Uniqueness and symmetry of minimizers of Hartree type equations with external Coulomb potential

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1 Introduction

In a recent paper [5], Georgiev and Venkov establish first radial symmetry and then uniqueness of minimizers to the action functional

$$\mathcal{S}_\omega(u) = \frac{1}{2}||\nabla u||_2^2 + \frac{1}{4}A(|u|^2) - \frac{1}{2}\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|}\, dx + \frac{\omega}{2}||u||_2^2$$

on $H^1(\mathbb{R}^3)$ and for $\omega \in (\frac{1}{16}, \frac{1}{4})$. Here the convolution term

$$A(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v(x)v(y)}{|x-y|}\, dxdy$$

reflects the nonlocal effect of the Coulomb potential. If the convolution term A had the opposite sign, as already remarked in [5], one could use symmetrization results to prove the symmetry of minimizers by variational arguments. This was in fact done in [8]. Instead, Georgiev and Venkov used a variant of the reflection method to prove symmetry, and then they analyzed the Euler equation, which was now an ordinary differential equation in $r = |x|$ to establish uniqueness. For the symmetry proof they had to assume $\omega > 1/16$.

In the present paper we prove first the uniqueness of possible positive minimizers (for any $\omega \in \mathbb{R}$, although positive minimizers can only exist for $0 \leq \omega < 1/4$) by revealing a hidden convexity property of the underlying functional. It turns out, that the "bad" sign in front of the convolution term is in fact "very good" because it has a strict convexity property. Then symmetry follows from the simple observation that uniqueness fails if there is a nonradial minimizer, because it could be rotated and give rise to a second minimizer.

Incidentally, the sign in front of the convolution term prevented also [11] from using symmetrization (as well as reflection) arguments. In Section 5 of [11] they pose the symmetry question for minimizers of

$$E(U) = \frac{1}{2}||\nabla u||_2^2 + A(|u|^2) + \int_{\mathbb{R}^3} F(u(x))\ dx$$

subject to $||u||_2^2 = \lambda > 0$ as an open problem, in particular for $F(u) = -C|u|^{8/3}$. While our arguments show that positive minimizers are symmetric if $F(u)$ is convex in u^2, they do not apply to $F(u) = -C|u|^{8/3}$ with $C > 0$.

2 Main result

Since $S_\omega(u) = S_\omega(|u|)$, if there exists a minimizer, there is also a nonnegative one, and it satisfies the associated nonlinear and nonlocal Euler equation

$$-\Delta u(x) + \omega u(x) + \int_{\mathbb{R}^3} \frac{|u(y)|^2 dy}{|x-y|}\ u(x) = \frac{u(x)}{|x|} \geq 0 \tag{2.1}$$

both in a weak and classical sense, classical except possibly at zero. Hence, by the strong maximum principle, nonnegative solutions are positive everywhere except possibly at zero.

Theorem 2.1. *Almost everywhere positive minimizers of S_ω are unique, because $S_\omega(u) = T_\omega(|u^2|)$ if $u > 0$ a.e., and the functional*

$$T_\omega(v) := \int_{\mathbb{R}^3} \frac{|\nabla v|^2}{4v}\ dx + \frac{1}{4}A(v) + \int_{\mathbb{R}^3} \left(-\frac{1}{2|x|} + \frac{\omega}{2}\right) v\ dx$$

is strictly convex on the convex set $\mathcal{V} := \{v = |u|^2 \mid u \in H^1(\mathbb{R}^3),\ u > 0 \text{ a.e.}\}$.

Remark 2.2. *If the functional is subject to the physically relevant constraint $\|u\|_2^2 = 1$, our argument still gives uniqueness. In terms of v, this constraint amounts to $\int_{\mathbb{R}^3} v\,dx = 1$, which is affine in v.*

For the proof we show in Lemmata 2.3 and 2.5 that all three terms are convex and that at least one of them is strictly convex in v. The last term is linear in v, so it is convex. The convexity of the convolution term, which is a quadratic form in v, has long been known in potential theory, see [7]. Finally, the convexity of the first term and of more general functionals was stated as Proposition 4 in [6], and inspired by [2]. As was kindly pointed out to us by the referee, in the one-dimensional case the convexity of the first term had already been observed in [3].

For the reader's convenience, let us also show that the functional S_ω is well-defined and finite for every $u \in H^1(\mathbb{R}^3)$.

Lemma 2.3. *Let $N = 3$. Then $H^1(\mathbb{R}^N)$ is continuously embedded in $L^{\frac{4N}{N+2}}(\mathbb{R}^N)$, and the functional*

$$A(|u|^2) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-2}}\,dx\,dy \tag{2.2}$$

is well-defined for $u \in H^1(\mathbb{R}^N)$.
Moreover, $A(v)$ is strictly convex on $\{v = |u|^2 \mid u \in H^1(\mathbb{R}^N)\}$.

Proof. For $0 < \alpha < N$ and $x \in \mathbb{R}^N$ let

$$k_\alpha(x) := \frac{1}{C_\alpha}\frac{1}{|x|^{N-\alpha}}, \quad \text{with the constant } C_\alpha := \pi^{\frac{N}{2}} 2^\alpha \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N-\alpha}{2})} > 0,$$

denote the kernel of the Riesz potential, whose Fourier transform is given by $\xi \mapsto |2\pi\xi|^{-\alpha}$ (see [12] or [7], e.g.). For the integrability properties of convolutions with k_α we recall the Hardy-Littlewood-Sobolev inequality for Riesz potentials: For $0 < \alpha < N$, $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, and for $u \in L^p(\mathbb{R}^N)$, we have that

$$\|k_\alpha * u\|_{L^q(\mathbb{R}^N)} \le C\,\|u\|_{L^p(\mathbb{R}^N)}$$

with some constant $C > 0$ (where $*$ denotes convolution). In particular, this implies that $v \cdot (k_2 * v) \in L^1$ and $k_1 * v \in L^2$ if $v \in L^{\frac{2N}{N+2}}$, the former by Hölder's inequality. By Sobolev's embedding theorem, if $u \in H^1(\mathbb{R}^N)$ and $N = 3$, then $u^2 \in L^{\frac{N}{N-2}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) = L^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and a fortiori $|u|^2 \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) = L^{\frac{6}{5}}(\mathbb{R}^3)$. This shows that $A(|u|^2)$ is well defined on $H^1(\mathbb{R}^3)$.

To show convexity, we essentially follow the proof of [7, Theorem 1.15]. Observe that A is a quadratic form and that it suffices to show that $A(v) \ge 0$ for every v, with strict inequality if $v \ne 0$. Since $k_\alpha * k_\beta = k_{\alpha+\beta}$ for arbitrary $\alpha, \beta > 0$ with $\alpha + \beta < N$, we obtain that

$$\begin{aligned} C_2 A(v) &= \int_{\mathbb{R}^N} v \cdot (k_2 * v)\,dx \\ &= \int_{\mathbb{R}^N} v \cdot \big((k_1 * k_1) * v\big)\,dx \\ &= \int_{\mathbb{R}^N} (k_1 * v)^2\,dz \ge 0, \end{aligned}$$

where we also used the fact that k_1 is symmetric with respect to the origin.

Moreover, the last inequality is strict unless $k_1 * v = 0$, and this may happen only if the Fourier transform of $k_1 * v$ vanishes, i.e., $|2\pi\xi|^{-1}\,\hat{v}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^N$. Clearly, this is possible only if $\hat{v} = 0$ and thus $v = 0$. □

In the next lemma we show that the term containing the weight $1/|x|$ is well defined on $H^1(\mathbb{R}^N)$, because the right hand side in (2.3) is dominated by $||u||^2_{H^{1,2}(\mathbb{R}^N)}$.

Lemma 2.4 (cf. Remark 6 in [10])**.** *For $N \geq 2$ and every $u \in H^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx \;\leq\; \frac{2}{N-1} \int_{\mathbb{R}^N} |u(x)\nabla u(x)|\, dx. \tag{2.3}$$

Moreover, equality holds in (2.3) *if and only if u is radially symmetric and $u(x)\nabla u(x)\cdot x \leq 0$ for a.e. $x \in \mathbb{R}^N$.*

Proof. We proceed as in the proof of Hardy's inequality in [4]. Clearly, it suffices to show (2.3) for $u \in C^1(\mathbb{R}^N)$ with compact support. Since

$$\frac{d}{dt}\, u(tx)^2 \;=\; 2u(tx)\nabla u(tx)\cdot x,$$

we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx &\leq \int_{\mathbb{R}^N} \int_1^\infty 2\, |u(tx)\nabla u(tx)|\, dt\, dx \\
&= \int_1^\infty 2t^{-N} \int_{\mathbb{R}^N} |u(x)\nabla u(x)|\, dx\, dt \\
&= \frac{2}{N-1} \int_{\mathbb{R}^N} |u(x)\nabla u(x)|\, dx.
\end{aligned}$$

Moreover, equality holds if and only if $-u(y)\nabla u(y)\cdot y = u(y)\,|\nabla u(y)|\,|y|$ for a.e. $y \in \mathbb{R}^N$. This is the case precisely if u is radially symmetric and $u\frac{\partial u}{\partial r} \leq 0$. ☐

It remains to establish the convexity of the first term in T_ω. This will follow from the following lemma.

Lemma 2.5. *Let $f : \mathbb{R}^N \times (0,\infty) \to [0,\infty)$ be defined by $f(\xi,\mu) := \dfrac{|\xi|^2}{\mu}$. Then f is convex in (ξ,μ).*

Proof. The Hessian of f is given by

$$H := D^2_{(\xi,\mu)} f(\xi,\mu) = \frac{2}{\mu^2} \begin{pmatrix} \mu & & & -\xi_1 \\ & \ddots & & \vdots \\ & & \mu & -\xi_N \\ -\xi_1 & \cdots & -\xi_N & \frac{|\xi|^2}{\mu} \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}.$$

It suffices to check that H is non-negative definite. This is easily done by noticing that $(Hy,y) \geq 0$ for every $y \in \mathbb{R}^{N+1}$ and that it vanishes only for $y = (\xi,\mu)$. ☐

Remark 2.6. *Incidentally, according to [1] "uniqueness of the minimum easily follows from nodal line properties", but this argument appears to be wrong. If there* *are two genuinely different positive minimizers of a quadratic functional, which solve a linear equation, a suitable linear combination will be another minimizer that has nodal sets and changes sign. For nonlinear equations one cannot expect the same behaviour unless one proves a convexity property of the underlying functional. This is what we did in Theorem 2.1.*

We conclude with brief variational arguments for nonexistence of positive solutions. Existence proofs for $\omega \in [0, 1/4)$ can be found in [10] (for $\omega = 0$, instead of $H^1(\mathbb{R}^3)$ a more natural, larger space is used). Let $I_\omega := \inf_{u \in H^1(\mathbb{R}^3)} S_\omega(u)$.

If $\omega < 0$ then $I_\omega = -\infty$. To see this one observes that for fixed $u \in H^1(\mathbb{R}^3)$ and $u_\delta(x) := \delta^{\frac{3}{2}} u(\delta x)$, $\delta > 0$,

$$\lim_{\delta \to 0^+} S_\omega(u_\delta) = \lim_{\delta \to 0^+} \frac{\omega}{2} \|u_\delta\|^2_{L^2(\mathbb{R}^3)} = \frac{\omega}{2} \|u\|^2_{L^2(\mathbb{R}^3)} .$$

On the other hand, if $\omega \geq \frac{1}{4}$, then $u = 0$ is the unique solution, because we then have that $I_\omega = S_\omega(0) = 0$, and $S_\omega(u) > 0$ for $u \neq 0$. This follows from the observation that by Lemma 2.4,

$$S_\omega(u) \geq \frac{1}{2}\left(\omega - \frac{1}{4}\right) ||u||^2_{L^2(\mathbb{R}^3)} + A(|u|^2).$$

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