Variational versus PDE-based Approaches in Mathematical Image Processing

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Abstract

In mathematical image processing we are often presented with amazing examples of image enhancement algorithms. Yet, when applied to different noisy images, they can produce unwanted effects. The analysis of such algorithms lags behind their intuitive development. Two essentially different models have found wide recognition: a variational approach according to Mumford and Shah and an approach via nonlinear diffusion equations. One of these equations is nonparabolic and was suggested by Perona and Malik. In this short survey I point out a connection between these two seemingly unrelated approaches and explain some connections with total variation flow.

Keywords: Perona-Malik equation, Mumford-Shah functional, Blake-Zisserman functional, total variation flow, staircasing

Math. Sci. Classification: 35, 49

1 Introduction

Suppose $u_0 : \Omega \to [0,1]$ represents the grey-scale distribution of a noisy image. **Mumford and Shah** suggested studying

$$J(v) = \alpha \int_\Omega |v - u_0|^2 \, dx + \beta \int_{\Omega \setminus S_v} |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v). \tag{1.1}$$

Here α and β are suitable positive constants which weigh the contributions of the three terms in the functional. A minimizer of this problem, if it exists, should have few sets of discontinuity, a relatively smooth appearance elsewhere and still resemble the original picture. One can replace the quadratic terms in the functional by strictly convex terms such as $|v-u_0|^p$ or $\sqrt{1+|\nabla v|^2}$

without changing essential features of the problem. For reasons of exposition, I will adhere to the simple functional above. Nowadays it is minimized on $SBV(\Omega)$, a space of special functions of bounded variation. One of the analytical problems is a lack of semicontinuity of $\mathcal{H}^{n-1}(S)$ with respect to the Hausdorff topology.

To demonstrate the lack of semicontinuity consider the following sequence S_i of subsets of the unit interval $S_0 = (0, 1)$. The first element of the sequence is $S_1 := (0, 0.5)$, the second $S_2 := (0,\ 0.25) \cup (0.5,\ 0.75)$, $S_3 := (0,\ 0.125) \cup (0.25,\ 0.375) \cup (0.5,\ 0.6125) \cup (0.75,\ 0.875)$ etc.. Each element in this sequence has 1-dimensional Hausdorff measure 0.5, but S_i has a Hausforff-limit S_0, and $\mathcal{H}^1(S_0) = 1$. Incidentally, this "divide and conquer"-algorithm is frequently used by politicians to prove that one can cut costs of public services in half by creating sufficiently small cost units, each of which is just a little short of what it really needs to function. Another application was given in [27], where a circle of radius 1 was apparently surrounded by a fence of total length $1 + \pi$.

Perona and Malik suggested taking the noisy image u_0 as initial datum for a diffusion equation such as

$$u_t - \text{div}\ \left(\frac{\nabla u}{(1 + |\nabla u|^2)^2}\right) = 0, \tag{1.2}$$

or, more generally

$$u_t - \text{div}\ \left(a(|\nabla u|^2)\nabla u\right) = 0, \tag{1.3}$$

with $a(s)$ positive and decreasing to zero as $s \to \infty$, and under no-flux boundary conditions. Small diffusion near discontinuities in u_0 was supposed to lead to edge preservation, while large diffusion elsewhere would somehow mollify the brightness function and take out noise.

If $a(s) = s^{-1/2}$, then formally equation (1.3) becomes

$$u_t - \text{div}\ \left(\frac{\nabla u}{|\nabla u|}\right) = 0. \tag{1.4}$$

This is commonly called total variation flow, **TV-flow** for short.

Let me explain why the Perona-Malik equations are called anisotropic. Upon differentiation, (1.3) becomes

$$u_t - a(|\nabla u|^2)\Delta u - 2a'(|\nabla u|^2)\nabla u D^2 u \nabla u = 0, \tag{1.5}$$

with D^2u denoting the Hessian matrix of second derivatives. We want to rewrite these second derivatives in intrinsic coordinates and set $\nu = -\frac{\nabla u}{|\nabla u|}$. Then

$$\Delta u = u_{\nu\nu} + (n-1)Hu_\nu = u_{\nu\nu} + \Delta_{n-1}u,$$

where H is the mean curvature of a level surface of u, or where Δ_{n-1} is the (n-1)-dimensional Laplace operator tangent to this level surface. Therefore (1.5) can be rewritten as

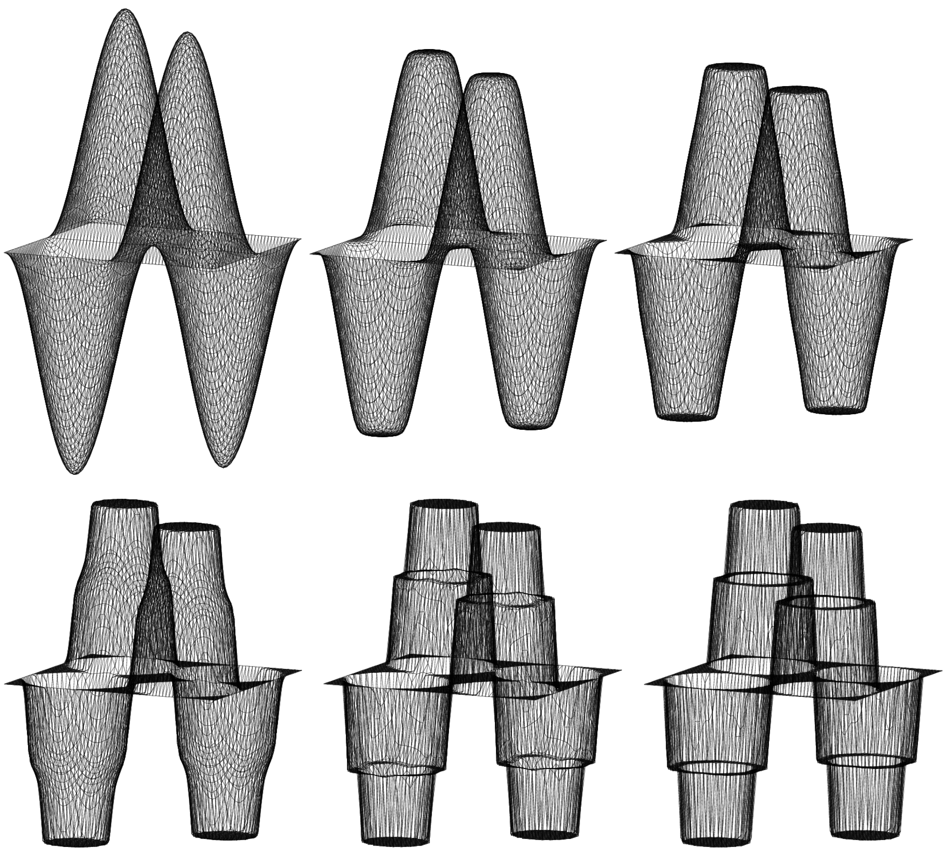
$$u_t - a(|\nabla u|^2)\Delta u - 2a'(|\nabla u|^2)|\nabla u|^2 u_{\nu\nu} = 0,$$

or even as

$$u_t - b(|\nabla u|^2)u_{\nu\nu} - a(|\nabla u|^2)\Delta_{n-1} u = 0.$$

Thus, for $a \neq b$ the diffusion in direction ν differs from the diffusion in the other directions and this is an anisotropic behaviour. Moreover, in many cases $b(s)$ can even become negative for large values of s. Then there is a backward diffusion effect which leads to a steepening of profiles in direction of the gradient of u, but at the same time there is forward diffusion along level surfaces of u. Little is known about such diffusion equations.

The following figure shows the evolution of an academically chosen gray scale $u_0(x, y) = \sin x \sin y$ *under Perona Malik flow. One can see a clear formation of edges, the so-called staircasing effect.*



This lecture is organized as follows. I present the Ambrosio-Tortorelli approximation of the Perona-Malik functional in Section 2. This approximation is amenable to numerical minimization algorithms, which in turn are related to the Perona-Malik equation. While such a formal analogy was also observed in [47], Richardson and Mitterer could not reconcile the significant difference between the Perona-Malik equation and their equation (8) which reads

$$v_t - \beta \text{ div} \left(\eta^{\,2} \nabla v\right) = -\alpha(v - u_0)$$

in my notation, see also (2.3) below. I wish to point out that this problem was only overcome in [29]. In Section 3 I briefly survey some older but not as well-known results from [26] on forward-backward diffusion equations like the Perona-Malik equation. In Section 4 I comment on related but somehow more benign degenerate equations.

2 Mumford Shah

While the Mumford-Shah functional (1.1) is well-defined for piecewise smooth functions u whose sets S_u of discontinuities are again piecewise smooth, it is difficult to evaluate it numerically. Any conceivable space of nonconforming finite elements would need to keep track of these discontinuities in its very definition. And to measure $\mathcal{H}^{n-1}(S_u)$ one would need to put a grid directly on the set S_u.

How can one approximate $\mathcal{H}^{n-1}(S_u)$? Modica & Mortola showed in [38] and [39] that $H_0(u) = \mathcal{H}^{n-1}(u = 0)$ is the Γ-limit of the following family of functionals

$$H_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \tfrac{1}{4\varepsilon}(u^2 - 1)^2 \; dx.$$

This suggests that the Hausdorff measure of a suitable level set can be approximated (in the sense of Γ–convergence) by a sequence of domain integrals.

And in fact, Ambrosio & Tortorelli showed in [4] that the Mumford-Shah functional

$$J_0(v) := \int_\Omega \alpha|v - u_0|^2 \; dx + \beta \int_{\Omega \setminus S_v} |\nabla v|^2 \; dx + \mathcal{H}^{n-1}(S_v) \tag{2.1}$$

is the Γ-limit of

$$J_\varepsilon(v, \eta) := \int_\Omega \alpha|v - u_0|^2 + \beta\eta^2|\nabla v|^2 + \varepsilon|\nabla\eta|^2 + \tfrac{1}{4\varepsilon}(\eta - 1)^2 \; dx, \tag{2.2}$$

where $\{\eta = 0\}$ stands for S_v. Therefore the set where η is close to zero can be used an an edge detector. It indicates where $|\nabla v|$ is large. Notice that in contrast to the Mumford-Shah functional (2.1) the integration of the gradient term extends now over all of Ω and not over $\Omega \setminus S_v$.

For the reader's convenience I recall the definition of Γ-convergence. Let X be a metric space and $F_\varepsilon : X \mapsto [0, \infty]$ a family of mappings. Then F **is the Γ-limit of** F_ε as $\varepsilon \to 0$, iff a) and b) hold.

a) For every $u \in X$ and every sequence $u_\varepsilon \to u$ in X

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq F(u) \ .$$

b) For every $u \in X$ there exists a sequence u_ε such that $u_\varepsilon \to u$ in X and

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = F(u) \ .$$

A convenient result in the theory of Γ–convergence states that minimizers of a Γ–convergent sequence of functionals converge to a minimizer of the limit functional.

Theorem 2.1 *If F is the Γ–limit of F_ε as $\varepsilon \to 0$ and u_ε minimizer of F_ε, then every cluster point u of the family $\{u_\varepsilon\}_{\varepsilon > 0}$ minimizes F.*

This theorem provides a certain amount of confidence that for small ε a minimizer of the Ambrosio-Tortorelli functional (2.2) will be at least close to a minimizer of the original Mumford Shah functional (2.1)..

In fact one can show the following: If $(v_\varepsilon, \eta_\varepsilon)$ minimize J_ε, then (after possibly passing to a subsequence) $v_\varepsilon \to v$ and $\eta_\varepsilon \to 1$ in $L^2(\Omega)$ as $\varepsilon \to 0$, but $|\eta_\epsilon| << 1$ near S_{v_ε}. The associated Euler equations for minimizers of the Ambrosio-Tortorelli functional are obtained by taking first variations with respect to v and η. They are given by

$$\alpha(v_\varepsilon - u_0) - \beta \ \mathrm{div} \left(\eta_\varepsilon^2 \nabla v_\varepsilon\right) = 0 \tag{2.3}$$

and

$$\beta \eta_\varepsilon |\nabla v_\varepsilon|^2 - \varepsilon \Delta \eta_\varepsilon + \frac{1}{4\varepsilon}(\eta_\varepsilon - 1) = 0,$$

or, upon rearranging the last equation, by

$$-\Delta \eta_\varepsilon + \frac{1 + 4\beta\varepsilon|\nabla v_\varepsilon|^2}{4\varepsilon^2} \left(\eta_\varepsilon - \frac{1}{1 + 4\beta\varepsilon|\nabla v_\varepsilon|^2} \right) = 0. \tag{2.4}$$

"Now a miracle occurs ..." Let us pretend for the moment, that the bracket in (2.4) is nonzero as $\varepsilon \to 0$. Then the second term in (2.4) is of order ε^{-2}. This is consistent with standard estimates in singular perturbation theory. However, Vese and Chan [50] set the round bracket equal to zero, i.e. $\eta = \frac{1}{1+4\beta\varepsilon|\nabla v|^2}$

and neglect the term $\varepsilon|\nabla\eta|^2$ in J_ε. In this way they arrive at a modified Ambrosio-Tortorelli functional

$$\tilde{J}_\varepsilon(v) := \int_\Omega \alpha|v-u_0|^2 + \tilde{A}_\varepsilon(|\nabla v|^2)\ dx \tag{2.5}$$

with $\tilde{A}_\varepsilon(|\nabla v|^2)$ given below in (2.6). On a continuous level this is an analytical crime, because one knows for instance that $\int_\Omega \varepsilon|\nabla v_\varepsilon|^2\ dx$ is of order one, $O(1)$ and not $o(1)$. However, since the gray scale function v_ε can only take values in $[0,1]$, upon discretization with grid-size h, its gradient cannot exceed $O(h^-1)$. Therefore, on a fixed grid one may indeed send ε to zero and justify the approximation of J_ε by $\tilde{J}_\varepsilon$ for $\varepsilon << h^2$.

Now I turn my attention to the modified functional (2.5) with

$$\tilde{A}_\varepsilon(s^2) := \frac{s^2}{1+4\beta\varepsilon s^2}\ . \tag{2.6}$$

The Euler equation for this modified functional reads

$$-\text{div}\left(\frac{\nabla v}{1+4\beta\varepsilon|\nabla v|^2}\right) + \frac{\alpha}{4\beta\varepsilon}(v-u_0) = 0\ , \tag{2.7}$$

and this looks formally like the first step in an explicit Euler scheme for the the Perona-Malik equation

$$v_t - \text{div}\left(\frac{\nabla v}{1+4\beta\varepsilon|\nabla v|^2}\right) = 0 \tag{2.8}$$

with initial datum u_0. Alternatively it can be interpreted as a stationary solution of the ROF-model [48, 21] which was apparently first suggested by Nordström [44]:

$$v_t - \text{div}\left(\frac{\nabla v}{1+4\beta\varepsilon|\nabla v|^2}\right) = -\lambda(v-u_0)\ . \tag{2.9}$$

Remark 2.2 It is in this sense that the variational approach and the Perona-Malik appproach are deeply interconnected. The diffusion approach is simply a Ljapunov flow for for the nonconvex functional (2.5) and a solution of the Perona Malik equation will lower the associated functional.

But there is more to learn from the functional (2.5). For nonegative s the map $s \mapsto \tilde{A}_\varepsilon(s^2)$ is a monotone increasing nonconvex function of s which begins convex at zero and approaches a constant $(4\beta\varepsilon)^{-1}$ as $s \to \infty$. Calculus of variations tells us that an infimum of $\tilde{J}_\varepsilon$ must also minimize the convex hull of $\tilde{J}_\varepsilon$, but the convex hull of $\tilde{J}_\varepsilon$ would be just the first term in (2.5).

Again it helps to realize that due to discretization we actually try to infimize $\tilde{J}_\varepsilon$ on those functions in $W^{1,\infty}(\Omega)$ the gradient of which does not exceed $1/h$ in modulus. The convex relaxation B_ε of $\tilde{A}_\varepsilon$ on $[0,\ 1/h]$ is then nontrivial, strictly convex near zero and affine near $1/h$. Consequently a function that realizes the infimum of (2.5) must have the modulus of its gradient either inside the interval $[0, s_1]$ where $\tilde{A}_\varepsilon$ conicides with the B_ε or at $1/h$. Therefore the gradient oscillates on a small scale between a number from $[0, s_1]$ and $1/h$; this effect is known as staircasing, see for example [34].

Remark 2.3 Staircasing is an effect that is caused by a nonconvex underlying functional which the solution tries to minimize.

Finally I wish to point out that there are many other functionals which are being used in image processing. Many of them lead to a qualitatively similar behaviour. It is clear that the quadratic term $\alpha|v - u_0|^2$ can be replaced by a general power $p > 1$ as, for instance in [21], or that the gradient term involving $\tilde{A}_\varepsilon$ term can be replaced $\phi(s^2) = \min\{s^2, \beta\}$. Then we speak of the Blake-Zisserman functional [9]. Its convex relaxation on $[0, 1/h]$ is quadratic for small values of s and affine for s close to $1/h$. Even total variation falls into this class provided we are willing to accept that this functional is strictly convex only on the interval consisting of the single point zero.

Remark 2.4 This observation explains why the behaviour of image processing via variational methods is fairly stable under a change of functionals and why many functionals work reasonably well as soon as one of them does. A word of caution though: tuning of constants like α, β and ε is required to obtain optimal results for a given set of images.

3 Perona Malik

Consider the initial-boundary value problem

$$\begin{aligned} u_t - \operatorname{div}\,(a(|\nabla u|^2)\nabla u) &= 0 \quad \text{in } \Omega \times (0,T), \\ a(|\nabla u|^2)\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{on } \Omega, \end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, and where u_0 represents a blurred image. Our structural assumptions on the coefficient a are that $a \in C^1([0,\infty))$, and $a(s) > 0$, and that the ellipticity function $b(s) := a(s) + 2sa'(s)$ is positive for s near 0 and changes sign exactly once at $s_0^2 > 0$. This includes the model cases: $a(s) = e^{-s}$ or $a(s) = (1+s)^{-1}$ that were studied by Perona and Malik.

As explained in the introduction, one speaks of the corresponding differential equation as anisotropic diffusion. In particular, if $\Omega \subset \mathbb{R}^2$ and if u is a classical solution and $|\nabla u| \neq 0$, one can rewrite the differential equation from (3.1) in intrinsic coordinates $\nu = -\nabla u/|\nabla u|$ and τ as

$$u_t - b(|\nabla u|^2)u_{\nu\nu} - a(|\nabla u|^2)u_{\tau\tau} = 0. \tag{3.2}$$

Recall that the diffusion coefficient in direction τ is different from direction ν and that b changes sign while a remains positive.

What type of differential equation is this? For small values of $|\nabla u|$ it is a regular parabolic diffusion equation with a classical solution, see [30]. The stationary version resembles the classical Tricomi problem, a differential equation that switches type from elliptic to hyperbolic. If the last term is missing, it looks like a forward-backward diffusion equation. The last case occurs if $n = 1$ and several results in [26] deal with this case.

The following questions can be raised:

• Can one describe initial or final data? In general, there is a negative answer [34].

• Can one prove existence? The answer to this question depends on what we understand by a solution and is in general negative. There are not even weak solutions [34]; but yes, there are some generalized solutions [53]; and if $|u_{0_x}| < s_0$ they are even regular [30].

• Can one prove uniqueness? This depends on the regularity of solutions. The answer is negative for $C^{0,1}$-solutions [53] and sometimes positive for C^1 solutions [26] or for C^2-solutions [35].

• Is there a maximum or comparison principle? This was dismissed in [1] because there is none for the backward heat equation. I shall now address these questions for C^1-solutions because for those we can evaluate the (sign of) the diffusion coefficients in every point (x,t) of the space-time cylinder.

Theorem 3.1 (Maximum Principle in $\mathbb{R}^n$) *Suppose that $u \in C^{0,1}$ is a weak solution to the anisotropic diffusion problem. Then*

$$\max_{\overline{\Omega}\times[0,T]} |u(x,t)| = \max_{\overline{\Omega}} |u_0(x)|.$$

Proof: For any $p \in [2,\infty)$ multiply the differential equation (3.1) with $|u|^{p-2}u$

and integrate over Ω. Integration by part gives

$$\frac{d}{dt}\int_\Omega |u(x,t)|^p dx = p\int_\Omega |u|^{p-2}u\ u_t dx \tag{3.3}$$
$$= p\int_\Omega |u|^{p-2}u\ \mathrm{div}(a(|\nabla u|^2)\nabla u)dx$$
$$= -p(p-1)\int_\Omega |u|^{p-2}a(|\nabla u|^2)|\nabla u|^2 \leq 0.$$

Therefore all $L^p(\Omega)$ norms are decreasing in t and after sending $p \to \infty$, so does the $L^\infty(\Omega)$ norm. To conclude the proof I should mention that Theorem 3.1 was also found by Weickert [51] independently from us.

From now on I assume that $n = 1$. Assuming that a C^1 solution exists, we can partition $\Omega| \times \mathbb{R}^+ = (-1,1) \times \mathbb{R}^+$ into forward- and backward regimes, depending on the sign of $u_x^2 - s_0^2$.

Theorem 3.2 (Shrinking of backward regime, preservation of shape) *If u is a weak C^1 solution of (3.1) and if u_0 switches only finitely often from forward to backward intervals or vice versa, then the backward regime of u shrinks in time and does not migrate.*

While a proof of this result follows from a delicate analysis of the level lines $\{u_x = \pm s_0\}$ in [26], there p.119, Theorem 3.2 can be interpreted as follows: In image enhancement edges are characterized as backward regimes. Therefore edges do not move and become enhanced through anisotropic diffusion. This is the desired effect in image enhancement. Moreover, no new edges can appear as time evolves. To give you an idea of how this result is shown, take the differential equation

$$u_t - b(u_x^{)}u_{xx} = 0, \tag{3.4}$$

Differentiate with respect to x

$$u_{xt} - b(u_x^2)u_{xxx} - 2b(u_x^2)u_x u_{xx}^2 = 0 \tag{3.5}$$

and substitute $v =: u_x$

$$v_t - b(v^2)v_{xx} - c(v)v_x^2 = 0. \tag{3.6}$$

Now v satisfies a forward diffusion equation in the forward regime and a backward difusion equation in the backward regime. Should a new backward regime be born in the evolution process, it would, without loss of generality, be a set where $u_x > s_0$ and it would be bounded by a set where $u_x = s_0$. Thus at some initial time in this set v would attain its minimum, a contradiction to the miniimum principle for backward diffusion equations, which states that solutions attain their minimum at the final time.

In [26] N. Kutev and I gave an explicit counterexample to a general comparison principle in which $u_0(x) \leq v_0(x)$, and u_0 and v_0 touch each other in some point with slope $> s_0$. For any small but positive t the corresponding solutions u and v have crossed each other at and near this point, so that $u(t,x) \leq v(t,x)$ fails to hold in Ω. Nevertheless, one can sometimes compare different solutions, and a discrete version of the following theorem from [26] appeared only recently in [20].

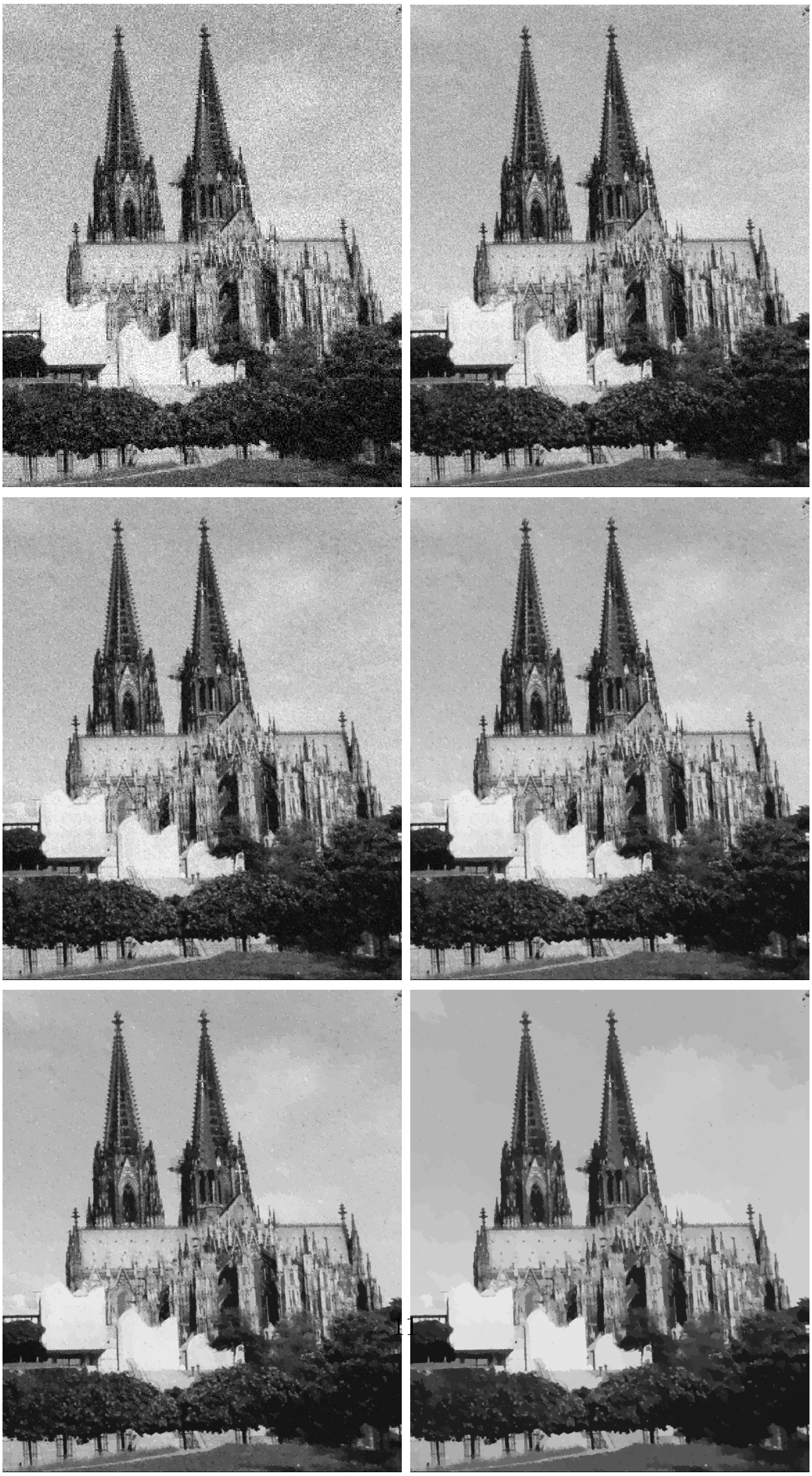
Theorem 3.3 (Restricted Comparison Principle) *Suppose that u and v are C^1 solutions of (3.1) on $Q_T := (-1,1) \times (0,T)$ with C^2 initial data $u_0(x) \leq v_0(x)$ satisfying a) or b):*
a) the backward regimes of u_0 and v_0 have empty intersection, or
b) there exists w_0 between u_0 and v_0 s.th. $|w_{0_x}| < s_0$, i.e. backw. reg. of $w_0 = \emptyset$.
Then $u(x,t) \leq v(x,t)$ in Q_T.

Again the proof is based on a simple idea. If $v - u$ has negative min in (y,t), then $v_x(y,t) = u_x(y,t)$ and by Theorem 3.2 y will be in the forward regime of both v and u. But then one is in a forward parabolic situation, where comparison results hold.

Theorem 3.4 (Uniqueness) *Suppose that u and v are weak C^1 solutions of (3.1) in $Q_T := (-1,1) \times (0,T)$ with identical and analytical initial data u_0. Moreover, assume that $(u_0)_x^2 - s_0^2$ has only simple zeroes and that the diffusion coefficient a is analytic. Then $u(x,t) \equiv v(x,t)$ in Q_T.*

The proof follows essentially from showing that the forward and backward regimes of u und v must coincide. There are many technicalities, e.g. on what happens in the interface where $u_x = \pm s_0$. In particular we use (and derive) the fact that the number of "edges" stays invariant.

In the next figure we see a noisy image of the Cologne cathedral under Perona-Malik flow taken from [28]. The image undergoes first a sharpening then a segmenation in which information gets lost. This illustrates the importance of choosing the right time to stop the flow. The stopping time is chosen interactively, like in the focussing of an overhead projector or beamer.



Our last sequence from [28] shows what the edge detector finds under Perona Malik flow. When there are too many edges the image looks noisy, and when there are too few edges essential features are lost.

Spoiling the beauty of these theoretical results on the Perona Malik equation, I should also mention that C^1 solutions must cease to exist as C^1 solutions after finite time. This is expressed in the following result from [26]

Theorem 3.5 (Nonexistence of global C^1-solutions) *Suppose that* $u_0(x)$ *satisfies certain technical but typical assumptions. Then there is no global weak solution in* $C^1((-1,1) \times \mathbb{R}^+)$.

The proof of this result uses a representation of u along level lines $\Gamma(t)$ of u_x and by showing that $I(t) = \int_0^{\Gamma(t)} u(y,t)dy$ blows up as $t \to \infty$.

4 TV flow and related equations

Since there is such a wide range of nonlinear diffusion equations that are used in mathematical image processing, there is not a single “right” equation. A popular equation is total variation flow as in (1.4)

$$w_t - \text{div}\left(\frac{\nabla w}{|\nabla w|}\right) = 0. \tag{4.1}$$

If the intitial datum u_0 is chosen as the characteristic function of a disk, then its height decreases with constant speed proportional to the curvature of the disk [17]; and if the intial datum is the characteristic function of a convex set with corners, then the solution decays faster near the corners and its shape is rounded off, as displayed in [5]. How can one explain this behaviour heuristically? The Ansatz $w(t,x) = T(t)u(x)$ with nonnegative T leads to

$$T'(t)u(x) - \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0\ ,$$

so that $T = -\lambda$ must be constant and $T(t) = \{T(0) - \lambda t\}_+$ decays linearly to zero in finite time. The constant λ is then supposed to solve the "eigenvalue" problem

$$\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda u = 0. \tag{4.2}$$

In a radially decreasing setting this means $\frac{-1}{r} + \lambda u = 0$ or $\kappa = -\lambda u$. Here κ is the curvature of a level set. Thus on the line $\partial\{u > t\}$ the decay rate λ is proportional to curvature. This is why corners are rounded off as time evolves.

Some explicit solutions for (4.2) were calculated in [7], but notice that equation (4.2) is not homogeneous of degree one, so that a multiple cu of an eigenfunction u is no longer an eigenfunction for the eigenvalue λ.

For this aesthetic reason I suggest considering a general equation

$$w_t - Aw = 0$$

with A to be determined later, for which the Ansatz $w(t,x) = T(t)u(x)$ leads to

$$T'(t)u(x) - A[T(t)u(x)] = 0\ ,$$

and if A is homogeneous of degree d, to

$$T'(t)u(x) - T^d(t)Au(x) = 0.$$

Now separation of variables gives

$$T^d T'(t) = -\lambda \quad \text{and} \quad Au + \lambda u = 0.$$

The "eigenfunction" for A should only be assigned that name if A is homogeneous of degree 1. For general $d \geq 0$ one easily calculates that

$$T(t) = \begin{cases} [(T^{1-d}(0) - (1-d)\lambda t)_+]^{\frac{1}{1-d}}, & \text{if } d \in [0,1), \\ T(0)\, e^{-\lambda t}, & \text{if } d = 1, \\ [T^{1-d}(0) + (d-1)\lambda t]^{\frac{-1}{d-1}}, & \text{if } d \in (0,\infty), \end{cases} \tag{4.3}$$

so that for $d \in [0,1)$ there is finite lifetime (as in the TV case with $d = 0$), while for $d \geq 1$ there is polynomial decay, and for $d = 1$ even exponential decay of a solution.

Clearly there is more than one way to define an operator A that is homogeneous of decay 1. If for $p \in [1,\infty)$ the operator A is given by $Au = |u|^{2-p}\Delta_p u := |u|^{2-p}\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$, then the eigenvalue problem reads

$$\Delta_p u + \lambda |u|^{p-2} u = 0, \tag{4.4}$$

and at least for the first eigenvalue this is a well-understood problem, even when $p \to \infty$ and $p \to 1$, see for instance [32, 8, 33].

For reasons that will become clear below I now define

$$A_p u := \frac{1}{p}|\nabla u|^{2-p}\Delta_p u$$

and recall that $\Delta u = u_{\nu\nu} + (n-1)Hu_\nu$ and $\Delta_p u = |\nabla u|^{p-2}\left\{(p-1)u_{\nu\nu} + (n-1)Hu_\nu\right\}$ so that $A_p u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \frac{p-1}{p}u_{\nu\nu} + \frac{n-1}{p}Hu_\nu$. Thus

$$A_p u = \frac{1}{p}A_1 u + \frac{p-1}{p}A_\infty u$$

is a convex combination of A_1 and A_∞.

Special cases of $u_t - A_p u$ are then

a) the case in which $p = \infty$. In this case $u_t - A_\infty u = u_t - u_{\nu\nu}$, an equation that is used in image processing and analyzed in [25].

b) the case that $p = 2$, where $u_t - A_2 u = u_t - \frac{1}{2}\Delta u$, that is the linear heat equation, and

c) the case $p = 1$, when $u_t - A_1 u = u_t - |\nabla u|\text{div}\left(\nabla u/|\nabla u|\right) = 0$ represents the level set formulation of mean curvature flow, see [22].

Notice that all these equations are all **scaling invariant**, i.e. that brightness of initial data does not matter when it comes to determining the evolution of shapes. I take this and the fact that important special cases are included as an indication of the importance of this operator. For $p \in (1,\infty)$ the operator

$$A_p u := \frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \sum_{i,j} a_{ij}(x)u_{x_i x_j}$$

is almost linear and has coefficients

$$a_{ij} = \frac{1}{p}\left(\delta_{ij} + (p-2)\frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)$$

i.e. a bounded coefficient matrix

$$a = \frac{1}{p}\ I + \frac{p-2}{p}\ \frac{Du \otimes Du}{|Du|^2}$$

that is positive definite for $p \in (1, \infty)$. In fact

$$\min\{\tfrac{1}{p},\ \tfrac{p-1}{p}\}\ I \leq a \leq \max\{\tfrac{1}{p},\ \tfrac{p-1}{p}\}\ I.$$

In that respect the equation $u_t - A_p u = 0$ is benign, but quite degenerate. Nevertheless, my student K.Does has been able to show in [18] that viscosity solutions to this equation exist under initial datum u_0 and various boundary conditions. These solutions enjoy also comparison principles and a decay behaviour that one expects from nondegenerate equations.

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References

[1] Alvarez, L., Guichard, F., Lions, P.l. & Morel, J.M. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.* 1993; **123**:199–257.

[2] Andreu, F., Ballester, C, Caselles, & Mazón, J.M. The Dirichlet problem for total variation flow. *J. Functional Anal.* 2001; **180**:347–403.

[3] Andreu, F., Caselles,V., Diaz, J.I. & Mazón, J.M. Some qualitative properties for the total variation flow. *J. Functional Anal.* 2002; **188**:516–547.

[4] Ambrosio, L. & Tortorelli, V. M. Approximation of functionals depending on jumps by elliptic functionals via Γ- convergence. *Comm. Pure Appl. Math.* 1990; **43**:999–1036.

[5] Alter, F., Caselles, V. & Chambolle, A. Evolution of convex sets in the plane by minimizing total variation flow. *Interfaces and Free Boundaries* 2005; **7**:29–53.

[6] Aubert, G. & Kornprobst, P. *Mathematical Problems in Image Processing* Springer, New York Berlin Heidelberg, 2002

[7] Bellettini, G., Caselles, V. & Novaga, M. Explicit solutions of the eigenvalue problem $-div(Du/|Du|) = u$. *SIAM J. Math. Anal.* 2005; **36**:1095-1129.

[8] Belloni, M., Kawohl, B. & Juutinen, P. The p-Laplace eigenvalue problem as $p \to \infty$ in a Finsler metric, *J. Europ. Math. Soc.* 2006; **8**:123–138.

[9] Blake, A. & Zisserman, A. *Visual Reconstruction*, The MIT Press, Cambridge, Mass., 1987.

[10] Chambolle, A. Image segmentation by variational methods: Mumford and Shah functionals and the discrete approximations. *SIAM J. Appl. Math.* 1995; **55**:827–863.

[11] Chambolle, A. & Lions P.L. Image recovery via total variation minimization and related problems. *Numer. Math.* 1997; **76**:167–188.

[12] Chan, T.F., Shen, J. & Vese, L. Variational PDE models in image processing. *Notices Amer. Math. Soc.* 2003; **50**:14–26.

[13] Chan, T.F. & Esedoglu, S. Aspects of total variation regularized L^1 function approximation. *SIAM J. Appl. Math.* 2005; **65**:1817–1837.

[14] Chan, T. & Vese, L. Active contours without edges, *IEEE Trans. Image Processing* 2001; **10**:266–277.

[15] Chipot, M., March, R., Rosati, M. & Vergara Caffarelli, G. Analysis of a nonconvex problem related to signal selective smoothing. *Math. Models Meth. Appl. Sci* 1997; **7**:313–328.

[16] Dal Maso, G. *An Introduction to* Γ*–convergence*. Birkhäuser Verlag, Basel, 1993.

[17] Dibos, F. & Koepfler, G. Global total variation minimization. *SIAM J. Numer. Anal.* 2000;**37** 646–664.

[18] Does, K. On a class of degenerate parabolic equations. doctoral thesis, in preparation

[19] Esedoglu, S. An analysis of the Perona-Malik scheme. *Comm. Pure Appl. Math.* 2001; **54**:1442–1487.

[20] Esedoglu, S. Stability properties of Perona-Malik scheme. *SIAM J. Numer. Anal.* 2006; **44**:1297–1313

[21] Esedoglu, S. & Osher, S.. Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. *Comm. Pure Appl. Math.* 2004; **57**:1609–1626.

[22] Evans, L.C. & Spruck, J. Motion of level sets by mean curvature I. *J. Differ. Geom.* 1991; **33**:635–681.

[23] Handlovičová, A. & Mikula, K. Analysis of a semidiscrete scheme for solving image smoothing equations of mean curvature flow type. *Acta Math. Univ. Comeniae* 2002; **71**:93–111.

[24] Höllig, K. Existence of infinitely many solutions for a forward-backward heat equation. *Trans. Amer. Math. Soc.* 1983; **278**:299–316.

[25] Juutinen,P. & Kawohl, B. On the evolution governed by the infinity Laplacian, *Math. Annalen* 2006; **335**:819–851.

[26] Kawohl, B. & Kutev, N. Maximum principle and comparison theorems for anisotropic diffusion. *Mathematische Annalen.* 1998; **311**:107–123.

[27] Kawohl, B. The opaque square and the opaque circle, in: General Inequalities VII, Int. Ser. Numer. Math. 1997; **123** (1997):339–346.

[28] Kawohl, B. & Mester, M. Mathematische Bildverarbeitung, in: *Kooperative Informationsverarbeitung an der Universität zu Köln, Bericht für das Jahr 2003.* Cologne 2003; 86–91.

[29] Kawohl, B. From Mumford-Shah to Perona-Malik in image processing, *Math. Methods Appl. Sci.* 2004; **27**:1803–1814.

[30] Kawohl, B. & Stará, J. Observations on a nonlinear evolution equation, *Applicable Analysis,* 2006; **85**:143-152.

[31] Kawohl,B. & Schuricht,F. Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. *Commun. Contemp. Math.*, to appear

[32] Kawohl, B. & Lindqvist, P. Positive eigenfunctions for the p–Laplace operator revisited. *Analysis (Munich)*, 2006; **26**, to appear

[33] Kawohl & Novaga, M. The p-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric, J. Convex Anal., to appear

[34] Kichenassamy, S. The Perona-Malik Paradox. *SIAM J. Appl. Math.* 1997; **57**:1328–1342.

[35] Lair, A. V. Uniqueness for a forward backward diffusion equation. Trans. Amer. Math. Soc. 1985; **291**:311–317.

[36] March, R. Visual reconstruction with discontinuities using variational methods. *Image and Vision Computing* 1992; **10**:30–38.

[37] Mester, M. Mathematische Methoden zur Bildbearbeitung, doctoral thesis, Cologne 2004

[38] Modica, L. & Mortola, S. Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. (Italian) Boll. Un. Mat. Ital. A (5) 14 (1977), no. 3, 526–529.

[39] Modica, L. Γ-convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, (1979):223–244.

[40] Morel, J.-M. & Solimini, S. *Variational methods in image segmentation* Birkhäuser Verlag, Boston, 1995

[41] Morini,M. & Negri, M. Mumford Shah functional as limit of discrete Perona Malik energies. *Math. Models and Meth. in Appl. Sci.* 2003; **13**:785–805.

[42] Mumford, D. & Shah, J. Optimal aproximations by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* 1989; **42**:577–685.

[43] Negri, M. & Paolini, M. Numerical minimization of the Mumford-Shah functional. *Calcolo* 2001; **38**:67–84.

[44] Nordström, K.N. Biased anisotropic diffusion – a unified regularization and diffusion approach to edge detection. *Image Vision Comput.* 1990; **8** 318–327.

[45] Osher, S. & Fedkiw, M: *Level set methods and dynamic implicit surfaces* Springer, New York Berlin Heidelberg, 2003

[46] Perona, P. & Malik, J. Scale space and edge detection using anisotropic diffusion, *IEEE Trans. Pattern Anal. Machine Intell.* 1990; **12**:629–639.

[47] Richardson, T.J. & Mitter, S.K. A variational formulation-based edge focussing algorithm. *Sadhana* 1997 **22**:553–574.

[48] Rudin, L.I., Osher, S. & Fatemi, E. Nonlinear total variation based noise removal algorithms. *Physica D* 1992; **60**:259–268 .

[49] Sapiro, G. *Geometric partial differential equations and image analysis* Cambridge University Press, Cambridge, 2001

[50] Vese, L. & Chan, T. Reduced non-convex functional approximations for image restoration and segmentation. UCLA CAM Report 97-56, 1997;

[51] Weickert, J. *Anisotropic Diffusion in Image Processing*, Teubner Verlag, Stuttgart, 1998.

[52] Weickert, J. Applications of nonlinear diffusion in image processing and computer vision. *Acta Math. Univ. Comenianae* 2001; **70**:33–50.

[53] Zhang, K. Existence of infinitely many solutions for the one-dimensional Perona-Malik model. Calc. Var. Partial Differential Equations 2006; **26**:171–199.

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