Best constants in some exponential Sobolev inequalities

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Abstract

A Pohozaev identity is used to classify the radial solutions of a quasilinear equation with exponential nonlinearity. The results are applied to find the infimum of the non-local functional

$$\mathcal{F}(\lambda, u) = \frac{1}{n} \int_\Omega |\nabla u|^n dx - \lambda F\left(\fint_\Omega e^u dx \right), \quad u \in W_0^{1,n}(\Omega),$$

for various nonlinearities F, where Ω is a bounded domain of $\mathbb{R}^n$ and λ a real parameter. Our results generalize the case when $F(s) = \log s$.

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1 Introduction

Given a bounded domain Ω of $\mathbb{R}^n$, consider the Sobolev space $W_0^{1,n}(\Omega)$ defined as the completion with respect to the norm $\|\nabla u\|_n = \left(\int_\Omega |\nabla u|^n\right)^{\frac{1}{n}}$ of the class of smooth functions in Ω having compact support. It is well-known that this Sobolev space embeds in all spaces $L^q(\Omega)$ with $q \in [1, \infty)$. The independent works of Judovič [11], Pohozaev [20] and Trudinger [27] have extended the classical

Sobolev inequalities by proving the existence of constants $\mu, C_n > 0$ depending only on the dimension n such that

$$\fint_\Omega e^{\mu\left(\frac{|u|}{\|\nabla u\|_n}\right)^{\frac{n}{n-1}}} \le C_n, \quad \forall u \in W_0^{1,n}(\Omega) \setminus \{0\}, \tag{1.1}$$

where $\fint_\Omega f := \frac{1}{|\Omega|}\int_\Omega f dx$. A sharp form of this inequality has been obtained later by Moser [18], who showed that (1.1) holds if and only if $\mu \le \mu_n := n\omega^{\frac{1}{n-1}}$ where ω denotes the measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Inequality (1.1), commonly called Moser-Trudinger inequality, implies the following estimate (as indicated in Section 5):

$$\fint_\Omega e^u \le e^{A_n} e^{\alpha_n \|\nabla u\|_n^n}, \quad \forall u \in W_0^{1,n}(\Omega), \tag{1.2}$$

where $A_n \ge 0$ is some universal constant and α_n is given by

$$\alpha_n = (n-1)^{n-1} n^{1-2n} \omega^{-1}. \tag{1.3}$$

In two dimensions, this inequality has revealed itself very useful in statistical physics ([5], [14], [1]) and is also related to the geometrical problem of prescribing Gauss curvature (see [18]). For this dimension the critical value (1.3) is given by $\alpha_2 = \frac{1}{16\pi}$.

Similarly to what has been done by Talenti [25] for the Sobolev inequalities $\|\nabla u\|_p \le C\|u\|_{p^*}$ with $p < n$, one may ask for the best constants C_n and A_n in the inequalities (1.1) and (1.2). By using Schwarz symmetrization it is enough to treat this question when Ω is a ball. For this domain, Carleson and Chang [6] proved the existence of an extremal function for the Moser-Trudinger inequality (1.1) and that the best constant C_n is strictly greater than $1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}$. Concerning the related inequality (1.2) it was already noted in [5] that for a ball in the plane, there are no functions realizing equality in (1.2). Furthermore Carleson and Chang [6] found that $A_n = \displaystyle\sum_{k=1}^{n-1}\frac{1}{k}$ is the optimal constant in (1.2) when the domain is a ball in $\mathbb{R}^n$. In other words any $u \in W_0^{1,n}(\Omega)$ satisfies the inequality

$$\frac{1}{n}\int_\Omega |\nabla u|^n dx - (n\alpha_n)^{-1}\log\left(\fint_\Omega e^u dx\right) > -(n\alpha_n)^{-1}\sum_{k=1}^{n-1}\frac{1}{k}, \tag{1.4}$$

and the constant on the right hand-side of (1.4) is optimal when the domain is a ball.

Similar inequalities hold on compact manifolds. For example on the two-dimensional canonical sphere, it was proved by Onofri [19] (see also [10]) that any $u \in W^{1,2}(S^2)$ of average zero satisfies

$$\frac{1}{2}\int_{S^2} |\nabla u|^2 \;-\; 8\pi \log\left(\fint_{S^2} e^u \right) \;\geq\; 0. \tag{1.5}$$

Using a stereographical transformation, Beckner [4] proved that such an inequality is equivalent to the following one on a disk B of $\mathbb{R}^2$

$$\frac{1}{2}\int_{B} |\nabla u|^2 \;-\; 8\pi \left\{ \log\left(\fint_{B} e^u \right) + \left(\fint_{B} e^u \right)^{-1} \right\} \;\geq\; -8\pi, \tag{1.6}$$

in the space of non-negative functions of $W_0^{1,2}(B)$. Motivated by this equivalence, Kim [15] has studied inequality (1.6) in higher dimension and proved that

$$\frac{1}{n}\int_{\Omega} |\nabla u|^n \;-\; (n\alpha_n)^{-1} \left\{ \log\left(\fint_{\Omega} e^u \right) + \left(\fint_{\Omega} e^u \right)^{-1} \right\} \;\geq\; -(n\alpha_n)^{-1} \sum_{k=1}^{n-1} \frac{1}{k}, \tag{1.7}$$

for any $u \in W_0^{1,n}(\Omega)$ with $u \geq 0$.

Note that the constants appearing on the right hand-side of (1.7) and (1.4) are exactly the same. This is not a just a simple coincidence, and the aim of the present paper is to exhibit a general setting in which both these sharp inequalities are covered. More precisely given a function $F : (0, \infty) \to \mathbb{R}$, and $\lambda \in (0, \infty)$ we shall consider the general *non-local* functional

$$\mathcal{F}(\lambda, u) := \frac{1}{n}\int_{\Omega} |\nabla u|^n dx \;-\; \lambda F\left(\fint_{\Omega} e^u dx \right), \quad u \in W_0^{1,n}(\Omega). \tag{1.8}$$

Under the assumptions

$$F \in C^1(0, \infty), \quad \lim_{s\to\infty} \{F(s) - \log s\} = 0, \quad F' \geq 0, \tag{1.9}$$

inequality (1.2) easily implies that the functional (1.8) is well-defined, admits a minimizer when $\lambda < (n\alpha_n)^{-1}$, is bounded from below at $\lambda = (n\alpha_n)^{-1}$, while beyond that value appropriate test functions show that $\mathcal{F}(\lambda, \cdot)$ is unbounded from below. Further any critical point of $\mathcal{F}(\lambda, \cdot)$ satisfies in the weak sense the non-local quasilinear equation

$$-\Delta_n u \;=\; \frac{\lambda}{|\Omega|} f\left(\fint_{\Omega} e^u \right) e^u, \quad u \in W_0^{1,n}(\Omega), \tag{1.10}$$

where $\Delta_n u := \mathrm{div}\,(|\nabla u|^{n-2}\nabla u)$ denotes the n-Laplacian and $f := F'$.

Thanks to the assumption $f \geq 0$ we may assume one of the minimizer to be non-negative and therefore by using Schwarz symmetrization, we are reduced to consider Problem (1.10) in the class of radially symmetric functions define on a ball of same volume as Ω. This restriction on the volume is actually irrelevant since both the functional $\mathcal{F}(\lambda, \cdot)$ and Problem (1.10) are invariant under scaling. The study of (1.10) in the class of radial functions on a ball will be based on the generalized Pohozaev identity as established through the works of Pohozaev [20], Pucci and Serrin [21], Degiovanni *et al.* [8]. Since this integral identity will play a central role in all our arguments, we will give an independent proof in Section 2. This identity has mainly be used to prove non-existence of solutions in several nonlinear problems. In the present paper we apply it to classify the radial solutions of the equations $-\Delta_n v = \pm e^v$ and of its non-local variant (1.10). Our first main result stated in a simpler form is the following:

Theorem 1.1. *Let Ω be ball of $\mathbb{R}^n$. Then Problem* (1.10) *admits a positive radial solution if and only if there exists $\sigma > 1$ satisfying*

$$n\alpha_n \lambda f(\sigma) = \frac{1}{\sigma}\left|1 - \frac{1}{\sigma}\right|^{n-1}. \tag{1.11}$$

A similar result holds for the existence of negative radial solutions and we refer to Theorem 4.1 for a complete statement. By applying our results to the special case where $f(s) = \frac{1}{s}\left|1-\frac{1}{s}\right|^{k-1}\left(1-\frac{1}{s}\right)$ $(k \geq 1)$, we generalize and complete the results obtained previously in [15] for $k = 2$ (see our Proposition 4.2). In particular inequality (1.6) can be extended as follows:

Theorem 1.2. *Assume $F(s) = \int_1^s \frac{1}{\tau}\left(1-\frac{1}{\tau}\right)^{n-1} d\tau$. Then the associated functional $\mathcal{F}(\lambda, \cdot)$ admits a radial critical point $u_0 > 0$ if and only if $\lambda = (n\alpha_n)^{-1}$. Furthermore, for each $\lambda \leq (n\alpha_n)^{-1}$ we have*

$$\mathcal{F}(\lambda, u) \;\geq\; \mathcal{F}\big([n\alpha_n]^{-1}, u_0\big) \;=\; 0, \quad \forall u \in W_0^{1,n}(\Omega),\, u \geq 0. \tag{1.12}$$

Another generalization is obtained by considering functions f that satisfy the condition:

$$s \mapsto (n\alpha_n)^{-1}\frac{\left(1-\frac{1}{s}\right)^{n-1}}{sf(s)} \quad \text{is strictly increasing in } (1,\infty). \tag{1.13}$$

For such nonlinearities and if Ω is a ball, Problem (1.10) admits radial positive solutions if and only if λ belongs to the range of the map defined in (1.13). Under the additional requirement (1.9) we can prove that this range is the interval $(0, [n\alpha_n]^{-1})$ and explicitly calculate the infimum of the functional (1.8) when $\lambda \uparrow (n\alpha_n)^{-1}$. As a consequence both inequalities (1.4) and (1.7) can be generalized as follows:

Theorem 1.3. *Assume that* (1.9), (1.13) *hold and let* B *be a ball of* $\mathbb{R}^n$. *Then for any* $u \in W_0^{1,n}(\Omega)$ *it holds*

$$\mathcal{F}\big([n\alpha_n]^{-1}, u\big) \; > \inf_{u \in W_0^{1,n}(B)} \mathcal{F}\big([n\alpha_n]^{-1}, u\big) \; = \; -(n\alpha_n)^{-1} \sum_{k=1}^{n-1} \frac{1}{k}. \tag{1.14}$$

In the ball B *the functional* $\mathcal{F}(\lambda, \cdot)$ *admits for each* $\lambda < (n\alpha_n)^{-1}$ *a unique minimizer, whereas the infimum in* (1.14) *is not achieved.*

Our paper is organized as follows. Section 2 provides an alternative proof of the Pohozaev identity for radial functions solving $-\Delta_n v = g(v)$. By applying this identity to the particular nonlinearity $g(s) = e^s$, we are able in Section 3 to classify the radial solutions of the quasilinear equation $-\Delta_n v = \pm e^v$ in $\mathbb{R}^n$. In Section 4 we use a suitable substitution to reduce the nonlocal equation (1.10) to the local equation $-\Delta_n v = \pm e^v$ and to classify the solutions of the nonlocal equation (1.10). In Section 5, after relating $\mathcal{F}(\lambda, u)$ to the Moser-Trudinger inequality, we prove the lower bounds stated in Theorems 1.2 and 1.3.

2 Pohozaev identity for radial solutions

Given a ball $B = B(0, R)$ of $\mathbb{R}^n$ and a function $g \in C^0(\mathbb{R})$, we assume in this section the existence of a solution v to the problem

$$-\Delta_n v = g(v), \quad v \in C^1(\bar{B}), \quad v \text{ radial}, \tag{2.1}$$

and derive some integral identities for v.

For semilinear elliptic differential equations involving the Laplace operator, Pohozahev [20] derived an integral formula that has been widely used to prove non-existence of solutions. This identity has later been generalized by Pucci, Serrin [21] to C^2-solutions of very general quasilinear elliptic equations. However, solutions satisfying non-linear equations involving the p-Laplace operator are not expected to be better than $C^{1,\alpha}$ (see [26]). The work of Degiovanni *et al.* [8] has relaxed the C^2–assumption and extended the Pucci-Serrin identity by assuming only the solution to be of class $C^1(\overline{\Omega})$. For Problem (2.1), by setting $G(s) := \int_0^s g(t)dt$, this general identity reads

$$\fint_{B_r} G(v)dx - G(v(r)) = \frac{n-1}{n} \left| \frac{dv}{dr} \right|^n (r). \tag{2.2}$$

Henceforth we set

$$\omega = \text{ measure of the unit sphere } S^{n-1} \subset \mathbb{R}^n,$$

and give an independent proof of (2.2) in the following equivalent form. Note that by integrating (2.1) one can easily check that (2.2) and (2.3) are equivalent.

Proposition 2.1. *Let $g \in C^0(\mathbb{R})$ and set $G(s) := \int_0^s g(t)dt$. Then any solution v of Problem* (2.1) *satisfies the following identities:*

$$\frac{\omega}{n} r^n G(v(r)) = \int_{B_r} G(v)\,dx - \frac{n-1}{n^2}\omega^{\frac{1}{1-n}} \left| \int_{B_r} g(v)\,dx \right|^{\frac{n}{n-1}}, \quad (2.3)$$

$$\frac{r}{n}\frac{d}{dr}\left\{ \int_{B_r} G(v)\ dx \right\} = \int_{B_r} G(v)\,dx - \frac{n-1}{n^2}\omega^{\frac{1}{1-n}} \left| \int_{B_r} g(v)\,dx \right|^{\frac{n}{n-1}}. \quad (2.4)$$

Proof: On each ball $B_r := B(0,r) \subseteq B$ define

$$\mathcal{G}(r) := \int_{B_r} g(v)\,dx, \quad r \in [0,R].$$

Since $g \circ v \in L^\infty(B)$ we easily check that the function $\mathcal{G} : [0,R] \to \mathbb{R}$ is Lipschitz continuous on $[0,R]$. Therefore $r \mapsto |\mathcal{G}(r)|$ is Lipschitz continuous, absolutely continuous and differentiable a.e in $[0,R]$.

Integrating (2.1) on the ball B_r and applying the divergence Theorem yield

$$\int_{\partial B_r} \langle -|\nabla v|^{n-2}\nabla v, \nu \rangle \, d\sigma = \int_{B_r} g(v)\,dx,$$

where ν denotes the outward normal derivative on ∂B_r. Since v is radial we deduce

$$\mathcal{G}(r) = \left|\frac{dv}{dr}\right|^{n-2} \left(-\frac{dv}{dr}\right) |\partial B_r| \quad \text{and} \quad \mathrm{sgn}\,(\mathcal{G}) = \mathrm{sgn}\left(-\frac{dv}{dr}\right), \quad (2.5)$$

where by definition $\mathrm{sgn}(h)(r) := \frac{h(r)}{|h(r)|}$ if $h(r) \neq 0$, and $\mathrm{sgn}(h)(r)$ is zero otherwise. Moreover

$$\begin{aligned} \frac{d|\mathcal{G}|}{dr}(r) &= \frac{d}{dr}\left| \int_{B_r} g(v)\,dx \right| = \mathrm{sgn}(\mathcal{G}) \int_{\partial B_r} g(v)\,d\sigma \\ &= \mathrm{sgn}\left(-\frac{dv}{dr}\right) g(v(r))|\partial B_r|. \quad (2.6) \end{aligned}$$

Hence by (2.5) and (2.6) we obtain

$$\begin{aligned}\frac{n-1}{n}\frac{d|\mathcal{G}|^{\frac{n}{n-1}}}{dr}(r) &= |\mathcal{G}|^{\frac{1}{n-1}}\frac{d|\mathcal{G}|}{dr}(r)\\ &= \left|\frac{dv}{dr}\right| \, |\partial B_r|^{\frac{1}{n-1}} \operatorname{sgn}\left(-\frac{dv}{dr}\right)\, g(v)\, |\partial B_r|\\ &= -\frac{dv}{dr}g(v)|\partial B_r|^{\frac{n}{n-1}}.\end{aligned}$$

Therefore,

$$\frac{n-1}{n}\frac{d|\mathcal{G}|^{\frac{n}{n-1}}}{dr}(r) = -\omega^{\frac{n}{n-1}} r^n \frac{d}{dr}[G(v)](r). \tag{2.7}$$

By integrating equation (2.7) on the interval $[0, r]$ and recalling that $|\mathcal{G}|$ is absolutely continuous we get

$$\begin{aligned}\frac{n-1}{n}\left|\int_{B_r} g(v)\,dx\right|^{\frac{n}{n-1}} &= \omega^{\frac{n}{n-1}}\left\{-G(v)r^n + n\int_0^r G(v)\rho^{n-1}\,d\rho\right\}\\ &= \omega^{\frac{n}{n-1}}\left\{-G(v)r^n + \frac{n}{\omega}\int_{B_r} G(v)\,dx\right\}.\end{aligned} \tag{2.8}$$

From (2.8) we immediately obtain (2.3). Equality (2.4) is now a consequence of (2.3) by noting that:

$$G(v(r)) = \frac{1}{\omega r^{n-1}}\int_{\partial B_r} G(v)\,d\sigma = \frac{1}{\omega r^{n-1}}\frac{d}{dr}\int_{B_r} G(v)\,dx\ .$$

□

Remark 2.2. *In dimension two and with the aim of deriving a priori estimates for radial subsolutions of $-\Delta v \le e^v$, Bandle [3] or Suzuki [23] derive first a differential inequality for the function $\int_{B_r} e^v\,dx$. In the proof of Proposition 2.1 we have extended this basic idea to higher dimension and general nonlinearity that may change sign.*

From now on, we find it more convenient to rewrite (2.3) and (2.4) in terms of the constant α_n defined in (1.3):

$$\frac{\omega}{n}r^n G(v(r)) = \int_{B_r} G(v)\,dx \;-\; (n\alpha_n)^{\frac{1}{n-1}}\left|\int_{B_r} g(v)\,dx\right|^{\frac{n}{n-1}}, \tag{2.9}$$

$$\frac{r}{n}\frac{d}{dr}\left\{\int_{B_r} G(v)\right\} = \int_{B_r} G(v)\,dx \;-\; (n\alpha_n)^{\frac{1}{n-1}}\left|\int_{B_r} g(v)\,dx\right|^{\frac{n}{n-1}}. \tag{2.10}$$

Notice that when the function G is a multiple of g, i.e. when $g(s) = \lambda e^s$, equation (2.10) gives an ODE for the function $\mathcal{G}(r) = \int_{B_r} g(v)\,dx$. This property will be exploited in the next section.

3 Radial solutions for a quasilinear Liouville equation

In this section we consider the special case of (1.10) in which $\frac{\lambda}{|\Omega|} f \equiv \pm 1$ in a ball B centered at the origin and restrict the study to the class of radial solutions, i.e. we now study

$$\begin{cases} -\Delta_n v = \varepsilon e^v, & \varepsilon = \pm 1, \\ v \in W^{1,n}(B), & v \text{ radial.} \end{cases} \tag{3.1}$$

In the next section we shall reduce the general non-local Problem (1.10) to the local equation (3.1).

In dimension two the nonlinear equation in (3.1) is also called "Liouville equation", because in [16] Liouville gave a representation formula of the solutions in terms of meromorphic functions (on any simply connected domain). When $n \geq 2$ the study of (3.1) (with $\varepsilon = 1$) can be found in Clément et *al.* [[7], Section 6], where the solutions of (3.1) with zero Dirichlet data are explicitly given. Using a different approach, we shall see how the classification of solutions to (3.1) can be obtained directly from the Pohozaev identity (2.10). We start with the following result:

Proposition 3.1. *Let v be a solution of* (3.1) *and set $M(r) := \int_{B_r} e^v dx$ for $r \leq R$. Then the following relations hold*

$$e^{v(r)} = \frac{M(r)}{|B_r|}\left(1 - \varepsilon\,[n\alpha_n M(r)]^{\frac{1}{n-1}}\right), \tag{3.2}$$

$$e^{v(0)} = \frac{M(r)}{|B_r|}\left(\frac{1}{1 - \varepsilon[n\alpha_n M(r)]^{\frac{1}{n-1}}}\right)^{n-1}, \quad \forall r \in [0, R], \tag{3.3}$$

$$M(r) = |B_r| \frac{e^{v(0)}}{\left(1 + \varepsilon[n\alpha_n e^{v(0)}|B_r|]^{\frac{1}{n-1}}\right)^{n-1}}, \quad \forall r \in [0, R]. \tag{3.4}$$

As a consequence

$$e^{v(0)} e^{(n-1)v(R)} = \left(\fint_B e^v\, dx\right)^n. \tag{3.5}$$

Proof: Equation (3.2) follows from applying (2.9) to $g(s) = G(s) = \varepsilon e^s$. In order to prove (3.3) we first use (2.10) to obtain

$$(n\alpha_n)^{\frac{1}{1-n}} \frac{r}{n} \frac{dM}{dr} = M\left\{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}\right\},$$

or equivalently

$$\frac{(n\alpha_n)^{\frac{1}{1-n}}}{M\left\{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}\right\}} \frac{dM}{dr} = \frac{n}{r}. \tag{3.6}$$

By noting that

$$\frac{(n\alpha_n)^{\frac{1}{1-n}}}{M\left\{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}\right\}} = \frac{1}{M} + \frac{\varepsilon M^{\frac{1}{n-1}-1}}{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}} \ ,$$

the differential equation (3.6) can explicitly be integrated on the interval (r_0, r). We then derive:

$$\log\left(\frac{M(r)}{M(r_0)}\right) - (n-1)\log\left(\frac{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r)}{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r_0)}\right) = \log\left(\frac{r}{r_0}\right)^n,$$

and therefore

$$M(r)\left(\frac{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r_0)}{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r)}\right)^{n-1} = \left(\frac{r}{r_0}\right)^n M(r_0)\ . \tag{3.7}$$

By sending r_0 to 0 in (3.7) we get

$$M(r)\left(\frac{(n\alpha_n)^{\frac{1}{1-n}}}{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r)}\right)^{n-1} = \frac{\omega}{n} r^n e^{v(0)}, \quad \forall r \in [0, R],$$

which implies

$$\frac{M^{\frac{1}{n-1}}(r)}{(n\alpha_n)^{\frac{1}{1-n}} - \varepsilon M^{\frac{1}{n-1}}(r)} = (\alpha_n \omega)^{\frac{1}{n-1}}\ e^{\frac{v(0)}{n-1}}\ r^{\frac{n}{n-1}}. \tag{3.8}$$

Equation (3.8) readily gives (3.3) and implies furthermore

$$M^{\frac{1}{n-1}}(r) \quad = \quad (n\alpha_n)^{\frac{1}{1-n}} \frac{(\alpha_n \omega e^{v(0)})^{\frac{1}{n-1}} r^{\frac{n}{n-1}}}{1 + \varepsilon(\alpha_n \omega e^{v(0)})^{\frac{1}{n-1}} r^{\frac{n}{n-1}}}.$$

The conclusion (3.4) follows from this last equality. Finally relation (3.5) is a direct consequence of (3.3) applied at $r = R$ together with (3.2). □

Based on our previous results, we have the following classification result :

Proposition 3.2. *The radial solutions to* (3.1) *are given by the* 1*-parameter family:*

$$v(x) = \log \frac{(\alpha_n \omega)^{-1} \mu^{n-1}}{\left(1 + \varepsilon\mu |x|^{\frac{n}{n-1}}\right)^n}, \quad \mu > 0. \tag{3.9}$$

If $\varepsilon = -1$*, the solution exists in* $\bar{B}$ *if and only if*

$$n\alpha_n |B| e^{v(0)} < 1. \tag{3.10}$$

Furthermore the n*-Dirichlet integral of* v *is given by*

$$\int_B |\nabla v|^n = \frac{1}{\alpha_n} \left| \int_1^{1+\varepsilon\mu R^{\frac{n}{n-1}}} \frac{1}{\tau} \left(1 - \frac{1}{\tau}\right)^{n-1} d\tau \right|. \tag{3.11}$$

Proof: Let v be a solution of (3.1). Then by plugging (3.4) in (3.2) we see that v has the form (3.9). Moreover, a straightforward calculation shows that the functions defined by (3.9) solve (3.1), which in radial coordinates reads:

$$-(n-1) \left| \frac{dv}{dr} \right|^{n-2} \left(\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right) = \varepsilon e^v.$$

We can now explicitly calculate the Dirichlet integral $\int_B |\nabla v|^n \, dx$ of any radial solution of (3.1). Using (3.9) we get

$$\frac{dv}{dr} = -\frac{n^2}{n-1} \frac{\varepsilon\mu r^{\frac{n}{n-1}-1}}{1 + \varepsilon\mu r^{\frac{n}{n-1}}}.$$

We then have

$$\int_B |\nabla v|^n \, dx \quad = \quad \omega \left(\frac{n^2}{n-1} \right)^n \int_0^R \left| \frac{\varepsilon\mu r^{\frac{n}{n-1}}}{1 + \varepsilon\mu r^{\frac{n}{n-1}}} \right|^n \frac{dr}{r}. \tag{3.12}$$

To calculate the integral in (3.12), we make the change of variable $t = 1 + \varepsilon\mu r^{\frac{n}{n-1}}$. Then a straight calculation yields

$$\begin{aligned} \int_0^R \left| \frac{\varepsilon\mu r^{\frac{n}{n-1}}}{1 + \varepsilon\mu r^{\frac{n}{n-1}}} \right|^n \frac{dr}{r} &= \frac{n-1}{n} \int_1^{1+\varepsilon\mu R^{\frac{n}{n-1}}} \left| \frac{\tau - 1}{\tau} \right|^n \frac{d\tau}{\tau - 1} \\ &= \frac{n-1}{n} \left| \int_1^{1+\varepsilon\mu R^{\frac{n}{n-1}}} \frac{1}{\tau} \left(1 - \frac{1}{\tau}\right)^{n-1} d\tau \right|. \end{aligned} \tag{3.13}$$

Using (3.12), (3.13) and the definition of α_n in (1.3) we obtain (3.11). □

4 Application to a non-local equation on a ball

Given $f : (0, \infty) \to \mathbb{R}$ and a ball $B = B(0, R)$ in $\mathbb{R}^n$, the results obtained so far will be applied to the non-local problem:

$$-\Delta_n u = \frac{\lambda}{|B|} f\left(\fint_B e^u \, dx \right) e^u, \quad u \in W_0^{1,n}(B), \quad u \text{ radial.} \tag{4.1}$$

When the function f satisfies suitable growth assumptions, the above problem is the Euler-Lagrange equation of the functional (1.8). The present section will be helpful later to study the infimum of this functional.

Note first that the Moser-Trudinger inequality (1.2) and the regularity result of [26] imply that the solutions satisfying (4.1) are $C^1(\bar{B})$. Furthermore, by the maximum principle the solutions of (4.1) cannot change sign, and by applying the strong maximum principle we see that for a solution u of (4.1) the following alternative holds:

$$\begin{cases} (a) \quad \lambda f(\fint_B e^u \, dx) > 0 \quad \text{and} \quad u > 0, \\ (b) \quad \lambda f(\fint_B e^u \, dx) < 0 \quad \text{and} \quad u < 0, \\ (c) \quad \lambda f(\fint_B e^u \, dx) = 0 \quad \text{and} \quad u \equiv 0. \end{cases} \tag{4.2}$$

Note that in the cases (a) and (b) the function

$$v := u + \log \left(\frac{1}{|B|} \left| \lambda f(\fint_B e^u \, dx) \right| \right) \tag{4.3}$$

solves Problem (3.1) with $\varepsilon = \text{sgn}(u)$. We have actually the following result which is a more precise statement of Theorem 1.1 stated in the introduction.

Theorem 4.1. *Let u be a solution of Problem (4.1). Then $\sigma := \fint_B e^u \, dx$ satisfies*

$$n\alpha_n \lambda f(\sigma) = \frac{\varepsilon}{\sigma} \left| 1 - \frac{1}{\sigma} \right|^{n-1} \quad \textit{with} \quad \varepsilon := sgn(\sigma - 1). \tag{4.4}$$

Conversely,

(a) if there exists $\sigma_0 \in (1, \infty)$ satisfying (4.4) *then Problem* (4.1) *admits a positive solution,*

(b) if there exists $\sigma_0 \in (0, 1)$ satisfying (4.4) *then Problem* (4.1) *admits a negative solution,*

(c) if $\sigma_0 = 1$ solves (4.4) then $u \equiv 0$ is a solution of (4.1).

In cases (a), (b) the solution is $u = v - v(R)$ where v is given by (3.9) with $\mu = |\sigma_0 - 1|$.

Proof: Assume Problem (4.1) admits a solution u. If $u \equiv 0$ solves (4.1), then $\sigma = 1$, $f(1) = 0$ and identity (4.4) holds in this case. Assume now $u \not\equiv 0$. Applying (2.3) with $G(s) = g(s) = \frac{\lambda}{|B|} f(\sigma) e^s$ we get

$$\lambda f(\sigma) = \lambda f(\sigma)\sigma - (n\alpha_n)^{\frac{1}{n-1}} |\lambda f(\sigma)\sigma|^{\frac{n}{n-1}}. \tag{4.5}$$

Since $u \not\equiv 0$ equation (4.1) shows that $\lambda f(\sigma) \neq 0$, and therefore (4.5) implies:

$$n\alpha_n \frac{\big|\lambda f(\sigma)\big|^n}{\big(\lambda f(\sigma)\big)^{n-1}} = \frac{1}{\sigma}\left(1 - \frac{1}{\sigma}\right)^{n-1},$$

which is equivalent to

$$n\alpha_n \big|\lambda f(\sigma)\big| = \left(\frac{\big|\lambda f(\sigma)\big|}{\lambda f(\sigma)}\right)^{n-1} \frac{1}{\sigma}\left(1 - \frac{1}{\sigma}\right)^{n-1}. \tag{4.6}$$

Using (4.2) note that $\operatorname{sgn}\big(\lambda f(\sigma)\big) = \operatorname{sgn}(u) = \operatorname{sgn}(\sigma - 1)$. Therefore (4.6) yields:

$$n\alpha_n \varepsilon \lambda f(\sigma) = \frac{1}{\sigma}\left|1 - \frac{1}{\sigma}\right|^{n-1}.$$

Hence (4.4) also holds for $u \not\equiv 0$.

Conversely, assume that (4.4) admits a solution $\sigma_0 \in (0, \infty) \setminus \{1\}$. By setting $\varepsilon := \operatorname{sgn}(\sigma_0 - 1)$ consider a solution v of the problem:

$$-\Delta_n v = \varepsilon e^v, \qquad e^{v(0)} = \frac{|\sigma_0 - 1|^{n-1}}{n\alpha_n |B|}. \tag{4.7}$$

By Proposition 3.2 such a function v exists and $v \in C^1(\bar{B})$ (since condition (3.10) holds). We claim that

$$u(r) := v(r) - v(R) \tag{4.8}$$

solves Problem (4.1). Indeed we have

$$-\Delta_n u = \varepsilon e^{v(R)} e^u, \quad u \in W_0^{1,n}(B),$$

and so we only need to verify that

$$\varepsilon e^{v(R)} = \frac{\lambda}{|B|} f\left(\fint_B e^{[v-v(R)]}\, dx \right). \tag{4.9}$$

Using (3.2) and (3.4), we write $e^{v(R)}$ and $\int_B e^v\, dx$ as a function of $e^{v(0)}$, which can in turn be expressed in terms of σ_0 using (4.7):

$$\varepsilon e^{v(R)} = \frac{\varepsilon e^{v(0)}}{\left(1+\varepsilon\left[n\alpha_n|B|e^{v(0)}\right]^{\frac{1}{n-1}}\right)^n} = \frac{\varepsilon}{n\alpha_n|B|}\frac{1}{\sigma_0}\left|1-\frac{1}{\sigma_0}\right|^{n-1},$$

$$e^{-v(R)}\fint_B e^v = 1+\varepsilon\left[n\alpha_n|B|e^{v(0)}\right]^{\frac{1}{n-1}} = 1+\varepsilon|\sigma_0-1| = \sigma_0.$$

Therefore (4.9) is equivalent to

$$\frac{\varepsilon}{\sigma_0}\left|1-\frac{1}{\sigma_0}\right|^{n-1} = n\alpha_n\lambda f(\sigma_0).$$

Finally if $\sigma_0 = 1$ is a solution of (4.4) then either $\lambda = 0$ or $f(1) = 0$, and in both cases $u \equiv 0$ solves (4.1). □

Theorem 4.1 shows that radial solutions of the Dirichlet Problem (4.1) are completely classified by the real numbers $\sigma_0 > 0$ solving (4.4). For $\sigma_0 \in (0,\infty)\setminus\{1\}$ using (4.7), (4.8) and (3.9), the solutions of (4.1) are explicitly given by:

$$u(x) = n\log\left(\frac{\sigma_0}{1+[\sigma_0-1]|x/R|^{\frac{n}{n-1}}}\right), \quad |x|<R. \tag{4.10}$$

For example in the case

$$f(s) = \frac{1}{s}\left|1-\frac{1}{s}\right|^{k-1}\left(1-\frac{1}{s}\right), \quad k \geq 1, \tag{4.11}$$

the function $u \equiv 0$ trivially solves (4.1) and we can prove:

Proposition 4.2. *Consider Problem* (4.1) *with f given by* (4.11)*.*

(a) For $k = n-1$, Problem (4.1) *admits a non-trivial solution if and only if $\lambda = (n\alpha_n)^{-1}$. In this case the family of positive (resp. negative) solutions is given by* (4.10) *with $\sigma_0 \in (1,\infty)$ (resp. $\sigma_0 \in (0,1)$).*

(b) For $k \neq n-1$, Problem (4.1) *admits positive solutions if and only if*

$$\begin{cases} \textit{either } 1 \leq k < n-1 \textit{ and } \lambda \in (0, (n\alpha_n)^{-1}), \\ \textit{or } k > n-1 \textit{ and } \lambda \in ((n\alpha_n)^{-1}, \infty), \end{cases}$$

and in this case the positive solution is unique.

(c) For $k \neq n-1$, Problems (4.1) *admits negative solutions if and only if $\lambda > 0$, and in this case the negative solution is unique.*

Proof: Assume the existence of a solution $u \not\equiv 0$. For the function f we are considering, equation (4.4) reads:

$$n\alpha_n \lambda = \left|1 - \frac{1}{\sigma}\right|^{n-1-k}, \quad \sigma \in (0, \infty) \setminus \{1\}. \tag{4.12}$$

(a) If $k = n-1$ equation (4.12) is solvable if and only if $\lambda = (n\alpha_n)^{-1}$. In this case any $\sigma \in (0, \infty) \setminus \{1\}$ is a solution, which is positive (reps. negative) if $\sigma > 1$ (resp. $\sigma < 1$).

(b) If $1 \leq k < n-1$, (4.12) admits a solution $\sigma > 1$ (which is unique) if and only if $n\alpha_n\lambda \in (0, 1)$. Whereas for $k > n-1$, (4.12) admits a solution $\sigma > 1$ (again unique) if and only if $n\alpha_n\lambda > 1$.

(c) Finally we easily see that (4.12) admits a solution $\sigma \in (0, 1)$ for each $\lambda > 0$ which is unique whenever $k \neq n-1$.

We conclude by applying Theorem 4.1. □

Another immediate consequence of Theorem 4.1 is the following:

Proposition 4.3. *Assume that the mapping $f_0 : (1, \infty) \to \mathbb{R}$ defined by*

$$s \mapsto f_0(s) := \frac{(n\alpha_n)^{-1}}{sf(s)} \left(1 - \frac{1}{s}\right)^{n-1} \tag{4.13}$$

is strictly monotone. Then Problem (4.1) *admits a positive solution if and only if $\lambda \in Range(f_0)$ and in this case the solution is unique.*

If condition (4.13) is stated on the interval $(0, 1)$ then we get a similar result for the existence of negative solutions. An example of a function f satisfying (4.13) is given by $f(s) = \frac{1}{s}$, for which Problem (4.1) reads

$$-\Delta_n u = \lambda \frac{e^u}{\int_B e^u \, dx}, \quad u \in W_0^{1,n}(B). \tag{4.14}$$

For this nonlinearity f and for each λ equation (4.4) admits the unique solution:

$$\sigma_\lambda = \left(1 - \varepsilon[n\alpha_n|\lambda|]^{\frac{1}{n-1}}\right)^{-1}.$$

Thus Theorem 4.1 shows that Problem (4.14) admits a positive solution (resp. negative) iff $\lambda \in (0, (n\alpha_n)^{-1})$ (resp. $\lambda < 0$). The solution is unique and referring to (4.10), it can explicitly be written in term of the parameter λ:

$$u_\lambda(x) = -n \log\left(1 - \varepsilon\big[n\alpha_n|\lambda|\big]^{\frac{1}{n-1}} + \varepsilon\big[n\alpha_n|\lambda|\big]^{\frac{1}{n-1}} \left|\frac{x}{R}\right|^{\frac{n}{n-1}}\right). \tag{4.15}$$

When $n = 2$ the solution (4.15) becomes

$$u_\lambda(x) = \log\left(\frac{8\pi}{8\pi - \lambda + \lambda|x/R|^2}\right)^2, \tag{4.16}$$

which can already be found in Aly [1] or in Suzuki [[24], Theorem 3.1]. Let us emphasize that these authors obtained (4.16) by making a suitable change of variable, while our method of proof relies on the Pohozaev identity stated in Proposition 2.1 and applies to any dimension $n \geq 2$.

We conclude this section with

Proposition 4.4. *Let u be a solution of Problem (4.1) and set $\sigma := \fint_B e^u\,dx$. Then*

$$\int_B |\nabla u|^n\,dx = \frac{1}{\alpha_n}\left|\int_1^\sigma \frac{1}{\tau}\left(1 - \frac{1}{\tau}\right)^{n-1} d\tau\right|. \tag{4.17}$$

Proof: Consider v defined by (4.3) which solves $-\Delta_n v = \operatorname{sgn}(u)e^v$. By applying (3.3) we easily obtained

$$e^{u(0)} = \sigma^n. \tag{4.18}$$

Formula (4.17) is now obtained from (3.11) if we note that $\mu = \left(\alpha_n \omega e^{v(0)}\right)^{\frac{1}{n-1}}$. Indeed, using successively (4.18) and (4.4), we get

$$\begin{aligned}
\operatorname{sgn}(u)\mu R^{\frac{n}{n-1}} &= \operatorname{sgn}(u)\big(\alpha_n \omega R^n e^{v(0)}\big)^{\frac{1}{n-1}} = \operatorname{sgn}(u)\left(n\alpha_n\big|\lambda f(\sigma)\big|e^{u(0)}\right)^{\frac{1}{n-1}} \\
&= \operatorname{sgn}(u)\big(n\alpha_n\big|\lambda f(\sigma)\big|\sigma^n\big)^{\frac{1}{n-1}} = \operatorname{sgn}(u)\left|1 - \frac{1}{\sigma}\right|\sigma \\
&= \operatorname{sgn}(u)|\sigma - 1| = \sigma - 1.
\end{aligned} \tag{4.19}$$

Plugging (4.19) in (3.11) gives the identity (4.17). □

5 Optimal constants

The aim of this section is to study the infimum of the functional $\mathcal{F}(\lambda, \cdot)$ defined in (1.8). We start by recalling that the Moser-Trudinger inequality (1.1) implies for each $\lambda \in [0, (n\alpha_n)^{-1}]$ the inequality

$$\frac{1}{n}\int_\Omega |\nabla u|^n \, dx \; - \lambda \log\left(\fint_\Omega e^u \, dx\right) \geq -\frac{1}{n\alpha_n} \log C_n, \quad \forall u \in W_0^{1,n}(\Omega), \tag{5.1}$$

where the constant C_n is the one appearing in (1.1). Indeed by using Young's inequality $ab \leq \frac{1}{n}|a|^n + \frac{n-1}{n}|b|^{\frac{n}{n-1}}$, we get

$$\begin{aligned} |u| \; &= \; \underbrace{\left(\frac{n-1}{n\mu_n}\right)^{\frac{n-1}{n}} \|\nabla u\|_n}_{a} \; \underbrace{\left(\frac{n\mu_n}{n-1}\right)^{\frac{n-1}{n}} \frac{|u|}{\|\nabla u\|_n}}_{b} \\ &\leq \; \frac{1}{n}\left(\frac{n-1}{n\mu_n}\right)^{n-1} \|\nabla u\|_n^n + \mu_n \left(\frac{|u|}{\|\nabla u\|_n}\right)^{\frac{n}{n-1}}. \end{aligned}$$

Therefore, since $\mu_n = n\omega^{\frac{1}{n-1}}$ and using the constant α_n defined by (1.3), we deduce

$$\log\left(\fint_\Omega e^u \, dx\right) \; \leq \; \alpha_n \|\nabla u\|_n^n + \log C_n. \tag{5.2}$$

Inequality (5.2) readily implies (5.1) as well as (1.2). We can slightly extend this result as follows:

Proposition 5.1. *Let $\Omega \subset\subset \mathbb{R}^2$ and $F \in C^0(0, \infty)$ satisfying*

$$\limsup_{s \to 0^+} F(s) < \infty \qquad \text{and} \qquad \limsup_{s \to \infty} \{F(s) - \log s\} < \infty. \tag{5.3}$$

Then

(a) $\mathcal{F}(\lambda, \cdot)$ is bounded from below for each $\lambda \in (0, (n\alpha_n)^{-1}]$,

(b) $\mathcal{F}(\lambda, \cdot)$ admits a minimizer for each $\lambda \in (0, (n\alpha_n)^{-1})$,

(c) $\mathcal{F}(\lambda, \cdot)$ is unbounded from below for $\lambda > (n\alpha_n)^{-1}$.

Proof: Using (5.1) it is known that the mapping $W_0^{1,n}(\Omega) \to L^1(\Omega)$, $u \mapsto e^u$ is compact. Claims (a) and (b) follow now easily. To prove the last statement we construct appropriate trial functions. Similarly to what has been done in [22] (on a two-dimensional torus), we consider for each $(a, \mu) \in \mathbb{R}^n \times [1, \infty)$ the functions

$$\delta_{a,\mu}(x) := \log \frac{(\alpha_n \omega)^{-1} \mu^{n-1}}{\left(1 + \mu |x - a|^{\frac{n}{n-1}}\right)^n}. \tag{5.4}$$

As a consequence of Proposition 3.2 these functions are radial solutions of $-\Delta_n v = e^v$ on $\mathbb{R}^n$, and by letting $r \to \infty$ in (3.4) we see

$$\int_{\mathbb{R}^n} e^{\delta_{a,\mu}(x)}\, dx = (n\alpha_n)^{-1}. \tag{5.5}$$

Given a fixed ball $B(a,r) \subset\subset \Omega$, set $c_{a,\mu}$ to be the value of $\delta_{a,\mu}$ on $\partial B(a,r)$ and define the function

$$\widetilde{\delta}_{a,\mu}(x) := \begin{cases} \delta_{a,\mu}(x) - c_{a,\mu} & \text{if } x \in B(a,r), \\ 0 & \text{otherwise.} \end{cases}$$

Using (5.5) and (3.11), we deduce

$$\log\left(⨍_\Omega e^{\widetilde{\delta}_{a,\mu}}\, dx\right) = \log\mu + O(1), \tag{5.6}$$

$$\int_\Omega |\nabla \widetilde{\delta}_{a,\mu}|^n\, dx = n(n\alpha_n)^{-1}\log\mu + O(1). \tag{5.7}$$

In particular, we have

$$\frac{1}{n}\int_\Omega |\nabla u|^n\, dx - \lambda \log\left(⨍_\Omega e^u\, dx\right) = \big((n\alpha_n)^{-1} - \lambda\big)\log\mu + O(1). \tag{5.8}$$

Hence as $\lambda > (n\alpha_n)^{-1}$ we see that (5.8) tends to $-\infty$ for $\mu \to \infty$. $\square$

When the domain is a ball B, the energy $\mathcal{F}(\lambda, u)$ of any radially symmetric critical point u can be expressed as a function of $\sigma := ⨍_B e^u\, dx$. Indeed by applying (4.17) and (4.4) we have

$$\begin{aligned} \mathcal{F}(\lambda,u) &= (n\alpha_n)^{-1}\left|\int_1^\sigma \left(1-\frac{1}{\tau}\right)^{n-1}\frac{d\tau}{\tau}\right| - \lambda F(\sigma) \\ &= (n\alpha_n)^{-1}\left\{\left|\int_1^\sigma \left(1-\frac{1}{\tau}\right)^{n-1}\frac{d\tau}{\tau}\right| - \operatorname{sgn}(\sigma-1)\frac{\left|1-\frac{1}{\sigma}\right|^{n-1}}{\sigma f(\sigma)}F(\sigma)\right\}. \end{aligned} \tag{5.9}$$

We first consider the case when $F(s) = \int_1^s \frac{1}{\tau}\left(1-\frac{1}{\tau}\right)^{n-1} d\tau$.

Proof of Theorem 1.2:
Proposition 4.2 applied to $f(s) = \frac{1}{s}\left(1-\frac{1}{s}\right)^{n-1}$ shows that

(i) for $\lambda \neq (n\alpha_n)^{-1}$ the function $u \equiv 0$ is the unique non-negative radial critical point of $\mathcal{F}(\lambda, \cdot)$,

(ii) $\mathcal{F}\big([n\alpha_n]^{-1}, \cdot\big)$ admits a family of radial critical points $u_\mu > 0$ and (5.9) implies

$$\mathcal{F}\big([n\alpha_n]^{-1}, u_\mu\big) = 0. \tag{5.10}$$

To prove (1.12) we define

$$\widetilde{F}(s) = \begin{cases} \int_1^s \frac{1}{\tau}\left(1 - \frac{1}{\tau}\right)^{n-1} d\tau & \text{if } s \geq 1, \\ 0 & \text{if } s \in (0,1), \end{cases} \tag{5.11}$$

and denote by $\widetilde{\mathcal{F}}(\lambda, \cdot)$ the functional associated with (5.11). By Proposition 5.1, $\widetilde{\mathcal{F}}(\lambda, \cdot)$ admits a a minimizer u_λ for each $\lambda \in [0, (n\alpha_n)^{-1})$. We check easily that $u_\lambda \geq 0$ and therefore using Schwarz symmetrization we deduce that u_λ must be radial. Then Proposition 4.2 shows that $u_\lambda \equiv 0$. Hence for each $\lambda < (n\alpha_n)^{-1}$ we get

$$\widetilde{\mathcal{F}}(\lambda, u) \geq \widetilde{\mathcal{F}}(\lambda, u_\lambda) \geq 0.$$

By considering $\lambda \uparrow (n\alpha_n)^{-1}$ together with (5.10) we get (1.12). □

When F satisfies the requirements (1.9) and (1.13) we can also calculate the infimum of the associated functional $\mathcal{F}\big((n\alpha_n)^{-1}, \cdot\big)$:

Proof of Theorem 1.3:
Note first that our assumptions readily imply $f > 0$ and we claim:

$$L := \liminf_{s \to \infty} \left\{ \big(sf(s) - 1\big) \log s \right\} = 0, \tag{5.12}$$

$$\lim_{s \to \infty} sf(s) = \lim_{s \to \infty} \frac{(1 - \frac{1}{s})^{n-1}}{sf(s)} = 1. \tag{5.13}$$

To prove (5.12) assume $L > 0$. Then for $a > 1$ large enough we have:

$$\begin{aligned} F(s) - \log s &= F(a) - \log a + \int_a^s \frac{\big(tf(t) - 1\big)\log t}{t \log t} dt \\ &> F(a) - \log a + \frac{L}{2} \int_a^s \frac{dt}{t \log t}, \quad \forall s > a. \end{aligned}$$

The last inequality would then imply $\lim_{s \to \infty} \{F(s) - \log s\} = +\infty$, in contradiction to our assumption (1.9). If $L < 0$ we get a similar contradiction. Hence $L = 0$

and (5.12) is established. Concerning (5.13), we first note that both functions appearing in (5.13) admit and have the same limit as $s \to \infty$. Indeed

$$\lim_{s\to\infty} sf(s) = \lim_{s\to\infty} \frac{sf(s)}{\left(1-\frac{1}{s}\right)^{n-1}} \left(1-\frac{1}{s}\right)^{n-1} = \lim_{s\to\infty} \frac{sf(s)}{\left(1-\frac{1}{s}\right)^{n-1}},$$

and this latter limit exists due to (1.13) since the function $sf(s)\left(1-\frac{1}{s}\right)^{1-n}$ is decreasing and positive. Using (5.12) we immediately conclude $\lim_{s\to\infty} sf(s) = 1$, and (5.13) follows.

Let $\lambda < (n\alpha_n)^{-1}$ and u_λ be a minimizer of the functional $\mathcal{F}(\lambda, \cdot)$ (which exists by Proposition 5.1). Using Schwarz symmetrization (see [12]), we may assume without loss of generality that Ω is a ball B and u_λ is a radial positive function. Clearly for any $u \in W_0^{1,n}(B)$ we have

$$\mathcal{F}\big((n\alpha_n)^{-1}, u\big) = \liminf_{\lambda\uparrow(n\alpha_n)^{-1}} \mathcal{F}(\lambda, u) \geq \liminf_{\lambda\uparrow(n\alpha_n)^{-1}} \mathcal{F}(\lambda, u_\lambda). \tag{5.14}$$

By setting $\sigma = \sigma(\lambda) := \fint_B e^{u_\lambda}\,dx$, we remind the reader now that the value $\mathcal{F}(\lambda, u_\lambda)$ can be written in terms of σ as in (5.9). Hence

$$\begin{aligned} n\alpha_n\mathcal{F}\big(\lambda, u_\lambda\big) &= \int_1^\sigma \frac{1}{\tau}\left(1-\frac{1}{\tau}\right)^{n-1} d\tau - \frac{\left(1-\frac{1}{\sigma}\right)^{n-1}}{\sigma f(\sigma)} F(\sigma) \\ &= \int_1^\sigma \left\{\left(1-\frac{1}{\tau}\right)^{n-1} - 1\right\}\frac{d\tau}{\tau} - \left\{\frac{\left(1-\frac{1}{\sigma}\right)^{n-1}}{\sigma f(\sigma)} F(\sigma) - \log\sigma\right\} \\ &= \int_0^{1-\frac{1}{\sigma}} \frac{s^{n-1}-1}{1-s}\,ds - \left\{\frac{\left(1-\frac{1}{\sigma}\right)^{n-1}}{\sigma f(\sigma)} F(\sigma) - \log\sigma\right\}. \end{aligned} \tag{5.15}$$

Since the value σ satisfies (4.4), the monotonicity assumption (1.13) with the property (5.13) imply that $\lim_{\lambda\uparrow(n\alpha_n)^{-1}} \sigma(\lambda) = \infty$. Therefore from (5.14) and (5.15) we get

$$\begin{aligned} n\alpha_n\mathcal{F}\big((n\alpha_n)^{-1}, u\big) &\geq -\sum_{k=1}^{n-1}\frac{1}{k} + \liminf_{\lambda\uparrow(n\alpha_n)^{-1}} \left\{\log\sigma - \frac{\left(1-\frac{1}{\sigma}\right)^{n-1}}{\sigma f(\sigma)} F(\sigma)\right\} \\ &= -\sum_{k=1}^{n-1}\frac{1}{k}, \end{aligned} \tag{5.16}$$

where the last equality follows by using (5.12) and (1.9). With (5.16) the proof of Theorem 1.3 follows. □

Remark 5.2. *In the particular case when $F(s) = \log s$, Proposition 4.4 readily shows that any solution (λ, u_λ) of* (4.14) *with $\lambda > 0$ satisfies*

$$\sigma = \fint_B e^{u_\lambda}\, dx = \left(1 - [n\alpha_n\lambda]^{\frac{1}{n-1}}\right)^{-1}. \tag{5.17}$$

In this case, by using (5.9)*, $\mathcal{F}(\lambda, u_\lambda)$ can be expressed as a function of λ:*

$$\begin{aligned}\mathcal{F}(\lambda, u_\lambda) &= \frac{1}{n}\int_B |\nabla u_\lambda|^n\, dx \;-\; \lambda \log\left(\fint_B e^{u_\lambda}\, dx\right)\\ &= -(n\alpha_n)^{-1}\left\{(1 - n\alpha_n\lambda)\log\left(1 - [n\alpha_n\lambda]^{\frac{1}{n-1}}\right) + \sum_{k=1}^{n-1}\frac{[n\alpha_n\lambda]^{\frac{k}{n-1}}}{k}\right\}.\end{aligned}$$

In dimension $n = 2$ this result can be found in [1].

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