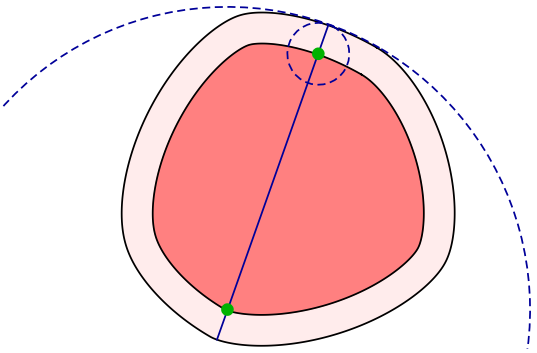

Figure 4: G and G_ε with the auxiliary circles

Letting $\omega_0 = \binom{\cos\varphi_0}{\sin\varphi_0} \in \mathbb{S}^1$ be some direction and $x_{\omega_0}^\varepsilon$ be the corresponding boundary point, one may parameterise the boundary locally by

$$t \mapsto X_\varepsilon(t) := x_{\omega_0}^\varepsilon + t\binom{-\sin\varphi_0}{\cos\varphi_0} - y(t)\binom{\cos\varphi_0}{\sin\varphi_0},$$

where y is a $C^{1,1}$-function with $y(0) = y'(0) = 0$ and y'' defined almost everywhere. One finds for the outside normal and curvature:

$$n(X_\varepsilon(t)) = \frac{1}{\sqrt{1+y'(t)^2}} \left(\begin{pmatrix} \cos\varphi_0 \\ \sin\varphi_0 \end{pmatrix} + y'(t) \begin{pmatrix} -\sin\varphi_0 \\ \cos\varphi_0 \end{pmatrix} \right), \tag{25}$$

$$\kappa(X_\varepsilon(t)) = \frac{1}{\left(1+y'(t)^2\right)^{3/2}} y''(t) \in \left[(2r+\varepsilon)^{-1}, \varepsilon^{-1} \right] \text{ a.e.} \tag{26}$$

Then the angle φ is parameterised by

$$\varphi = \Phi_\varepsilon(t) := \varphi_0 + \arcsin\left(\frac{y'(t)}{\sqrt{1+y'(t)^2}} \right)$$

and $\Phi_\varepsilon \in C^{0,1}$ is differentiable almost everywhere with

$$\Phi_\varepsilon'(t) = \frac{y''(t)}{1+y'(t)^2} \text{ a.e.},$$

which is strictly positive and bounded for t small by (26). So locally $\Phi_\varepsilon^{\text{inv}}$ exists, lies in $C^{0,1}$, is differentiable almost everywhere and locally near φ_0 with a strictly positive and bounded derivative where it is defined. By choosing several φ_0 we may parameterise ∂G_ε by a function $\varphi \mapsto \boldsymbol{x}_\varepsilon(\varphi) : [0, 2\pi] \to \partial G_\varepsilon$ that lies in $C^{0,1}$ and is diffferentiable almost everywhere.

The construction of $\boldsymbol{x}_\varepsilon$ is such that

$$(\boldsymbol{x}_\varepsilon(\varphi) - \boldsymbol{x}_\varepsilon(\varphi+\pi)) \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = |\boldsymbol{x}_\varepsilon(\varphi) - \boldsymbol{x}_\varepsilon(\varphi+\pi)| = 2r + 2\varepsilon, \tag{27}$$

$$(\boldsymbol{x}_\varepsilon(\varphi) - \boldsymbol{x}_\varepsilon(\varphi+\pi)) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = 0 \quad \text{and} \tag{28}$$

$$\boldsymbol{x}_\varepsilon'(\varphi) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = |\boldsymbol{x}_\varepsilon'(\varphi)| \text{ a.e.} \tag{29}$$

By differentiating (28) and using (27) we find that

$$(\boldsymbol{x}_\varepsilon'(\varphi) - \boldsymbol{x}_\varepsilon'(\varphi+\pi)) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = 2r + 2\varepsilon \text{ a.e.}$$

and hence by (29)

$$|\boldsymbol{x}_\varepsilon'(\varphi+\pi)| + |\boldsymbol{x}_\varepsilon'(\varphi)| = (\boldsymbol{x}_\varepsilon'(\varphi) - \boldsymbol{x}_\varepsilon'(\varphi+\pi)) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = 2r + 2\varepsilon \text{ a.e.}$$

Since $\boldsymbol{x}_\varepsilon(\Phi_\varepsilon(t)) = X_\varepsilon(t)$, we find for $\varphi = \Phi_\varepsilon(t)$ that

$$|\boldsymbol{x}_\varepsilon'(\varphi)| = \frac{|X_\varepsilon'(t)|}{|\Phi_\varepsilon'(t)|} = \frac{\sqrt{1+y'(t)^2}}{\frac{y''(t)}{1+y'(t)^2}} = \kappa(\boldsymbol{x}_\varepsilon(\varphi))^{-1} \in [\varepsilon, 2r+\varepsilon] \text{ a.e.}$$

Hence

$$\kappa(\boldsymbol{x}_\varepsilon(\varphi))^{-1} + \kappa(\boldsymbol{x}_\varepsilon(\varphi+\pi))^{-1} = 2r + 2\varepsilon \text{ a.e.}$$

The function $a \in L^\infty(0, 2\pi)$ is then well-defined by:

$$a(\varphi) = \frac{1}{2}\left(|\boldsymbol{x}_\varepsilon'(\varphi+\pi)| - |\boldsymbol{x}_\varepsilon'(\varphi)|\right) = \frac{1}{2}\left(\frac{1}{\kappa(\boldsymbol{x}_\varepsilon(\varphi+\pi))} - \frac{1}{\kappa(\boldsymbol{x}_\varepsilon(\varphi))} \right)$$

and satisfies

$$\|a\|_\infty \leq \frac{1}{2}(2r + \varepsilon - \varepsilon) = r.$$

This definition implies directly that $a(\varphi+\pi) = -a(\varphi)$ and, moreover, that

$$\begin{aligned} r + \varepsilon - a(\varphi) &= r + \varepsilon - \frac{1}{2}\left(|\boldsymbol{x}_\varepsilon'(\varphi+\pi)| - |\boldsymbol{x}_\varepsilon'(\varphi)|\right) \\ &= r + \varepsilon - \frac{1}{2}\left(2r + 2\varepsilon - 2|\boldsymbol{x}_\varepsilon'(\varphi)|\right) = |\boldsymbol{x}_\varepsilon'(\varphi)| \end{aligned}$$

and hence

$$(r + \varepsilon - a(\varphi)) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = |\boldsymbol{x}_\varepsilon'(\varphi)| \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = \boldsymbol{x}_\varepsilon'(\varphi),$$

which in turn implies that

$$\boldsymbol{x}_\varepsilon(\varphi) = \boldsymbol{x}_\varepsilon(0) + \int_0^s (r+\varepsilon-a(s)) \binom{-\sin s}{\cos s} ds.$$

Since

$$\begin{aligned}\boldsymbol{x}_\varepsilon(0) - \binom{2r+2\varepsilon}{0} = \boldsymbol{x}_\varepsilon(\pi) &= \boldsymbol{x}_\varepsilon(0) + \int_0^\pi (r+\varepsilon-a(s)) \binom{-\sin s}{\cos s} ds \\ &= \boldsymbol{x}_\varepsilon(0) - \binom{2r+2\varepsilon}{0} - \int_0^\pi a(s) \binom{-\sin s}{\cos s} ds,\end{aligned}$$

the conditions in (14) hold as well. ∎

5 Applications

By the formula in (16) one may find rather straightforward approximation results for sets of constant width in 2 dimensions.

The function a that determines the set is in $L^\infty(0,2\pi)$ and can be approximated a) in the L^2-topology by step functions and b) in the weak L^2 topology by functions that oscillate finitely often between just two values $\pm r$. Moreover, it can be approximated in L^2 c) by functions of class C^∞ using convolution with Friedrichs mollifiers ζ_ε, or d) by analytic functions using partial sums of Fourier series. Any approximation of a in L^2 or weak L^2 implies an approximation of the underlying set defined by (16) in the Hausdorff-metric. In the following list of corollaries we approximate a along the lines of a) through d), but we must also make sure that our approximations satisfy (13),(14) and (15).

Blaschke proved in 1915 [4] that two-dimensional sets of constant width $2r$ can be approximated in C^0 by Reuleaux polygons of constant width $2r+2\varepsilon$, that is, by domains having a boundary consisting of circular arcs. Using (16) we can replace the geometric proof by an analytic one.

Corollary 6 *Let $\gamma \in \left(0, \frac{1}{2}\right)$. Any two-dimensional set of constant width can be approximated in $C^{0,\gamma}$-sense by a Reuleaux-polygon of constant width.*

Remark 6.1 *Let G_1 and G_2 be two bounded domains, $k \in \mathbb{N}$ and $\gamma \in [0,1]$. We say ∂G_1 and ∂G_2 are ε-close in X-sense, when there exist parameterisations $\gamma_1 : M \to \partial G_1$ and $\gamma_2 : M \to \partial G_2$ with $\|\gamma_1 - \gamma_2\|_X \leq \varepsilon$.*

Proof. Continuous functions are dense in $L^2(0,\pi)$. Choose $a_\varepsilon \in C[0,\pi]$ such that

$$\|a - a_\varepsilon\|_{L^2(0,\pi)} < \varepsilon,$$

then, since $\|a\|_\infty \leq r$, also

$$a_\varepsilon^*(x) = \begin{cases} \|a\|_\infty & \text{if } a_\varepsilon(x) > \|a\|_\infty, \\ a_\varepsilon(x) & \text{if } |a_\varepsilon(x)| \leq \|a\|_\infty, \\ -\|a\|_\infty & \text{if } a_\varepsilon(x) < -\|a\|_\infty, \end{cases}$$

is continuous, satisfies $\|a - a_\varepsilon\|_{L^2(0,\pi)} < \varepsilon$ and additionally $\|a_\varepsilon^*\|_\infty \leq r$. The function a_ε^* can be approximated uniformly by the step function

$$a_{\varepsilon,n}^*(x) = \sum_{k=0}^{2n-1} a_\varepsilon^*\left(\tfrac{k+1/2}{2n}\pi\right) \mathbf{1}_{\left[\frac{k}{2n}\pi, \frac{k+1}{2n}\pi\right)}(x).$$

Choosing n large enough we find $\left\|a - a_{\varepsilon,n}^*\right\|_{L^2(0,\pi)} < 2\varepsilon$ and also, for cs either cos or sin, that

$$c_{cs} = \int_0^\pi a_{\varepsilon,n}^*(x)\ cs(x)\ dx$$

satisfies

$$|c_{cs}| = \left| \int_0^\pi \left(a^*_{\varepsilon,n}(x) - a(x) \right) cs(x)\, dx \right| \le \left\| a - a^*_{\varepsilon,n} \right\|_{L^2(0,\pi)} \sqrt{\tfrac{1}{2}\pi} < \sqrt{2\pi}\varepsilon.$$

Then

$$\tilde{a}_\varepsilon(x) = a^*_{\varepsilon,n}(x) - \tfrac{1}{2}(c_{\sin} + c_{\cos}) \mathbf{1}_{[0,\frac{1}{2}\pi)}(x) - \tfrac{1}{2}(c_{\sin} - c_{\cos}) \mathbf{1}_{[\frac{1}{2}\pi,\pi)}(x)$$

is a stepfunction with $\|a - \tilde{a}_\varepsilon\|_{L^2(0,\pi)} < \left(2 + \sqrt{2}\pi\right)\varepsilon$ which satisfies (14). Moreover,

$$\|\tilde{a}_\varepsilon\|_\infty \le \left\| a^*_{\varepsilon,n} \right\|_\infty + \sqrt{2\pi}\varepsilon \le \|a\|_\infty + \sqrt{2\pi}\varepsilon.$$

Thus, by changing r to $r + \sqrt{2\pi}\varepsilon$ also (15) is satisfied. With a approximated in L^2-sense one finds that the curve defined in (16) is approximated in H^1-sense which, by Sobolev, is embedded in $C^{0,\gamma}$ for $\gamma < \frac{1}{2}$. ∎

The next result from 1951 is stated and proved by Jaglom and Boltjanskii [14, p. 63 ff] using a geometrical approach. It claims that the approximation by Reulaux polygons can be restricted to those polygons whose circular arcs have identical curvature $1/r$. Their lengthy geometric proof can be shortened using (16).

Corollary 7 *Every set of constant width d in 2 dimensions can be approximated in C^0-sense by sets of constant width d that have a boundary which consists of circular arcs with curvature $1/d$.*

Before proving Corollary 7 we need an auxiliary lemma.

Lemma 8 *Let $0 < \beta - \alpha < \pi$. For each $\rho \in L^\infty(\alpha, \beta)$ with $0 \le \rho \le 1$, there exists an interval $[\gamma, \delta] \subset [\alpha, \beta]$ such that*

$$\int_\alpha^\beta \rho(x)\, e^{ix} dx = \int_\gamma^\delta e^{ix} dx. \tag{30}$$

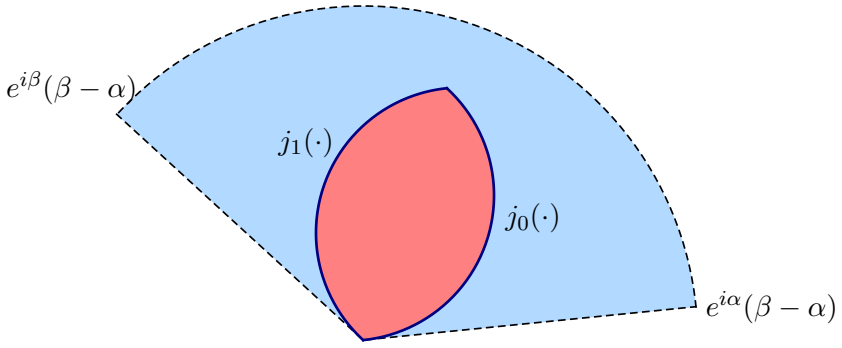

Figure 5: The dark (red) part contains the values $\int_\alpha^\beta \rho(s)\, e^{is} ds$ for all allowed ρ. The bounding curves $j_0(\cdot)$, $j_1(\cdot)$ correspond to the extreme cases in (33) as function of θ: $j_0(\theta) = \int_\alpha^{\alpha+\theta(\beta-\alpha)} e^{is} ds$ and $j_1(\theta) = \int_{\beta-\theta(\beta-\alpha)}^\beta e^{is} ds$.

Proof. Fix $\theta \in (0,1)$ and consider all ρ as in Lemma 8 with average θ, that is,

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \rho(x)\, dx = \theta. \tag{31}$$

For $\beta - \alpha < \pi$ one finds

$$\alpha_0 := \arg\left(\int_\alpha^\beta \rho_0(x)\, e^{ix} dx \right) \le \arg\left(\int_\alpha^\beta \rho(x)\, e^{ix} dx \right) \le \arg\left(\int_\alpha^\beta \rho_1(x)\, e^{ix} dx \right) =: \beta_0, \tag{32}$$

where

$$\rho_0(x) = \begin{cases} 1 & \text{for } x \le \alpha + \theta(\beta-\alpha), \\ 0 & \text{for } x > \alpha + \theta(\beta-\alpha), \end{cases} \quad \text{and } \rho_1(x) = \begin{cases} 0 & \text{for } x \le \beta - \theta(\beta-\alpha), \\ 1 & \text{for } x > \beta - \theta(\beta-\alpha). \end{cases} \tag{33}$$

Indeed Pontryagin's maximum principle [17], helps us to determine the minimum for

$$J := \int_\alpha^\beta \operatorname{Re}\left(e^{i(x-\alpha_0)}\rho(x)\right) dx = \int_\alpha^\beta \cos(x-\alpha_0)\,\rho(x)\,dx$$

among all ρ satisfying (31). We take

$$v'(t) = \rho(t) \text{ with } v(\alpha) = 0 \text{ and } v(\beta) = \theta(\beta-\alpha),$$
$$H(v(t), \rho(t), \lambda(t), t) = \lambda(t)\,\rho(t) + \cos(t-\alpha_0)\,\rho(t).$$

Pontryagin states that for the optimal trajectory (v^*, ρ^*, λ^*) that minimizes H with λ^* piecewise continous, it holds that

$$\lambda^*(t)\,\rho^*(t) + \cos(t-\alpha_0)\,\rho^*(t) \le \lambda^*(t)\,\rho(t) + \cos(t-\alpha_0)\,\rho(t) \text{ for all } \rho(t) \in [0,1].$$

So if $\rho^*(t) = 0$, then $\lambda^*(t) + \cos(t-\alpha_0) \ge 0$. If $\rho^*(t) > 0$ then we get a contradiction, unless $\rho^*(t) = 1$ and $\lambda^*(t) + \cos(t-\alpha_0) \le 0$. Hence $\rho^*(t) \in \{0,1\}$. A similar result shows the right hand side of (32). Through approximations by stepfunctions and rearrangement one finds that the optimal cases are as in (32). For these cases, that is ρ_0 and ρ_1 as in (33), one has

$$\int_\alpha^\beta \rho_0(x)\,e^{ix}dx = \int_\alpha^{\alpha+\theta(\beta-\alpha)} e^{ix}dx \text{ and } \int_\alpha^\beta \rho_1(x)\,e^{ix}dx = \int_{\beta-\theta(\beta-\alpha)}^\beta e^{ix}dx.$$

For all other ρ with $\int_\alpha^\beta \rho(x)\,dx = \theta(\beta-\alpha)$ one finds that $\alpha_\rho := \arg\left(\int_\alpha^\beta \rho(x)\,e^{ix}dx\right)$ lies in (α_0, β_0). For the norm with those ρ one finds

$$\left|\int_\alpha^\beta \rho(x)\,e^{ix}dx\right| \le \left|\int_\alpha^\beta \rho_0(x)\,e^{ix}dx\right|.$$

By rearrangement there is $\rho_* \in [0,1]$ such that

$$\int_\alpha^\beta \rho(x)\,e^{ix}dx = \rho_* \int_{\alpha_\rho - \frac{1}{2}\theta(\beta-\alpha)}^{\alpha_\rho + \frac{1}{2}\theta(\beta-\alpha)} e^{ix}dx$$

for some $\left[\alpha_\rho - \frac{1}{2}\theta(\beta-\alpha), \alpha_\rho + \frac{1}{2}\theta(\beta-\alpha)\right] \subset [\alpha,\beta]$. If $\rho_* = 1$, we are done. If $\rho_* < 1$, there exists $s > 0$ such that

$$\rho_* \int_{\alpha_\rho - \frac{1}{2}\theta(\beta-\alpha)}^{\alpha_\rho + \frac{1}{2}\theta(\beta-\alpha)} e^{ix}dx = \int_{\alpha_\rho - \frac{1}{2}\theta(\beta-\alpha)+s}^{\alpha_\rho + \frac{1}{2}\theta(\beta-\alpha)-s} e^{ix}dx.$$

Since $\theta \in (0,1)$ was arbitrary, the claim follows. ∎

Proof of Corollary 7. We divide the interval $[0,\pi]$ in $2n$ subintervals $\left\{\left[\frac{k}{2n}\pi, \frac{k+1}{2n}\pi\right]\right\}_{k=0}^{2n-1}$. By Lemma 8 there is $[c_k, d_k] \subset \left[\frac{k}{2n}\pi, \frac{k+1}{2n}\pi\right]$ with

$$\int_{\frac{k}{2n}\pi}^{\frac{k+1}{2n}\pi} (r - a(x)) \binom{\cos x}{\sin x} dx = \int_{c_k}^{d_k} 2r \binom{\cos x}{\sin x} dx.$$

So we define

$$a_n(x) = \begin{cases} -r & \text{if } x \in \bigcup_{k=1}^{2n} [c_k, d_k), \\ r & \text{elsewhere in } [0,\pi), \end{cases}$$

and $a_n(x) = -a_n(x-\pi)$ for $x \in [\pi, 2\pi)$. Since

$$\int_{\frac{k}{2n}\pi}^{\frac{k+1}{2n}\pi} a_n(x) \binom{\cos x}{\sin x} dx = \int_{\frac{k}{2n}\pi}^{\frac{k+1}{2n}\pi} a(x) \binom{\cos x}{\sin x} dx$$

holds, the conditions in (13) and (14) are satisfied. The r and $\tilde{\boldsymbol{x}}_0$ remain and since $|a_n(x)| = r$, the curve

$$\tilde{\boldsymbol{x}}_n(\varphi) = \tilde{\boldsymbol{x}}_0 + \int_0^{\varphi} (r - a_n(s)) \binom{-\sin s}{\cos s} ds$$

consists of circular arcs all with the same curvature $2r$. It remains to show the convergence. Since $\tilde{\boldsymbol{x}}_n(\varphi)$ and $\tilde{\boldsymbol{x}}(\varphi)$ coincide for $\varphi = \frac{k}{2n}\pi$, we find for $\varphi \in \left[\frac{k}{2n}\pi, \frac{k+1}{2n}\pi\right]$, using $|a_n(x)|, |a(x)| \leq r$, that

$$|\tilde{\boldsymbol{x}}_n(\varphi) - \tilde{\boldsymbol{x}}(\varphi)| \leq \int_{\frac{k}{2n}\pi}^{\varphi} |4r| \, ds \leq \frac{2\pi r}{n},$$

which shows the convergence in $C[0, 2\pi]$. ∎

Our third result concerns an approximation by smooth sets of constant width. On page 60 of [8] Chakerian and Groemer write that Tanno in [20] in 1976 was the first to prove that arbitrary sets of constant width in 2d can be approximated by sets of constant width with smooth boundaries. Using (16) we can give a direct proof.

Corollary 9 *Let $\gamma \in [0, 1)$. Any set of constant width in 2 dimensions can be approximated in $C^{0,\gamma}$-sense by sets of constant width with C^{∞}-boundary.*

Proof. Suppose that a, r and $\tilde{\boldsymbol{x}}_0$ as in Recipe 3 describes the boundary of the set of constant width G. Let ζ_ε be the usual mollifier:

$$\zeta_\varepsilon(s) = \begin{cases} \frac{1}{c\varepsilon} \exp\left(\frac{\varepsilon^2}{s^2 - \varepsilon^2}\right) & \text{for } |s| < \varepsilon \\ 0 & \text{for } |s| \geq \varepsilon \end{cases} \quad \text{with} \int_{-1}^{1} \zeta_1(s) \, ds = c.$$

We replace a by

$$a_\varepsilon = \zeta_\varepsilon * a - \frac{1}{2} \langle \sin, \zeta_\varepsilon * a \rangle \sin - \frac{1}{2} \langle \cos, \zeta_\varepsilon * a \rangle \cos, \tag{34}$$

where

$$\langle f, g \rangle = \int_0^{\pi} f(x) \, g(x) \, dx,$$

r by $r_\varepsilon = \max(r, \|a_\varepsilon\|_\infty)$ and $\tilde{\boldsymbol{x}}_0$ by $\tilde{\boldsymbol{x}}_0^\varepsilon + \binom{r_\varepsilon - r}{0}$. The construction in (34) implies that (13) and (14) are satisfied. Since $a \in L^\infty(\mathbb{R})$ one finds $a \in L^p(0, 2\pi)$ for all $p < \infty$ and

$$\|\zeta_\varepsilon * a - a\|_{L^p(0,2\pi)} \to 0 \text{ for } \varepsilon \downarrow 0.$$

Hence, also $\|a_\varepsilon - a\|_{L^p(0,2\pi)} \to 0$ for $\varepsilon \downarrow 0$. Moreover $\|\zeta_\varepsilon * a\|_\infty \leq \|a\|_\infty$ and hence with the L^1-convergende of a_ε to a, we also find $\|a_\varepsilon\|_\infty \to \|a\|_\infty$ for $\varepsilon \downarrow 0$. Comparing $\tilde{\boldsymbol{x}}(\varphi) = \tilde{\boldsymbol{x}}_0 + \int_0^{\varphi} (r - a(s)) \binom{-\sin s}{\cos s} ds$ with the ε-perturbed version, one finds by taking p large enough for $1 - \frac{1}{p} > \gamma$ to hold, that by a Sobolev imbedding

$$\begin{aligned} \|\tilde{\boldsymbol{x}}(\varphi) - \tilde{\boldsymbol{x}}_\varepsilon(\varphi)\|_{C^{0,\gamma}[0,2\pi]} &\leq c_{p,\gamma} \|\tilde{\boldsymbol{x}}(\varphi) - \tilde{\boldsymbol{x}}_\varepsilon(\varphi)\|_{W^{1,p}[0,2\pi]} \\ &\leq c_{p,\gamma} \|\tilde{\boldsymbol{x}}_0 - \tilde{\boldsymbol{x}}_0^\varepsilon\|_\infty + c'_{p,\gamma} \|a - a_\varepsilon\|_{L^p(0,2\pi)} \to 0 \text{ for } \varepsilon \downarrow 0, \end{aligned}$$

which completes the proof. ∎

One year after Tanno's approximation by C^∞ functions Weber [21] proved that one may approximate sets of constant width by those that have analytic boundary. Also this result is directly obtained using (16).

Corollary 10 *Let $\gamma \in [0, 1)$. Any set of constant width in 2 dimensions can be approximated in $C^{0,\gamma}$-sense by sets of constant width with analytic boundary.*

Proof. If a_ε is a smooth function approximating a as in Corollary 9, then, by taking a finite Fourier series $a_{N,\varepsilon}(x) := \sum_{k=-N}^{N} \frac{1}{2\pi} \langle e^{ik\cdot}, a_\varepsilon \rangle e^{ikx}$ one finds an analytic approximation of a_ε and hence of a. Note that the coefficients for $k \in \{0, 1\}$ are zero, so $a_{N,\varepsilon}$ satisfies (13) and (14). Moreover, $\|a_{N,\varepsilon} - a_\varepsilon\|_\infty$ can be made arbitrary small by taking N large. We do not get that $\|a_{N,\varepsilon}\|_\infty$ equals $\|a_\varepsilon\|_\infty = \|a\|_\infty$, but through replacing r by $r + \|a_{N,\varepsilon} - a_\varepsilon\|_\infty$ when $r = \|a\|_\infty$, one can still satisfy (15). ∎

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