The $p$-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric

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Dedicated to the memory of Thomas Lachand-Robert

Abstract

We consider the $p$-Laplacian operator on a domain equipped with a Finsler metric. After deriving and recalling relevant properties of its first eigenfunction for $p > 1$, we investigate the limit problem as $p \to 1$.

Keywords: $p$-Laplace, eigenfunction, Finsler metric, Cheeger set, anisotropic isoperimetric inequality

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1 Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial \Omega$ of a plane domain $\Omega$. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p \, dx$, then a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

on $W^{1,p}_0(\Omega)$ satisfies the Euler-Lagrange equation

$$-\Delta_p u = \lambda_p |u|^{p-2} u \quad \text{in } \Omega,$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the well-known $p$-Laplace operator. This eigenvalue problem has been extensively studied in the literature. As $p \to 1$, formally the limit equation reads

$$-\text{div} \left( \frac{\nabla u}{|\nabla u|} \right) = \lambda_1(\Omega) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$
For a precise interpretation of (1.2) see [23] or [33]. Naturally, here $\lambda_1(\Omega) := \lim_{p \to 1^+} \lambda_p(\Omega)$. A somewhat surprising recent result is that the family of eigenfunctions $\{u_p\}$ converges in $L^1(\Omega)$ cum grano salis to (a multiple of) the characteristic function $\chi_{C_\Omega}$ of a subset $C_\Omega$ of $\Omega$, a so called Cheeger-set, see [21]. A Cheeger set of $\Omega$ is characterized as a domain that minimizes

$$h(\Omega) := \inf_D \frac{\partial D}{|D|}$$

with $D$ varying over all smooth subdomains of $\Omega$ whose boundary $\partial D$ does not touch $\partial \Omega$, and with $|\partial D|$ and $|D|$ denoting $(n - 1)$- and $n$-dimensional Lebesgue measure of $\partial D$ and $D$. The existence, uniqueness, regularity and construction of such sets is discussed in [21] and [22] (and partly in [34]) and its continuous dependence on $\Omega$ in [18]. The paper [26] contains a numerical method for the calculation of $n$-dimensional Cheeger sets and some three-dimensional examples. Cheeger sets are of significant importance in the modelling of landslides, see [19], [20], or in fracture mechanics, see [24]. Notice that a set $D \subseteq \Omega$ is a Cheeger set if and only if it is a minimizer of

$$|\partial E| - h(\Omega)|E| \quad \text{for } E \subseteq \Omega. \quad (1.3)$$

Now suppose that the membrane is not isotropic. It is for instance woven out of elastic strings like a piece of material. Then the deformation energy can be anisotropic, see [5]. Another way to describe this effect is by stating that the Euclidean distance in $\Omega$ is somehow distorted. It is the purpose of the present paper to generalize the above result on eigenfunctions and their convergence as $p \to 1$ to the situation, where $\Omega \subset \mathbb{R}^n$ is no longer equipped with the Euclidean norm, but instead with a general norm $\phi$. In that case a Lipschitz continuous function $u : \Omega \to \mathbb{R}$ (in a convex domain $\Omega$) has Lipschitz constant $L = \sup_{z \in \Omega} \phi^*(\nabla u(z))$, where $\phi^*$ denotes the dual norm to $\phi$. Therefore the Rayleigh quotient studied in this paper is given by

$$R_p(u) := \frac{\int\int_{\Omega} (\phi^*(\nabla u))^p \, dx}{\int \int_{\Omega} |u|^p \, dx} \quad (1.4)$$

on $W^{1,p}_0(\Omega)$ and the Cheeger constant by

$$h(\Omega) := \inf_{D \subset \Omega} \frac{P_\phi(D)}{|D|}, \quad (1.5)$$

with $P_\phi$ denoting anisotropic perimeter in $\mathbb{R}^n$ (see (2.10) below). The minimizer $u_p$ of $R_p$ satisfies the Euler-Lagrange inclusion

$$-Q_p u := -\text{div} \left( (\phi^*(\nabla u))^{p-2} J(\nabla u) \right) \geq \lambda_p |u|^{p-2} u \quad \text{in } \Omega \quad (1.6)$$
in the weak sense \cite{8}, i.e.

\[
\int_{\Omega} \left( \phi^*(\nabla u_p) \right)^{p-2} \langle \eta, \nabla v \rangle \, dx = \lambda_p \int_{\Omega} |u_p|^{p-2} u_p \cdot v \, dx \tag{1.7}
\]

for any \( v \in W^{1,p}_0(\Omega) \) and for a measurable selection \( \eta \in J(\nabla u_p) \), where the function \( J: \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n) \) is defined as the subdifferential

\[
J(\xi) := \partial \left( \frac{\phi^*(\xi)^2}{2} \right) . \tag{1.8}
\]

Note that the function \( J \) is single-valued iff the norm \( \phi \) is strictly convex, i.e. if its unit sphere \( \{ x : \phi(x) = 1 \} \) contains no nontrivial line segments \cite[pag. 400]{39}. Note further that \( J(0) = 0 \) and that for the Euclidean norm the duality map reduces to the identity \( J(\nabla u) = \nabla u \).

The paper is organized as follows. In Section 2 we fix some notation. In Section 3 we recall and derive the existence, uniqueness, regularity and log-concavity of solutions for \( p > 1 \). In Section 4 we derive the limit equation for \( p \to 1 \). In Section 5, we discuss in detail the two-dimensional case, proving uniqueness of Cheeger sets in the convex case. In Section 6 we provide some instructive examples.

\section{Notation}

We say that the norm \( \phi \) is \textit{regular} if \( \phi^2, (\phi^*)^2 \in C^2(\mathbb{R}^n) \). This includes for instance \( \phi(x) = \|x\|_q \) with \( q \in (1, \infty) \) but excludes the crystalline cases \( q = 1 \) or \( q = \infty \), see Section 6.

Given \( E \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), we set

\[
dist_\phi(x, E) := \inf_{y \in E} \phi(x - y), \quad d^E_\phi(x) := \dist_\phi(x, E) - \dist_\phi(\mathbb{R}^n \setminus E, x).
\]

\( d^E_\phi(x) \) indicates the signed distance of \( x \) to \( \partial E \) and is positive outside \( E \). Notice that, at each point where \( d^E_\phi \) is differentiable, there holds

\[
\phi^*(\nabla d^E_\phi) = 1. \tag{2.9}
\]

Let us define the (anisotropic) perimeter of \( E \) as

\[
P_\phi(E) := \sup \left\{ \int_E \text{div} \eta \, dx \mid \eta \in C^1_c(\mathbb{R}^n), \phi(\eta) \leq 1 \right\} = \int_{\partial^* E} \phi^*(\nu^E) d\mathcal{H}^{n-1}, \tag{2.10}
\]

\[3\]
where \( \partial^* E \) and \( \nu^E \) denote the reduced boundary of \( E \) and the (Euclidean) unit normal to \( \partial^* E \).

Given an open set \( \Omega \subseteq \mathbb{R}^n \) we define the \( BV \)-seminorm of \( v \in BV(\Omega) \) as

\[
\int_{\Omega} \phi^*(Dv) := \sup \left\{ \int_{\Omega} v \text{div} \eta \, dx \mid \eta \in C^1_c(\mathbb{R}^n), \phi(\eta) \leq 1 \right\}.
\]

Given \( \delta > 0 \), we define

\[
E_\delta^+ := \{ x \in \mathbb{R}^n \mid d_E^\phi < \delta \} = E + \delta W_\phi,
\]

\[
E_\delta^- := \{ x \in \mathbb{R}^n \mid d_E^\phi < -\delta \},
\]

\[
E_\delta^\pm := (E_\delta^-)^\delta_+ \subseteq E,
\]

where \( W_\phi := \{ x \mid \phi(x) < 1 \} \), also called \textit{Wulff shape}, denotes the unit ball with respect to the norm \( \phi \).

Given a compact set \( E \subset \mathbb{R}^n \) with Lipschitz boundary, we denote by \( n_\phi : \partial E \rightarrow \mathbb{R}^n \) any Lipschitz vector field satisfying \( n_\phi \in J(\nabla d_E^\phi) \) a.e. on \( \partial E \). Moreover, we set

\[
\|\kappa_\phi\|_{L^\infty(\partial E)} := \inf_{n_\phi \in J(\nabla d_E^\phi)} \|\text{div}_\tau n_\phi\|_{L^\infty(\partial E)},
\]

which represents the \( L^\infty \)-norm of the \( \phi \)-mean curvature of \( \partial E \). Here \( \text{div}_\tau \) denotes the tangential divergence operator. We make the convention that \( \|\kappa_\phi\|_{L^\infty(\partial E)} = +\infty \) if the set \( E \) does not admit any Lipschitz vector field \( n_\phi \in J(\nabla d_E^\phi) \). We say that \( E \) is \( \phi \)-regular if \( \|\kappa_\phi\|_{L^\infty(\partial E)} < +\infty \).

Notice that in the Euclidean case \( E \) is \( \phi \)-regular iff \( \partial E \) is of class \( C^{1,1} \). Moreover, the unit ball \( W_\phi \) is always \( \phi \)-regular and \( \|\kappa_\phi\|_{L^\infty(\partial W_\phi)} = n - 1 \). To see this, it is enough to consider the vector field \( n_\phi(x) = x/\phi(x) \).

### 3 Existence, uniqueness, regularity and log-concavity of solutions

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. If we minimize the functional

\[
I_p(v) = \int_{\Omega} \phi^*(\nabla v)^p \, dx \quad \text{on} \quad K := \{ v \in W^{1,p}_0(\Omega) : \|v\|_{L^p(\Omega)} = 1 \}, \quad (3.1)
\]

then via standard arguments (see [6]) a minimizer \( u_p \) exists for every \( p > 1 \) and it is a weak solution to the equation (1.6), with \( \lambda_p = I_p(u_p) \). Note that \( \Lambda_p := I_p(u_p)^{1/p} \) is the minimum of the Rayleigh quotient

\[
R_p(v) := \frac{\int_{\Omega} (\phi^*(\nabla v))^p \, dx}{\|v\|_p^{1/p}} \quad (3.2)
\]
on $W^{1,p}_0(\Omega) \setminus \{0\}$. Without loss of generality we may assume that $u_p$ is non-negative. Otherwise we can replace it by its modulus.

Moreover, as shown in [6] any nonnegative weak solution of (1.6) is necessarily bounded and positive in $\Omega$. If $p > n$, then $u_p$ is also uniformly Hölder continuous because of the Sobolev-embedding theorem and the equivalence of the usual Sobolev norm with

$$
\|u\|_{1,p} := \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} + \left( \int_{\Omega} (\phi^*(\nabla u))^p \, dx \right)^{1/p}.
$$

(3.3)

If the norm $\phi$ is regular and $p > 1$, one can even show that $u_p \in C^{1,\alpha}(\Omega)$. Indeed, the function $u_p$ minimizes $J_p(v) := \int_{\Omega} (\phi^*(\nabla v))^p - \lambda_p |u|^p \, dx$, and the theory for quasiminima in [17] implies that minimizers are bounded (Thm. 7.5), Hölder continuous (Thm. 7.16) and satisfy a strong maximum principle (Thm. 7.12), because one can easily check that $u_p$ satisfies (7.71) in [17]. Therefore $u_p$ is positive. Once positivity is known, the uniqueness follows from a simple convexity argument, see [4] or [6]. Moreover, from the result in [12] one can conclude that $u_p \in C^{0,\beta}(\Omega)$ for any $\beta \in (0,1)$. Finally, if $\phi$ is regular, then $u_p \in C^{1,\alpha}(\Omega)$ according to [7], [28], [37], [38] or [13]. Let us summarize these statements.

**Theorem 3.1.** For every $p \in (1,\infty)$ the nonnegative minimizer $u_p$ of (3.1) is positive, unique, belongs to $C^{0,\beta}(\Omega)$ for any $\beta \in (0,1)$ and it solves (1.6) in the weak sense. Moreover, if the norm $\phi$ is regular then $u_p$ is of class $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Finally, if $\Omega$ is convex, then $u_p$ is log-concave and the level sets set $\{u_p > t\} \subseteq \Omega$ are convex for all $t > 0$.

**Proof.** To prove the last statement, we follow Sakaguchi’s approach from [31], first for strictly convex $\Omega$ and for a smooth norm $\phi$. The general case follows then from approximation arguments for $\Omega$ and $\phi$. Log-concavity of a sequence $u_{p,n}$ is preserved under pointwise limits as $n \to \infty$, because the inequality

$$
\log u_{p,n} \left( \frac{x_1 + x_2}{2} \right) \geq \frac{1}{2} \log u_{p,n}(x_1) + \frac{1}{2} \log u_{p,n}(x_2)
$$

in $\Omega \times \Omega$ is stable under such limits. If $u_p$ solves (1.6), then $v_p := \log u_p$ solves

$$
-\text{div} \left( (\phi^*(\nabla v))^{p-2} J(\nabla v) \right) = (p-1)\phi^*(\nabla v)^p + \lambda_p \quad \text{in } \Omega
$$

(3.4)

and this degenerate elliptic equation can be approximated by a nondegenerate one

$$
-\text{div} \left( (\varepsilon + (\phi^*(\nabla v))^2)^{\frac{p-2}{2}} J(\nabla v) \right)
$$
\[ \lambda_p(\Omega) \geq \left( \frac{h(\Omega)}{p} \right)^p. \] (4.1)

Here \( h(\Omega) \) is the Cheeger constant of \( \Omega \) as defined in (1.5). Moreover, as \( p \to 1 \), the eigenvalue \( \lambda_p(\Omega) \) converges to \( \lambda_1(\Omega) = h(\Omega) \).

**Proof.** In the Euclidean case this is Cheeger’s original estimate [11] when \( p = 2 \), and for general \( p \) it can be found in [27], [2], [29] and [36]. For a more general \( \phi \) one can easily modify their proofs by using the generalized coarea formula from [15] or [16]. To prove the limiting behaviour of \( \lambda_p(\Omega) \) as \( p \to 1 \) we proceed as in [21] and observe that (4.1) implies \( \liminf_{p \to 1} \lambda_p(\Omega) \geq h(\Omega) \). Therefore it suffices to find a suitable upper bound. Let \( \{D_k\}_{k=1,2,\ldots} \) be a sequence of regular domains for which \( P_\phi(D_k)/|D_k| \) converges to \( h(\Omega) \). We approximate the characteristic function of each \( D_k \) by a function \( w_k \) with the following properties: \( w \equiv 1 \) on \( D_k \), \( w \equiv 0 \) outside an \( \varepsilon \)-neighborhood of \( D_k \) and \( \phi^*(\nabla w_k) = 1/\varepsilon \) in an \( \varepsilon \)-layer outside \( D_k \). For small \( \varepsilon \) the function \( w_k \) is in \( W^{1,\infty}_0(\Omega) \) and provides the upper bound

\[ \lambda_p(\Omega) \leq \frac{P_\phi(D_k)}{|D_k|} \varepsilon^{1-p}. \] (4.2)
Now one sends first $p \to 1$, then $k \to \infty$ to complete the proof.

**Theorem 4.2.** (Convergence of eigenfunctions) As $p \to 1$, the eigenfunction $u_p$ converges, up to a subsequence, to a limit function $u_1 \in BV(\Omega)$, with $u_1 \geq 0$ and $\|u_1\|_1 = 1$. Moreover, almost all level sets $\Omega_t := \{u_1 > t\}$ of $u_1$ are Cheeger sets.

**Proof.** For every $p > 1$ the function $u_p$ minimizes

$$J_p(v) := \int_\Omega (\phi^*(\nabla v))^p - \lambda_p(\Omega) |v|^p \, dx$$

on $W^{1,p}_0(\Omega)$. If one extends $J_p$ to $L^1(\Omega)$ by setting it $+\infty$ on $L^1(\Omega) \setminus W^{1,p}_0(\Omega)$, the family $J_p$ $\Gamma$-converges (see [14]) with respect to the $L^1(\Omega)$-topology to

$$J_1(v) := \begin{cases} \int_\Omega \phi^*(Dv) - h(\Omega) \int_\Omega |v| \, dx & v \in BV(\Omega), \\ +\infty & v \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Indeed, since $J_1$ is lower semicontinuous on $L^1(\Omega)$, it is enough to prove the $\Gamma$-limsup inequality on the subset $C^1(\overline{\Omega}) \subset L^1(\Omega)$ (which is dense both in topology and in energy), where it becomes trivial.

Let us now prove the $\Gamma$-liminf inequality. Notice that, if $v_{p_n} \to u$ in $L^1(\Omega)$, then either there exists a subsequence $v_{p_{n_k}}$ which is equibounded in $BV(\Omega)$, or $J_{p_n}(v_{p_n})$ goes to $+\infty$. If $v_k := v_{p_{n_k}}$ is bounded in $BV(\Omega)$, letting $p_k := p_{n_k}$ and $J_k := J_{p_{n_k}}$, we have

$$J_1(v_k) = \int_\Omega \phi^*(\nabla v_k) - h(\Omega)|v_k| \, dx$$

$$\leq \left[ \int_\Omega (\phi^*(\nabla v_k))^{p_k} \, dx \right]^{\frac{1}{p_k}} |\Omega|^{\frac{p_k-1}{p_k}} \int_\Omega |v_k| \, dx$$

$$\leq \frac{1}{p_k} \int_\Omega (\phi^*(\nabla v_k))^{p_k} \, dx + \frac{p_k - 1}{p_k} |\Omega| - h(\Omega) \int_\Omega |v_k| \, dx$$

$$+ \lambda_{p_k}(\Omega) \int_\Omega |v_k|^{p_k} \, dx - \lambda_{p_k}(\Omega) \int_\Omega |v_k|^{p_k} \, dx$$

$$\leq J_k(v_k) + \frac{p_k - 1}{p_k} |\Omega| + \lambda_{p_k}(\Omega) \int_\Omega |v_k|^{p_k} \, dx - h(\Omega) \int_\Omega |v_k| \, dx$$

$$= J_k(v_k) + \omega_k,$$

where $\lim_{k \to \infty} \omega_k = 0$. It then follows

$$J_1(u) \leq \liminf_{k \to \infty} J_1(v_k) \leq \liminf_{k \to \infty} J_k(v_k).$$
Since $J_p \geq 0$ on $W^{1,p}_0(\Omega)$, we get $J_1 \geq 0$ on $BV(\Omega)$. Moreover $u_p$ forms a minimizing sequence for $J_1$ since, from the last inequality in (4.3), we have
\[ \int_\Omega \phi^*(\nabla u_p) \, dx \leq \frac{p-1}{p} |\Omega| + \lambda_p(\Omega), \]
where we have used the fact that $J_p(u_p) = 0$ and $\|u_p\|_p = 1$. As a consequence, the family $\{u_p\}_{p>1}$ is bounded in $BV(\Omega)$ and, after possibly passing to a subsequence, it converges strongly in $L^1(\Omega)$ to a limit function $u_1 \in BV(\Omega)$ such that $J_1(u_1) = 0$, $u_1 \geq 0$ and $\|u_1\|_1 = 1$. Using the coarea formula as in [21, Eq. (7)] (adapted to the anisotropic setting), one can see that for all $t \in [0, \max_{\Omega} u_1)$ the level set $\Omega_t := \{u_1 > t\}$ is a Cheeger set.

**Remark 4.3.** As a consequence of Theorem 4.2 and the logconcavity of $u_p$, for every convex set $\Omega$ (Theorem 3.1) there exists a convex Cheeger set. Moreover, it follows from the results of [10] that there exists a convex Cheeger set $D \subseteq \Omega$ which is maximal, in the sense that any other Cheeger set of $\Omega$ must be contained in $D$. Recently it was shown in [9] that for convex $\Omega$ the Cheeger set is uniquely determined. The uniqueness of Cheeger sets is in general not true for nonconvex domains (see [22]).

## 5 The planar case

In this section we derive further properties of the function $u_1$, under the additional assumption $n = 2$. Let us begin with the following theorem, which extends the analogous result in the Euclidean case [22, Th. 1].

**Theorem 5.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set. Then, there exists a unique Cheeger set $D \subseteq \Omega$. Moreover, $D$ is convex and we have
\[ h(\Omega) = \frac{1}{t^*}, \quad D = \Omega_{t^*}^+, \]
where $t^* > 0$ is the (unique) value $t$ such that $|\Omega_{t^*}^-| = t^2 |W_\phi|$.\(\)\(\)

**Remark 5.2.** In other words, the theorem states that the Cheeger set is the union of all the Wulff shapes of radius $t^*$ with center in $\Omega_{t^*}^-$, a level set of the distance function to $\partial \Omega$, that was called inner Cheeger set in [22].

**Proof.** Let $D$ be a Cheeger set of $\Omega$. Notice first that $D$ is a convex set, since otherwise we could replace it by its convex hull and reduce (1.3) (see [3, Th. 7.1]). Moreover, from the first variation of (1.3) it follows that the anisotropic curvature of $\partial D$ is bounded by $h(\Omega)$, and each connected component of $\partial D \cap \Omega$ is contained up to translation in $\frac{1}{h(\Omega)} \partial W_\phi$ (see [30, Theorem 4.5]). Let $\tilde{D}$ be
the open maximal Cheeger set of $\Omega$ (recall Remark 4.3), and let $\Gamma \subset \frac{1}{h(\Omega)} \partial W_{\phi}$ be a connected component of $\partial D \cap \tilde{D}$. We denote by $x, y \in \Gamma \cap \partial \tilde{D}$ the extremal points of $\Gamma$, and we let $\Gamma'$ be the arc of $\partial \tilde{D}$ with extrema $x, y$ and lying in the same halfplane of $\Gamma$ with respect to the straight line $r$ passing through $x, y$ (see Figure 1). Reasoning as in [3, Lemma 7.3], it is easy to show that both $\Gamma$ and $\Gamma'$ can be written as graphs on $r$ along some directions. More precisely, there exists a vector $v \in \mathbb{R}^2$, with $|v| = 1$, and two functions $f_1, f_2 : r \to \mathbb{R}$ such that $0 \leq f_1 \leq f_2$ on $[x, y]$, that $\min\{f_2(x), f_2(y)\} = 0$, and that $\Gamma = F_1([x, y])$ and $\Gamma' = F_2([x, y])$, with $F_i(x) := f_i(x)v$, for $i = 1, 2$. Without loss of generality, we shall assume that $v \perp r$. Since $D$ and $\tilde{D}$ are both minimizers of (1.3), it follows that both $f_1$ and $f_2$ are minimizers of

$$G(f) := \int_{[x,y]} \phi^*(f'(s), 1) - h(\Omega)f(s) \, ds. \quad (5.2)$$

If $\phi$ is a regular norm, then the functional $G$ is strictly convex, which implies $f_1 = f_2$, i.e. $D = \tilde{D}$. For a general norm, one has to be more careful, since the functional $G$ is not strictly convex, but only convex. However, reasoning as in [3, Lemma 8.2], the inclusion $\Gamma \subset \frac{1}{h(\Omega)} \partial W_{\phi}$ and the inequality $f_1 \leq f_2$ imply $\|\kappa_{\phi}\|_{L^\infty(\Gamma')} \geq h(\Omega)$, with equality iff $\Gamma = \Gamma'$, which proves the uniqueness of the Cheeger set $D$.

Let us now prove (5.1), reasoning as in [22, Th. 1]. It has been proved in [3] that the convex set $D = \Omega^{1/h(\Omega)}_x$ is a Cheeger set of $\Omega$, hence it is the
unique Cheeger set of $\Omega$. Therefore, it remains to prove that $t^* = 1/h(\Omega)$, i.e.

$$|\Omega_{\frac{1}{h(\Omega)}}| = \frac{|W_\phi|}{h(\Omega)^2}. $$

Let us recall from [1, Section 2.7],[32] the following Steiner-type formulae

$$|C^{\delta}| = |C| + \delta P_\phi(C) + \delta^2 |W_\phi|,$$
$$P_\phi(C^{\delta}) = P_\phi(C) + \delta P_\phi(W_\phi). \quad (5.3)$$

Incidentally, the second equation follows from the first one and, as in the Euclidean case, $P_\phi(W_\phi) = 2|W_\phi|$. This follows from integrating $\text{div} x$ on $W_\phi$.

Applying (5.3) to $C = D_{1/h(\Omega)}$ and recalling that $h(\Omega) = P_\phi(D)/|D|$, we get

$$|D_{1/h(\Omega)}| = \frac{|W_\phi|}{h(\Omega)^2}.$$ 

The claim now follows if we observe that

$$\Omega_{\frac{1}{h(\Omega)}} = D_{\frac{1}{h(\Omega)}}.$$

**Corollary 5.3.** If $n = 2$ and $\Omega$ is a bounded convex set, then the sequence of functions $u_p$ converges to a multiple of the characteristic function of $D$. Moreover, $D = \Omega$ if and only if

$$\|\kappa_\phi\|_{L^\infty(\partial \Omega)} \leq h(\Omega). \quad (5.4)$$

In particular, (5.4) always holds in the case $\Omega = W_\phi$.

**6 Example and concluding remarks**

If the norm under consideration for $x \in \Omega$ is the usual $l_q$- norm, i.e. for $\phi_q(x) = (\sum_{i=1}^n |x_i|^q)^{1/q}$, $q \geq 1$. When $q > 1$, the dual norm of $\phi_q$ is given by $\phi_q^* = \phi_{q'}$, with $q' = q/(q-1)$, and the duality map according to (1.8) is

$$J_i(y) = (|y|_{q'})^{2-q'}|y_i|^{q'-2}y_i.$$ 

Then the $p$-Laplace operator in this metric is given by (see [6])

$$Q_{p,q}u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \phi_{q'}(\nabla u)^{p-q'} \left| \frac{\partial u}{\partial x_i} \right|^{q'-2} \frac{\partial u}{\partial x_i} \right),$$
and for \( q = 2 = q' \) the norm \( \phi_{q'} \) is just the Euclidean norm and \( Q_{p,q} \) reduces to the well-known \( p \)-Laplace Operator

\[
Q_{p,q} u = \Delta_p u = \text{div}( |\nabla u|^{p-2} \nabla u ) .
\]

For general \( q \) and \( p \to 1 \) the operator \( Q_{1,q} \) is formally given by

\[
Q_{1,q} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left[ \frac{|u_{x_i}|}{\phi_{q'}(\nabla u)} \right]^{q'-2} \frac{u_{x_i}}{\phi_{q'}(\nabla u)} \right) .
\]

Again for \( q = 2 = q' \) this expression shrinks down to the customary

\[
Q_{1,2} u = \Delta_1 u = \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) .
\]

We complete this section with the construction of a particular Cheeger set for a nonregular anisotropy. Let us fix \( n = 2 \) and consider the norm \( \phi = \phi_1 \). Notice that in this case the Wulff Shape \( W_\phi \) has the shape of a rhombus. To be precise, it is square of sidelength \( \sqrt{2} \), centered in the origin and rotated by \( \pi/2 \) with respect to the coordinate axes. Moreover, the dual norm \( \phi^* \) is given by \( \phi^*(y) = \max\{||y_1|, |y_2|| \} \). To better illustrate the results of Section 5, let us compute the Cheeger set (and Cheeger constant) of a square \( Q \) of sidelength 1 (see Figure 2).

Since in this case \( |W_\phi| = 2 \) and \( Q^- \) is a square of sidelength \( 1 - 2t \), from Theorem 5.1 we get \( t^* = 1 - \sqrt{2}/2 \) and \( h(Q) = 2 + \sqrt{2} \). It is interesting to note that the Cheeger set of \( Q \) is a regular octahedron (and the inner Cheeger set \( Q^- \) defined in Remark 5.2 is a square). After this manuscript was submitted for publication, we learned that G. Strang independently discusses this example in [35].
Figure 2: The Cheeger set of a square with respect to the norm $\phi_1$.

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**References**


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