The p-Laplace eigenvalue problem as $p \to \infty$ in a Finsler metric*

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Abstract

We consider the p–Laplacian operator on a domain equipped with a Finsler metric. We recall relevant properties of its first eigenfunction for finite p and investigate the limit problem as $p \to \infty$.

Keywords: p-Laplace, eigenfunction, Finsler metric

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1 Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain Ω. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p \, dx$, then a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

on $W_0^{1,p}(\Omega)$ satisfies the Euler-Lagrange equation

$$-\Delta_p u = \lambda_p \, |u|^{p-2} u \quad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the well-known p–Laplace operator. This eigenvalue problem has been extensively studied in the literature. A somewhat surprising recent result is that (as $p \to \infty$) the limit equation reads

$$\min \{ \; |\nabla u| - \Lambda_\infty u, \; -\Delta_\infty u \; \} = 0. \tag{1.2}$$

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Here $\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i x_j}$, $\Lambda_\infty = \lim_{p\to\infty} \Lambda_p$ and $\Lambda_p = \lambda_p^{1/p}$ (see [19, 14]). Although the function $d(x, \partial\Omega)$ minimizes $||\nabla u||_\infty / ||u||_\infty$, it is not always a viscosity solution of (1.2), see [19].

Now suppose that the membrane is not isotropic. It is for instance woven out of elastic strings like a piece of material. Then the deformation energy can be anisotropic, see [5]. Another way to describe this effect is by stating that the Euclidean distance in Ω is somehow distorted. It is the purpose of the present paper to generalize the result on eigenfunctions for the p-Laplacian to the situation, where $\Omega \subset \mathbb{R}^n$ is no longer equipped with the Euclidean norm, but instead with a general norm $|\cdot|$, for instance with $|x| = (\sum_{i=1}^n |x_i|^q)^{1/q}$ and $q \in (1, \infty)$. In that case a Lipschitz continuous function $u : \Omega \mapsto \mathbb{R}$ (in a convex domain Ω) has Lipschitz constant $L = \sup_{z\in\Omega} |\nabla u(z)|^*$, where $|\cdot|^*$ denotes the dual norm to $|\cdot|$, because $|u(x) - u(y)| \le L\ |x - y|$ with this L. In order to give a meaningful definition of viscosity solutions, we assume throughout the paper that the dual norm $H : \mathbb{R}^n \mapsto [0, \infty)$ defined by $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$.

It is well-known, that the infinite-Laplacian operator Δ_∞ is closely related to finding a minimal Lipschitz extension of a given function $\phi \in C^{0,1}(\partial\Omega)$ into Ω. In [2] this result on minimal Lipschitz extensions was generalized from the Euclidean to a general norm, see also [26]. In [6] the eigenvalue problem was carried over to a general norm and studied for finite p, while in [5] the eigenvalue problem was investigated first for finite p and the special non-euclidean norm $|x| = (\sum_{i=1}^n |x_i|^{p'})^{1/p'}$ with p' conjugate to p, and then for the limit $p \to \infty$.

Moreover, the infinite-Laplacian operator plays an important role in problems of optimal transportation. For technical reasons it is often approximated by p–Laplacians with large p, see for instance [13], [8].

Our paper is organized as follows. In Section 2 we recall the existence, uniqueness and regularity of weak and viscosity solutions for finite p. In Section 3 we derive the limit equation for $p \to \infty$. In Section 4 we provide some instructive examples.

2 Existence, uniqueness and regularity of solutions

If we minimize the functional

$$I_p(v) = \int_\Omega (|\nabla u|^*)^p\ dx \quad \text{on} \quad K := \{\ v \in W_0^{1,p}(\Omega)\ |\ ||v||_{L^p(\Omega)} = 1\ \}, \tag{2.1}$$

then via standard arguments (see [6]) a minimizer u_p exists for every $p > 1$ and it is a weak solution to the equation

$$-Q_p u := -\mathrm{div}\left((|\nabla u_p|^*)^{p-2} J(\nabla u_p)\right) = \lambda_p |u_p|^{p-2} u_p \ , \tag{2.2}$$

that is

$$\int_\Omega (|\nabla u_p|^*)^{p-2} \langle J(\nabla u_p), \nabla v\rangle \ dx = \lambda_p \int_\Omega |u_p|^{p-2} u \cdot v \ dx \tag{2.3}$$

for any $v \in W_0^{1,p}(\Omega)$. Here $\lambda_p = I_p(u_p)$ and

$$J_i(\xi) := \frac{\partial}{\partial \xi_i}\left(\frac{(|\xi|^*)^2}{2}\right) \ . \tag{2.4}$$

Clearly (2.4) is well defined as long as the dual norm $H(\eta) = |\eta|^*$ is of class $C^1(\mathbb{R}^n \setminus \{0\})$. Recall that (2.4) is well defined (and single valued) if and only if the norm $|\cdot|$ is strictly convex, i.e. if its unit sphere $\{x : \ |x| = 1\}$ contains no nontrivial line segments, see [27] p.400. Note further that in this case $J(0) = 0$ and that for the Euclidean norm the duality map reduces to the identity $J(\nabla u) = \nabla u$. Note finally that $\Lambda_p := \lambda_p^{1/p}$ is the minimum of the Rayleigh quotient

$$R_p(v) := \frac{\left(\int_\Omega (|\nabla v|^*)^p \ dx\right)^{1/p}}{||v||_p} \tag{2.5}$$

on $W_0^{1,p}(\Omega) \setminus \{0\}$. Without loss of generality we may assume that u_p is nonnegative. Otherwise we can replace it by its modulus.

Moreover as shown in [6] any nonnegative weak solution of (2.3) is necessarily bounded and positive in Ω. If $p > n$, then u_p is Hölder continuous because of the Sobolev-embedding theorem and the equivalence of the usual Sobolev norm with

$$||u||_{1,p} := \left(\int_\Omega |u(x)|^p \ dx\right)^{1/p} + \left(\int_\Omega (|\nabla u(x)|^*)^p \ dx\right)^{1/p} \ . \tag{2.6}$$

But even for general $p \geq 2$, one can show its $C^{1,\alpha}$ regularity as in [6]. For the reader's convenience let us briefly repeat the arguments. The function u_p minimizes I_p in (2.1) and the theory for quasiminima in [15] implies that minimizers of I_p are bounded (Thm. 7.5), Hölder continuous (Thm. 7.6) and satisfy a strong maximum principle (Thm. 7.12). Therefore u_p is positive. Once positivity is known, the uniqueness follows from a simple convexity argument, see [4] or [6]. Moreover $u_p \in C^{1,\alpha}(\Omega)$ according to [24],[25] or [12]. Let us summarize these statements.

Theorem 2.1 *Suppose that $H(\eta) = |\eta|^*$ is of class $C^1(\mathbb{R}^n \setminus \{0\})$ or that the norm $|\cdot|$ is strictly convex. Then for every $p \in [2,\infty)$, the nonnegative minimizer u_p of (2.1) is unique, positive and of class $C^{1,\alpha}$. It solves (2.2) in the weak sense of (2.3).*

The next item will be viscosity solutions. As in [19] and [5] we plan to show that every weak solution is a viscosity solution. For every $z \in \mathbb{R}$, $q \in \mathbb{R}^n$ and for every real symmetric $n \times n$ matrix X we consider the function

$$\begin{aligned}\tilde{F}_p(z,\xi,X) = &-(p-2)\,(|\xi|^*)^{p-4}\,\langle XJ(\xi), J(\xi)\rangle \\ &-(|\xi|^*)^{p-2}\,X \otimes DJ(\xi) - \lambda_p |z|^{p-2} z.\end{aligned}$$

where $X \otimes DJ(\xi)$ is shorthand for $\sum_{i,j=1}^n X_{ij} \frac{\partial J_i}{\partial \xi_j}(\xi)$. Now $(|\xi|^*)^2/2$ is convex and homogeneous of degree 2 and its first derivative $J(\xi)$ is homogeneous of degree 1. Therefore its second derivative $DJ(\xi)$ exists almost everywhere and is essentially bounded. If we assume that $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$, then DJ is well-defined and continuous outside the origin, so that $\tilde{F}_p$ is well-defined and continuous for $\xi \neq 0$. To define F_p at $\xi = 0$ we use the homogeneity of the norm $|\cdot|^*$ and see that for any $t > 0$ and $\xi \neq 0$

$$J(t\xi) = tJ(\xi) \qquad \text{implies} \qquad DJ(\xi) = DJ(t\xi)\,.$$

So if we assume that the dual norm is of class C^2 outside the origin, then one easily sees that for $p > 2$ the function

$$\begin{aligned}\tilde{F}_p = \; & -(|\xi|^*)^{p-2}\left[(p-2)\left\langle XJ\left(\frac{\xi}{|\xi|^*}\right), J\left(\frac{\xi}{|\xi|^*}\right)\right\rangle + X \otimes DJ(\xi)\right] + \\ & -\lambda_p |z|^{p-2} z \end{aligned} \tag{2.7}$$

has a continuous extension to $\xi = 0$. So now we can define

$$F_p(z,\xi,X) := \begin{cases} \tilde{F}_p(z,\xi,X) & \text{if } \xi \neq 0, \\ -\lambda_p |z|^{p-2} z & \text{if } \xi = 0, \end{cases} \tag{2.8}$$

and the upper and lower semicontinuous envelopes F_p^* and F_{p*} of F_p coincide with F_p for $p > 2$. Notice that the case $p = 2$ is more delicate, because $\tilde{F}_2(z,\xi,X) = X \otimes DJ(\xi) - \lambda_2 z$ is not continuous at $\xi = 0$. This problem was overcome in [23] for $p \in (1,2)$ by multiplying F_p with $|\nabla u|$ and by studying the modified differential equation, but since we are interested in the limit $p \to \infty$ we do not investigate the range $p \in (1,2]$ any further.

Definition 2.2 Let F_p be as in (2.8). We call $u \in C(\Omega)$ a viscosity subsolution (resp. supersolution) of $F_p = 0$ if

$$F_p(\phi(x), D\phi(x), D^2\phi(x)) \leq 0 \quad (\text{ resp. } F_p(\phi(x), D\phi(x), D^2\phi(x)) \geq 0) \quad (2.9)$$

for every $\phi \in C^2(\Omega)$ with $u - \phi$ attaining a local maximum (resp. minimum) zero at x. We call u a viscosity solution of $F_p = 0$ if it is both a viscosity subsolution and a viscosity supersolution.

Lemma 2.3 *Suppose that $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. Then for $p > 2$ every (weak) solution of (2.3) is a viscosity solution of $F_p = 0$ with F_p given by (2.8).*

For the proof we omit the subscript p on u_p and check first if u is a viscosity subsolution. Without loss of generality fix $x_0 \in \Omega$ and choose $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$ for $x \neq x_0$. We want to show that

$$\begin{aligned} &-(p-2)\,(|\nabla\phi(x_0)|^*)^{p-4}\,\langle D^2\phi(x_0)J(\nabla\phi(x_0)), J(\nabla\phi(x_0))\rangle \\ &-(|\nabla\phi(x_0)|^*)^{p-2}\,D^2\phi(x_0) \otimes DJ(\nabla\phi(x_0)) - \lambda_p|\phi(x_0)|^{p-2}\phi(x_0) \leq 0 \end{aligned} \quad (2.10)$$

and argue by contradiction. Otherwise there exists a small ball $B_r(x_0)$, in which (2.10) is violated. Set $M = \sup\{\phi(x) - u(x) \mid x \in \partial B_r(x_0)\}$ and $\Phi = \phi - M/2$. Then $\Phi > u$ on $\partial B_r(x_0)$, $\Phi(x_0) < u(x_0)$ and

$$\begin{aligned} &-(p-2)\,(|\nabla\Phi|^*)^{p-4}\,\langle D^2\Phi J(\nabla\Phi), J(\nabla\Phi)\rangle \\ &-(|\nabla\Phi|^*)^{p-2}\,D^2\Phi \otimes DJ(\nabla\Phi) > \lambda_p\,|\phi|^{p-2}\,\phi \qquad \text{in } B_r(x_0). \end{aligned} \quad (2.11)$$

If we multiply (2.11) by $(u - \Phi)^+$ and integrate by parts, we obtain

$$\int_{\{u>\Phi\}} (|\nabla\Phi|^*)^{p-2}\,\langle J(\nabla\Phi), \nabla(u - \Phi)\rangle\,dx > \lambda_p \int_{\{u>\Phi\}} |\phi|^{p-2}\phi(u - \Phi)\,dx. \quad (2.12)$$

Now we exploit the fact that u is a weak solution of (2.3) and pick $v = (u-\Phi)^+$, extended by zero outside $B_r(x_0)$, as a test function in (2.3). Then

$$\int_{\{u>\Phi\}} (|\nabla u|^*)^{p-2}\,\langle J(\nabla u), \nabla(u - \Phi)\rangle\,dx = \lambda_p \int_{\{u>\Phi\}} |u|^{p-2}u(u - \Phi)\,dx. \quad (2.13)$$

Subtracting (2.12) from (2.13) we obtain

$$\begin{aligned} &\int_{\{u>\Phi\}} \langle[(|\nabla u|^*)^{p-2}\,J(\nabla u) - (|\nabla\Phi|^*)^{p-2}\,J(\nabla\Phi)], \nabla(u - \Phi)\rangle dx \\ &< \lambda_p \int_{\{u>\Phi\}} (|u|^{p-2}u - |\phi|^{p-2}\phi)(u - \Phi)\,dx. \end{aligned} \quad (2.14)$$

But the right hand side of (2.14) is nonpositive, while the left hand side is nonnegative because the functional $\int (|\nabla v|^*)^p \, dx$ is convex in v. So $u(x_0) \leq \Phi(x_0)$, a contradiction to $\Phi(x_0) < u(x_0)$. This proves that u is a viscosity subsolution. The proof that u is also a viscosity supersolution is left to the reader.

Note, that as a byproduct of this proof, there are no admissible test functions ϕ that touch u_p in a critical point from below. This shows that u_p is not of class C^2.

3 The limit eigenvalue equation for $p \to \infty$

In this chapter we study the sequence (Λ_p, u_p) of eigenvalues and normalized eigenfunctions as $p \to \infty$. In particular we will derive the equation which is satisfied by the cluster points u_∞ of u_p. Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$. The distance function to the boundary $\delta(x) := \inf_{y \in \partial\Omega} |x - y|$ is Lipschitz continuous, satisfies $|\nabla \delta(x)|^* = 1$ almost everywhere in Ω and it is equal to zero on the boundary of Ω. We have then for every $\varphi \in W_0^{1,\infty}(\Omega)$ and $y \in \partial\Omega$

$$|\varphi(x)| = |\varphi(x) - \varphi(y)| \leq || \, |\nabla \varphi|^* ||_\infty \delta(x),$$

which implies

$$\frac{1}{||\delta||_\infty} \leq \frac{|| \, |\nabla \varphi|^* ||_\infty}{||\varphi||_\infty}. \tag{3.1}$$

Now let us define

$$\Lambda_\infty := \frac{|| \, |\nabla \delta|^* ||_\infty}{||\delta||_\infty} \left(= \frac{1}{||\delta||_\infty} \right). \tag{3.2}$$

Therefore Λ_∞ is a geometric quantity related to Ω. It is the inverse of the radius of the largest (in general non-Euclidean) ball inside Ω. We can now prove the following Lemma, which explains the analytic meaning of Λ_∞.

Lemma 3.1 *The following limit holds*

$$\left(\lim_{p \to \infty} \lambda_p^{1/p} = \right) \lim_{p \to \infty} \Lambda_p = \Lambda_\infty.$$

Here $\Lambda_p = R_p(u_p)$ and the Rayleigh quotient R_p is given by (2.5).

From the definition of the Rayleigh quotient and $\delta(x)$ we get

$$\Lambda_p \leq \frac{|\Omega|^{1/p}}{||\delta||_p}$$

which implies

$$\limsup_{p\to\infty} \Lambda_p \leq \Lambda_\infty.$$

In order to obtain the opposite inequality, we observe that $||\nabla u_p||_p \leq C < \infty$ uniformly in p, because $\delta(x)$ can be used as a test function in any of the Rayleigh quotients. But then (see also [7] and [19]) Hölder's inequality allows us to conclude that $||\nabla u_p||_m \leq C < \infty$ for $p > m > n$. We can thus select a subsequence (still denoted by $\{u_p\}$) converging strongly in C^α and weakly in $W^{1,m}$ to a cluster point u_∞ of the original sequence. Without loss of generality we may assume that each u_p has L^∞ norm 1. Then by the convergence in C^α, $\lim u_p = u_\infty$ has L^∞ norm 1 and positive L^m-norm. From the lower semicontinuity of the Rayleigh quotient we get now

$$\frac{\left(\int_\Omega (|\nabla u_\infty|^*)^m \, dx\right)^{1/m}}{||u_\infty||_m} \leq \liminf_{p\to\infty} \frac{\left(\int_\Omega (|\nabla u_p|^*)^m \, dx\right)^{1/m}}{||u_p||_m}$$

Multiplying and dividing the last inequality by $||u_p||_p$, we get by Hölder's inequality that for $p > m$ we have

$$\frac{\left(\int_\Omega (|\nabla u_\infty|^*)^m \, dx\right)^{1/m}}{||u_\infty||_m} \leq \liminf_{p\to\infty} \left(\Lambda_p \frac{||u_p||_p}{||u_p||_m} |\Omega|^{\frac{p-m}{pm}}\right).$$

By taking first the limit in p and next the limit in m and using (3.1) we conclude that $\Lambda_\infty \leq \liminf_{p\to\infty} \Lambda_p$, which completes the proof of the Lemma.

Before we derive the limit equation, which a nontrivial cluster point u_∞ of the sequence u_p must satisfy, let us show that u_∞ is positive in Ω. The functions u_p are viscosity supersolutions of $H_p(\nabla u, D^2 u) = 0$, where

$$H_p(\xi, X) := -\langle XJ(\xi), J(\xi)\rangle - \frac{(|\xi|^*)^2}{p-2} X \otimes DJ(\xi)$$

is elliptic and continuous for $p > 2$ by assumption. Therefore by a well-known stability theorem [10] supersolutions converge to a supersolution of the limiting problem, i.e, to a supersolution u_∞ of the equation

$$H_\infty(\xi, X) = -\langle XJ(\xi), J(\xi)\rangle = 0$$

in the viscosity sense. As we saw above $u_\infty \not\equiv 0$. Now the positivity of u_∞ follows from a comparison result of Barles and Busca, see [3], Lemma 3.2.

Theorem 3.2 *If $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$ then every cluster point u_∞ of the sequence $\{u_p\}$ is a viscosity solution of the equation*

$$F_\infty(u, \nabla u, D^2 u) = \min\ \{\ |\nabla u|^* - \Lambda_\infty u,\ -Q_\infty u\} = 0$$

with $Q_\infty u = \langle D^2 u J(\nabla u), J(\nabla u)\rangle$ representing the infinite-Laplacian in the Finsler metric.

We show first the result for viscosity supersolutions. We consider a subsequence $\{u_p\}$ converging uniformly in Ω to a function u_∞. Let us fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) > \varphi(x)$ for $x \neq \xi$. Also fix $B_{2R}(\xi) \subseteq \Omega$. If $0 < r < R$ we have

$$\inf\{u_\infty(x) - \varphi(x) \mid x \in B_R(\xi) \backslash B_r(\xi)\} > 0.$$

The sequence $\{u_p\}$ converges uniformly, so for sufficiently large p we have

$$\inf\{u_p(x) - \varphi(x) \mid x \in B_R(\xi) \backslash B_r(\xi)\} > u_p(\xi) - \varphi(\xi).$$

For those p we have

$$\inf\{u_p(x) - \varphi(x) \mid x \in B_R(\xi)\} = u_p(x_p) - \varphi(x_p)$$

with $x_p \in B_r(\xi)$, and obviously $x_p \to \xi$ when $p \to \infty$. The function u_p is a viscosity solution of (2.2), therefore

$$\begin{aligned} &-(p-2)\left(|\nabla\varphi(x_p)|^*\right)^{p-4} \langle D^2\varphi(x_p) J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle \\ &-\left(|\nabla\varphi(x_p)|^*\right)^{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \geq \Lambda_p^p |\varphi(x_p)|^{p-2}\varphi(x_p)\ . \end{aligned} \tag{3.3}$$

Now $u_\infty(\xi) > 0$, but then also $\varphi(x_p) > 0$ for sufficiently large p and by (3.3) $\nabla\varphi(x_p) \neq 0$ for large p. Dividing both members of (3.3) by the term $(p-2)\left(|\nabla\varphi(x_p)|^*\right)^{p-4}$ we obtain

$$\begin{aligned} -\langle D^2\varphi(x_p) J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle - \tfrac{(|\nabla\varphi(x_p)|^*)^2}{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \\ \geq \tfrac{\Lambda_p^4 |\varphi(x_p)|^3}{p-2} \left(\tfrac{|\varphi(x_p)|\Lambda_p}{|\nabla\varphi(x_p)|^*}\right)^{p-4}. \end{aligned} \tag{3.4}$$

Let us take the limit for $p \to \infty$ in (3.4). We obtain the following necessary condition:

$$\frac{\Lambda_\infty \varphi(\xi)}{|\nabla\varphi(\xi)|^*} \leq 1, \tag{3.5}$$

and taking into account (3.5), letting $p \to \infty$ in (3.4) we obtain

$$-Q_\infty\varphi(\xi) = -\langle D^2\varphi(\xi) J(\nabla\varphi(\xi)), J(\nabla\varphi(\xi))\rangle \ \geq 0\ . \tag{3.6}$$

Inequalities (3.5) and (3.6) must hold together, and therefore the cluster points u_∞ of the sequence u_p must satisfy, in the viscosity sense, the following equation

$$\min\ \{\ |\nabla u(\xi)|^* - \Lambda_\infty u(\xi),\ -Q_\infty u(\xi)\} \geq 0. \tag{3.7}$$

This shows that u_∞ is a viscosity supersolution of

$$F_\infty(u, \nabla u, D^2 u) = \min\ \{\ |\nabla u|^* - \Lambda_\infty u,\ -Q_\infty u\} = 0.$$

Let us run the proof for subsolutions. Fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) < \varphi(x)$ for $x \neq \xi$. We have to show that

$$\min \{ \ |\nabla u(\xi)|^* - \Lambda_\infty u(\xi), \ -Q_\infty u(\xi)\} \leq 0.$$

Clearly if $|\nabla u(\xi)|^* - \Lambda_\infty u(\xi) \leq 0$, then there is nothing to prove. Therefore we assume $|\nabla u(\xi)|^* - \Lambda_\infty u(\xi) > 0$, i.e.

$$\frac{\varphi(\xi)\Lambda_\infty}{|\nabla\varphi(\xi)|^*} < 1 - \varepsilon. \tag{3.8}$$

By continuity, this inequality remains true (for every sufficiently large p) if Λ_∞ is replaced by Λ_p and ξ by x_p, and x_p is now the maximum point of $u_p(x) - \varphi(x)$. As in the supersolution case, repeating step by step the proof but reversing the inequality between left and right member, we get

$$\begin{aligned} -\langle D^2\varphi(x_p)J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle - \tfrac{(|\nabla\varphi(x_p)|^*)^2}{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \\ \leq \tfrac{\Lambda_p^4 \varphi(x_p)^3}{p-2} \left(\tfrac{|\varphi(x_p)|\Lambda_p}{|\nabla\varphi(x_p)|^*}\right)^{p-4}. \end{aligned} \tag{3.9}$$

Letting $p \to \infty$ and taking into account (3.8) we get

$$-Q_\infty \varphi(\xi) \leq 0,$$

which ends the proof.

We do not know how to prove uniqueness of solutions to the Dirichlet problem for $F_\infty(u, \nabla u, D^2u) = 0$, but as in [19], we are able to obtain a comparison result. In the setting of viscosity solutions given in [11], the function F_∞ is degenerate elliptic but not proper. Therefore the standard theory cannot be applied directly. The strict positivity of u_p for $1 < p \leq \infty$ allows us to consider in place of $F_\infty(u, \nabla u, D^2u) = 0$ a new equation satisfied by $w_\infty = \log u_\infty$ (see [5], [19]). Let us write

$$G_\infty(\nabla w, D^2 w) = 0, \tag{3.10}$$

where

$$G_\infty(\nabla w, \ D^2w) := \min \ \left\{ \ |\nabla w|^* - \Lambda_\infty, \ -Q_\infty w - (|\nabla w|^*)^4 \right\}$$

and Q_∞ is defined as before. We claim that if u is a viscosity supersolution (subsolution) of $F_\infty(u, \ \nabla u, \ D^2u) = 0$, then $w = \log u$ is a viscosity supersolution (subsolution) $G_\infty(\nabla w, \ D^2w) = 0$. Let us take $\xi \in \Omega$ and $\varphi \in C^2$ such that $\varphi(\xi) = w(\xi)$ and $\varphi(x) < w(x)$ for $x \neq \xi$. The function $\theta(x) = e^{\varphi(x)}$ is a good test function for u at ξ. Then we have

$$\min \ \{ \ |\nabla\theta(\xi)|^* - \Lambda_\infty\theta(\xi), \ -Q_\infty\theta(\xi) \ \} \geq 0.$$

We write the last inequality in terms of $\varphi(x)$ as

$$\min \left\{ e^{\varphi} \left(|\nabla\varphi|^* - \Lambda_\infty\right)(\xi),\ -e^{3\varphi} \left(Q_\infty \varphi + \langle \nabla\varphi, J(\nabla\varphi)\rangle^2\right)(\xi) \right\} \geq 0,$$

and the claim follows from the observation that $\langle y, J(y)\rangle = (|y|^*)^2$. The proof for subsolutions is symmetric.

Now we can study $G_\infty(\nabla w, D^2 w) = 0$, which (in contrast to $F_\infty = 0$) is now proper.

Theorem 3.3 *Let Ω be a bounded domain, and suppose that u is a uniformly continuous viscosity subsolution and v a uniformly continuous viscosity supersolution of (3.10) in Ω. Then the following equality holds:*

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) = \sup_{x\in\partial\Omega}(u(x) - v(x)). \tag{3.11}$$

There is no loss of generality if we assume $u, v \geq 0$. Otherwise we add constants to u and v. We proceed by contradiction. Suppose that (3.11) is false, then

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) > \sup_{x\in\partial\overline{\Omega}}(u(x) - v(x)). \tag{3.12}$$

To obtain a contradiction, we construct a new supersolution w having the following properties:

(i) $||v - w||_\infty$ is small enough to preserve the inequality (3.12);

(ii) w is a *strict* supersolution of (3.10). With those properties in mind, we introduce the following function (see [19])

$$f(z) = \frac{1}{\alpha} \log\left(1 + A\left(e^{\alpha z} - 1\right)\right),$$

where $\alpha, A > 1$. In [19] this function was shown to satisfy a) through d):

a) $f'(z) > 1$ for every $z > 0$;

b) f_A is invertible and $(f_A)^{-1} = (f_{A^{-1}})$ for every $z > 0$;

c) $1 - [f'(z)]^{-1} + [f'(z)]^{-2} f''(z) < 0$ for every $z > 0$;

d) $0 < f(z) - z < (A-1)/\alpha$ for every $z > 0$.

We define $w = f(v)$. Taking A sufficiently close to 1, property (i) holds easily. Let us check (ii). Let $\xi \in \Omega$ and $\varphi \in C^2$ such that $\varphi(\xi) = w(\xi)$ and $\varphi(x) \leq w(x)$ for $x \neq \xi$. Set $\theta = f^{-1}(\varphi)$. The function f^{-1} is monotone increasing, and so θ is a good test function for v at ξ. But v is a supersolution of (3.10), therefore

$$\min \left\{ |\nabla\theta(\xi)|^* - \Lambda_\infty,\ -Q_\infty\theta(\xi) - (|\nabla\theta(\xi)|^*)^4 \right\} \geq 0. \tag{3.13}$$

It follows from (3.13) that

$$|\nabla\theta(\xi)|^* - \Lambda_\infty \geq 0, \tag{3.14}$$

$$-Q_\infty\theta(\xi) - (|\nabla\theta(\xi)|^*)^4 \geq 0. \tag{3.15}$$

But if we write explicitly

$$\begin{aligned} \theta_{x_j} &= [f'(\theta)]^{-1}\varphi_{x_j} \\ \theta_{x_ix_j} &= [f'(\theta)]^{-1}\varphi_{x_ix_j} - [f'(\theta)]^{-3}f''(\theta)\varphi_{x_i}\varphi_{x_j} \end{aligned}$$

we get from (3.14)

$$|\nabla\varphi(\xi)|^* \geq f'(\theta(\xi))\Lambda_\infty \tag{3.16}$$

or

$$|\nabla\varphi(\xi)|^* - \Lambda_\infty \geq [f'(\theta(\xi)) - 1]\Lambda_\infty > 0. \tag{3.17}$$

With some calculus we obtain

$$D^2\varphi = f'(\theta)D^2\theta + f''(\theta)\nabla\theta \otimes \nabla\theta$$

so that (because J is homogeneous of degree one)

$$-Q_\infty\varphi = \langle D^2\varphi J(\nabla\varphi), J(\nabla\varphi)\rangle = -f'(\theta)^3 Q_\infty\theta - f''(\theta)f'(\theta)^2(|\nabla\theta|^*)^4\ .$$

This and (3.15) implies

$$-Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^4 \geq \left(f'^3 - f''f'^2 - f'^4\right)(\theta(\xi))(|\nabla\theta(\xi)|^*)^4$$

whose right hand side is positive because of d). Therefore we have shown

$$-Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^2 \geq f'^4\left(\frac{1}{f'} - \frac{f''}{f'^2} - 1\right)(v(\xi))\Lambda_\infty^4 \tag{3.18}$$

From a), (3.17) and (3.18) we conclude

$$\min\ \left\{\ |\nabla\varphi(\xi)|^* - \Lambda_\infty,\ -Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^4\right\} \geq \rho(\xi) > 0, \tag{3.19}$$

where we have defined

$$\rho(x) := \min\left\{[f'(v(x)) - 1]\Lambda_\infty,\ \left(\frac{1}{f'} - \frac{f''}{f'^2} - 1\right)(v(x))\Lambda_\infty^4\right\}.$$

Inequality (3.19) and properties a) and c) tell us that w is a strict supersolution.

Now the contradiction follows easily by standard techniques for viscosity solutions, see [11]. Let us sketch the conclusion. We consider (x_t, y_t) a maximum point of the function

$$u(x) - w(y) - \frac{t}{2}|x - y|^2$$

in $\overline{\Omega} \times \overline{\Omega}$. Up to a subsequence, we have that

$$x_t \to \xi \quad \text{and} \quad y_t \to \xi,$$

where $\xi \in \overline{\Omega}$ is a maximum point of $(u - w)$ in $\overline{\Omega}$. But inequality (3.12) holds, so ξ lies in the interior. We apply the max principle for semicontinuous function (see Chapter 3 in [11] for this result and for the definition of the semijets $\overline{J}^{2,+}(u(x_t))$ and $\overline{J}^{2,-}(w(x_t))$), which ensure the existence of real symmetric matrices $X_t,\ Y_t$ such that

$$\begin{aligned} (t(x_t - y_t);\ X_t) &\in \overline{J}^{2,+}(u(x_t)) \\ (t(x_t - y_t);\ Y_t) &\in \overline{J}^{2,-}(w(x_t)) \\ (X_t\nu, \nu) - (Y_t\mu, \mu) &\leq 3t|\nu - \mu|^2. \end{aligned}$$

Now u is a subsolution of $G_\infty = 0$, so

$$G_\infty\left(t(x_t - y_t);\ X_t\right) \leq 0. \tag{3.20}$$

Since w is a strict supersolution of $G_\infty = 0$, we get from (3.19)

$$G_\infty\left(t(x_t - y_t);\ Y_t\right) \geq \rho(x_t) > 0. \tag{3.21}$$

Now (3.20) and (3.21) give after some calculation

$$\rho(x_t) \leq 0,$$

which is obviously a contradiction. This completes the proof.

Remark 3.4 Theorem 3.3 also holds when one of the functions takes the value $-\infty$ on the whole boundary.

It is well-known that for any $1 < p < \infty$, the eigenvalue λ_p can be characterized by the property that $\lambda = \lambda_p$ is the only real number for which the equation

$$-\text{div}\left((|\nabla u_p|^*)^{p-2} J(\nabla u_p)\right) = \lambda |u_p|^{p-2} u_p$$

has a continuous positive solution with zero boundary value. We will show next that Λ_∞ has an analogous characterization.

Theorem 3.5 *Let Ω be any bounded domain and suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. If u is a continuous positive viscosity solution in Ω of*

$$\min\{|\nabla u|^* - \Lambda u, -Q_\infty u\} = 0$$

with zero boundary value, then $\Lambda = \Lambda_\infty$.

To prove this, we need the following gradient estimate. For the standard Euclidean norm this was derived in [22]. Using a perturbation argument due to Crandall, we show that the general case follows from the results in [2].

Theorem 3.6 *Suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. Let u be a nonnegative viscosity supersolution of $-Q_\infty u = 0$ in Ω, and let $\delta(x) = dist(x, \partial\Omega)$ for $x \in \Omega$. Then*

$$|\nabla u(x)|^* \leq \frac{u(x)}{\delta(x)} \qquad \textit{for a.e. } x \in \Omega. \tag{3.22}$$

In order to prove the assertion, it suffices to verify that u enjoys the following *comparison with cones from below* property in Ω (see [2]):

> Whenever $V \subset\subset \Omega$ is an open set and $C(x) = a|x - z| + b$ with $a, b \in \mathbb{R}$, $z \notin V$ is a cone function such that $u \geq C$ on ∂V, then $u \geq C$ in V.

Indeed, for functions that enjoy comparison with cones from below, (3.22) is Remark 2.17 in [2].

To show that viscosity supersolutions of $-Q_\infty u = 0$ enjoy comparison with cones from below, we argue as in the proof of Theorem 4.13 in [2]. Suppose u does not enjoy comparison with cones from below in Ω. Then there is an open set $V \subset\subset \Omega$ and a cone function $C(x) = a|x - z| + b$ with $a, b \in \mathbb{R}$, $z \notin V$ such that $u = C$ on ∂V and $u < C$ in V. If for each $\varepsilon > 0$ we can find a perturbation $P \in C^2(\overline{V})$ such that $|P| \leq \varepsilon$ in V and

$$-Q_\infty(C + P) \leq -\delta < 0 \text{ in } V, \tag{3.23}$$

we will be done. Indeed, for $\varepsilon > 0$ small enough, the function $u - (C + P)$ has an interior local minimum point $x_0 \in V$. Since u is a viscosity supersolution and $C + P \in C^2(V)$, this implies

$$-Q_\infty(C + P)(x_0) \geq 0,$$

contradicting (3.23).

Since we are assuming that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$, suitable perturbations can be explicitly constructed using this norm. Suppose, without loss of generality, that $z = 0$ and put $P = \gamma|x|^2$ and $\gamma > 0$. Then $C(x) + P(x) = g(|x|)$ where $g(s) = as + \gamma s^2 + b$. A direct computation shows that

$$-Q_\infty g(|x|) = -g'(|x|)^3 \langle D^2|x| J(\nabla|x|), J(\nabla|x|)\rangle + \\ - g''(|x|)g'(|x|)^2 \langle \nabla|x|, J(\nabla|x|)\rangle^2.$$

Since $\langle \nabla|x|, x\rangle = |x|$ by the homogeneity and $J(\nabla|x|) = x/|x|$ for $x \neq 0$, this reduces to

$$-Q_\infty g(|x|) = -(g')^3|x|^{-2}\langle D^2|x|x, x\rangle - g''(g')^2.$$

Next observe that by the linearity of $h(t) = |tx|$ we have $0 = h''(1) = \langle D^2|x|x, x\rangle$, so that

$$-Q_\infty(C + P)(x) = -g''(|x|)g'(|x|)^2 = -2\gamma(2\gamma|x| + a)^2.$$

This is strictly negative in V if either $a \geq 0$ or if $a < 0$ and $0 < \gamma$ is sufficiently small. If γ is sufficiently small we also attain $|P| \leq \epsilon$ in V.

For the proof of Theorem 3.5, we will also need the following auxiliary comparison result.

Lemma 3.7 *Suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. If u is a continuous positive viscosity solution of*

$$\min\{|\nabla u|^* - \Lambda u, -Q_\infty u\} = 0 \tag{3.24}$$

in a bounded domain Ω with zero boundary values, normalized so that $\sup u = \frac{1}{\Lambda}$, *then*

$$u(x) \leq dist(x, \partial\Omega) \qquad \text{for every } x \in \Omega.$$

Fix $z \in \partial\Omega$ and for $a > 1$, $\gamma > 0$ let $v(x) = a|x-z| - \gamma|x-z|^2$. Analogously to the proof of Theorem 3.6 above, we obtain $-Q_\infty v(x) > 0$ provided that $\gamma > 0$ is sufficiently small. Moreover,

$$|\nabla v(x)|^* = (a - 2\gamma|x-z|)|\nabla|x-z||^* = a - 2\gamma|x-z| > 1$$

if γ is small enough. Thus we have

$$\min\{|\nabla v|^* - 1, -Q_\infty v\} > 0. \tag{3.25}$$

Next notice that due to the assumption $\sup u = \frac{1}{\Lambda}$, (3.24) implies

$$\min\{|\nabla u|^* - 1, -Q_\infty u\} \leq 0 \qquad \text{in the viscosity sense.} \tag{3.26}$$

Since $v \in C^2$ and $v \geq u = 0$ on $\partial\Omega$ (if γ is small enough), it follows that $v \geq u$ in Ω. Indeed, otherwise $u - v$ would have an interior local maximum point at which v would be a test-function for u from above, contradicting (3.25) and (3.26).

We have thus shown that $u(x) \leq a|x-z| - \gamma|x-z|^2$ for every $z \in \partial\Omega$, $a > 1$ and $\gamma > 0$ sufficiently small. Hence

$$u(x) \leq \inf_{z \in \partial\Omega} |x-z| = \operatorname{dist}(x, \partial\Omega),$$

as desired.

Remark 3.8 Lemma 3.7 implies that if u is any positive viscosity solution to the eigenvalue equation $F_\infty(u, \nabla u, D^2u) = 0$ with zero boundary data, it cannot be differentiable at its maximum points. To see this, let us normalize u so that $\sup u = \frac{1}{\Lambda}$. Then if $u(x_0) = \sup_{x\in\Omega} u(x)$, it follows that $\delta(x_0) = \sup_{x\in\Omega} \delta(x)$. Since δ is not differentiable at x_0 and $u \leq \delta$, $u(x_0) = \delta(x_0)$, it is now clear that u is not differentiable at x_0.

Now we prove Theorem 3.5. Notice first that if $\Lambda \leq 0$, then the eigenvalue equation above reduces to the equation $-Q_\infty u = 0$, whose only solution with zero boundary values is $u \equiv 0$, see [2] or [3].

Let us normalize u so that $\sup u = \frac{1}{\Lambda}$. Then we obtain by Lemma 3.7 that $u(x) \leq \delta(x) := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$, which together with the gradient estimate (3.22) yields $|\nabla u(x)|^* \leq 1$ for a.e. $x \in \Omega$. Consequently,

$$\frac{\||\nabla u|^*\|_\infty}{\|u\|_\infty} \leq \frac{1}{\|u\|_\infty} = \Lambda.$$

Because

$$\Lambda_\infty = \inf \left\{ \frac{\||\nabla w|^*\|_\infty}{\|w\|_\infty} : w \in W_0^{1,\infty}(\Omega) \setminus \{0\} \right\}$$

by (3.1) and (3.2), we must have $\Lambda_\infty \leq \Lambda$.

To prove the reverse inequality, we approximate $v = \log u$ by its semiconcave inf-convolutions

$$v^\epsilon(x) = \inf_{y \in \overline{\Omega}_\sigma} \{v(y) + \tfrac{1}{2\varepsilon}|x-y|^2\}$$

for $\varepsilon > 0$ in the set $\Omega_\sigma = \{x \in \Omega : \delta(x) > \sigma\}$. Since $|\nabla v|^* \geq \Lambda$ in the viscosity sense by the assumptions and v^ε is twice differentiable a.e., it follows from the properties of the inf-convolution that $|\nabla v^\varepsilon(x)|^* \geq \Lambda$ for a.e. x in a smaller set $\Omega_{\sigma,\varepsilon} = \{x \in \Omega_\sigma : \operatorname{dist}(x, \partial\Omega_\sigma) > C\varepsilon\}$. Moreover, the function e^{v^ε} is a positive supersolution of $-Q_\infty w = 0$ in $\Omega_{\sigma,\varepsilon}$. Thus we obtain using the gradient estimate (3.22)

$$\Lambda \leq |\nabla v^\varepsilon(x)|^* = \frac{1}{e^{v^\varepsilon}} |\nabla(e^{v^\varepsilon(x)})|^* \leq \frac{1}{\operatorname{dist}(x, \partial\Omega_{\sigma,\varepsilon})}$$

for a.e. $x \in \Omega_{\sigma,\varepsilon}$, and so, letting $\varepsilon \to 0$, $\sigma \to 0$,

$$\Lambda \leq \frac{1}{\sup_{x\in\Omega} \delta(x)} = \Lambda_\infty.$$

This completes the proof.

4 Example and concluding remarks

If the norm under consideration for $x \in \Omega$ is the usual ℓ_q- norm, i.e. for $|x| = (\sum_{i=1}^n |x_i|^q)^{1/q}$ with $q \in (1, \infty)$, the duality map according to (2.4) is easily calculated as

$$J_i(y) = (|y|_{q'})^{2-q'} |y_i|^{q'-2} y_i,$$

with $q' = q/(q-1)$ as conjugate exponent. Notice that this differs from the J in [2], Example 5.2. Then the p-Laplace operator in this Finsler metric is explicitly given by, see [6]

$$Q_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|\nabla u|_{q'}^{p-q'} \left| \frac{\partial u}{\partial x_i} \right|^{q'-2} \frac{\partial u}{\partial x_i} \right).$$

For $p > 2$ this definition is meaningful and for $q = 2 (= q')$ it recovers the well-known p-Laplace Operator. The operator Q_2 is formally given by

$$Q_2 u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left[\frac{|u_{x_i}|}{|\nabla u|_{q'}} \right]^{q'-2} \frac{\partial u}{\partial x_i} \right).$$

However, $Q_2 u$ does not seem to be well-defined in critical points of u. The ∞-Laplace operator in the same Finsler metric is explicitly given by

$$Q_\infty u = |\nabla u|_{q'}^{4-2q'} \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i x_j} \left| \frac{\partial u}{\partial x_i} \right|^{q'-2} \frac{\partial u}{\partial x_i} \left| \frac{\partial u}{\partial x_j} \right|^{q'-2} \frac{\partial u}{\partial x_j} \right)$$

and for $q = 2$ this expression shrinks down to the customary

$$\Delta_\infty u = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

Remark 4.1 It should be remarked that the distance function minimizes the Rayleigh quotient R_∞, but that $\delta(x)$ is in general not a viscosity solution of the limiting eigenvalue problem, unless Ω is a "ball" in the Finsler metric, see [19], [20], [5].

Remark 4.2 If Ω is a "ball" in $\mathbb{R}^n$ and $p = n$, then all the level sets of solutions to (2.2)

$$-Q_n u = \lambda_n |u|^{n-2} u$$

are similar "balls", see [6].

Remark 4.3 The smoothness assumption made on the dual spheres in our paper is violated if the underlying norm is the ℓ_1 or ℓ_∞ norm. However, the pde $-Q_p = 1$ and its limit as $p \to \infty$ was studied even in this case in [16], see also [21], [7], [18] and [17] for the case of the Euclidean norm and for variants of this problem.

Remark 4.4 Clearly the eigenvalue λ_p depends on Ω. There is an analogue of the Faber-Krahn inequality which states that among all domains of given volume $\lambda_p(\Omega)$ becomes minimal if Ω is a "ball" in the Finsler metric. This result is formulated in [6], but it is based on a rearrangement inequality from [1]

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