On 'Anti'-eigenvalues for Elliptic Systems and a Question of McKenna and Walter

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ABSTRACT. We investigate for which range of b the biharmonic boundary value problem $\Delta^2 u + bu = f$ in Ω, with $\Delta u = u = 0$ on $\partial\Omega$, is positivity-preserving in the sense that $f \geq 0$ in Ω implies $u \geq 0$. We will also disprove a conjecture of McKenna and Walter on the isoperimetric nature of the upper bound $b_c(\Omega)$ for such b. The investigation gives rise to related questions for certain linear elliptic systems and to curious identities for sums of inverse eigenvalues.

1. INTRODUCTION

McKenna and Walter remarked in [13] that the following linear problem needs more study:

$$\begin{cases} \Delta^2 v + bv = f & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and f a given positive function, say $f \in L^p(\Omega)$ with $p > n$. Let us quote (using our notation):

> *... we should be able to estimate the value $b_c(\Omega)$ with the property that for $0 \leq b < b_c(\Omega)$ the inverse operator is positivity preserving (...). We conjecture that this constant $b_c(\Omega)$ is largest among all regions Ω with given volume when Ω is a ball.*

To study such questions one should note that fourth order equations in general do not satisfy a maximum principle. Rewriting (1.1) as a system of second

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order equations, namely

$$(1.2)\qquad \begin{cases} -\Delta u = f - \mu v & \text{in } \Omega, \\ -\Delta v = u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu = b$, one notices that the coupling for $\mu > 0$ is noncooperative such that indeed, see [14], the standard maximum principle cannot be used. Nevertheless, by the results in [14] one is able to estimate the value $b_c(\Omega)$ from below by using the so called 3G-Theorem from [3] which can be rephrased as a positivity preserving property for the auxiliary system:

$$(1.3)\qquad \begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where we take $\lambda = \sqrt{b}$. Except for the fact that the best constants in these systems are strictly positive and finite for smooth bounded domains, little is known. The aim of the present paper is to shed some more light on these numbers, in other words, give some answers to the following question:

> *Which are the best positive numbers such that the systems* (1.1), (1.2), *and* (1.3) *are positivity preserving?*

By positivity preserving we mean that $f \geq 0$ implies $v \geq 0$, respectively twice u, $v \geq 0$.

In particular we address the following issues:

- relations between these constants and eigenvalues will be recalled;
- exploiting these relations, a result of Xu in [23] will give us a counterexample to the conjecture of McKenna and Walter;
- in one dimension, the explicit numbers will be computed;
- in some special two-dimensional domains, one of these numbers will be compared with the first eigenvalues;
- a partial explanation will be given of a rather surprising inverse-sum-of-eigenvalues formula.

It should be remarked that, besides for large positive numbers, the positivity preserving property for (1.1) and (1.2) also breaks down if $-b$ attains the first eigenvalue of Δ^2 under the Navier boundary conditions in (1.1). This eigenvalue coincides with $\lambda_1(\Omega)^2$, where $\lambda_1(\Omega)$ is the first (and positive) Dirichlet-Laplace eigenvalue. The result will be that the system (1.1) is positivity preserving for $b \in \langle -\lambda_1(\Omega)^2, b_c(\Omega)]$.

Such a bounded interval is in marked contrast to the second order problem:

$$(1.4) \qquad \begin{cases} -\Delta v + dv = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The boundary value problem (1.4) preserves positivity for all $d \in \langle -\lambda_1(\Omega), \infty \rangle$. This behaviour, no bound from above, is a particular feature of equations of second order and cooperative systems of second order equations. Schröder [18] showed in one space dimension, that b_c is characterized as an eigenvalue of a related problem. Inspired by the fact that b_c is on the other side of zero, is not an eigenvalue for the original problem, but a threshold for the positivity preserving property like λ_1 and is (in one dimension) an eigenvalue for yet another problem, we call b_c an 'anti'-eigenvalue.

2. Estimates and a Counterexample

While McKenna and Walter were interested in the positivity of v, one may wonder about the sign of u in (1.2) as well. If u is nonnegative and nonzero, then v is positive because of the second equation in (1.2) and the strong maximum principle. Hence if (1.2) is positivity preserving, then (1.1) also has this property. However, u might be sign-changing, while v is still nonnegative.

The relation between (1.2) and (1.3) with respect to sign preserving is studied in [14]. From there it is known that, if (1.3) is positivity preserving with $\lambda = \sqrt{b}$, then so is (1.2) for $\mu = b$.

Finally, let us explain the connection of the critical constant in the last system with Brownian motion. See also [20]. This constant for (1.3) equals the inverse of the expectation of the lifetime of a conditioned Brownian motion. Let us denote the inverse Dirichlet Laplacian by $\mathcal{G} : L^p(\Omega) \to L^p(\Omega)$, that is, $w = \mathcal{G}f$ solves $-\Delta w = f$ in Ω, $w = 0$ on $\partial\Omega$. We will also use the corresponding iterated Green functions:

$$(2.1) \qquad (\mathcal{G}^k f) =: \int_\Omega G_k(x,y)\, f(y)\, dy.$$

Then (1.3) is positivity preserving if $\mathcal{G}f - \lambda\mathcal{G}^2 f \geq 0$ for all positive functions $f \in L^p(\Omega)$. This condition is equivalent with

$$\lambda \frac{G_2(x,y)}{G_1(x,y)} \leq 1 \quad \text{for all } x, y \in \Omega,$$

or in other words

$$(2.2) \qquad \lambda_c(\Omega) = \left(\sup_{x,y\in\Omega} \frac{G_2(x,y)}{G_1(x,y)} \right)^{-1}.$$

Cranston, Fabes, and Zhao [3] proved with their so-called 3G-Theorem that the supremum in (2.2) is bounded on bounded Lipschitz domains. By probability theory one has for $n \geq 2$

$$\frac{G_2(x,y)}{G_1(x,y)} = \mathbb{E}_x^y(\tau_\Omega), \tag{2.3}$$

where $\mathbb{E}_x^y(\tau_\Omega)$ is the expectation for the lifetime τ_Ω of Brownian motion that is conditioned to start in x, converge to y, and to be killed when reaching the boundary $\partial\Omega$. In a probabilistic setting estimates for $\mathbb{E}_x^y(\tau_\Omega)$ in two-dimensional domains are found in [4]; for higher dimensions see [5].

Next we recall and extend some results from [14, 19]. We will also state some estimates for the three critical numbers involved. In order to do so we need the eigenvalues of the Dirichlet Laplace operator which we denote by $\lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots$.

Theorem 2.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ Lipschitz. Then there exist positive numbers $b_c(\Omega)$, $\mu_c(\Omega)$, and $\lambda_c(\Omega)$ such that:*

1. (a) *if $-\lambda_1(\Omega)^2 < b \leq b_c(\Omega)$, then for all $f \geq 0$, the solution v of* (1.1) *satisfies $v \geq 0$;*
   (b) *if $b > b_c(\Omega)$, then there exists $f \in C(\bar{\Omega})$, with $f > 0$, such that the solution v of* (1.1) *changes sign.*
2. (a) *if $-\lambda_1(\Omega)^2 < \mu \leq \mu_c(\Omega)$, then for all $f \geq 0$, the solution u of* (1.2) *satisfies $u \geq 0$;*
   (b) *if $\mu > \mu_c(\Omega)$, then there exists $f \in C(\bar{\Omega})$, with $f > 0$, such that the solution u of* (1.2) *changes sign.*
3. (a) *if $\lambda \leq \lambda_c(\Omega)$, then for all $f \geq 0$, the solution u of* (1.3) *satisfies $u \geq 0$;*
   (b) *if $\lambda > \lambda_c(\Omega)$, then there exists $f \in C(\bar{\Omega})$, with $f > 0$, such that the solution u of* (1.3) *changes sign.*

These numbers satisfy the following estimates:

$$\lambda_c(\Omega)^2 \leq \mu_c(\Omega) \leq b_c(\Omega), \tag{2.4}$$

$$\mu_c(\Omega) \leq \lambda_1(\Omega)\lambda_2(\Omega), \tag{2.5}$$

$$\lambda_c(\Omega) \leq \left(\frac{1}{\lambda_1(\Omega)} + \frac{1}{\lambda_2(\Omega)}\right)^{-1}. \tag{2.6}$$

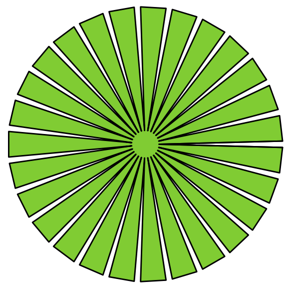
Let us remark in passing that λ_c, like the eigenvalues λ_i, is homogeneous of degree -2 under scaling, that is,

$$\lambda_c(t\Omega) = t^{-2}\lambda_c(\Omega)$$

for $t > 0$, while b_c and μ_c are homogeneous of degree -4. This gives some explanation of the exponents in (2.4), (2.5) and (2.6).

Next let us show how the link with Brownian motion allows us to find a counter example to the conjecture of McKenna and Walter. In [23, Theorem 3] it was shown that there exists a domain Ω of infinite area for which the expected lifetime of a conditioned Brownian motion is bounded. In [8, p.244] it was observed that this result implies that one may construct a planar domain of given area with the expected lifetime bounded by a number as small as desired. Since this bound is the inverse of λ_c, (2.4) implies that there exist a planar domain with bounded surface area and $b_c(\Omega)$ larger than $b_c(D)$. Here D denotes the disk with equal surface area. Such a domain supplies a counterexample to the conjecture of McKenna and Walter.

In fact, as just explained, it not only allows us to disprove that conjecture but even gives the following stronger result.



Corollary 2.2. *For any $M > 0$ there exist planar domains Ω of given area 1 such that $\lambda_c(\Omega) > M$ and a forteriori $b_c(\Omega) \geq \mu_c(\Omega) > M^2$.*

A possible domain with $b_c(\Omega) > b_c(D)$ and equal surface areas $|\Omega| = |D|$ is sketched.

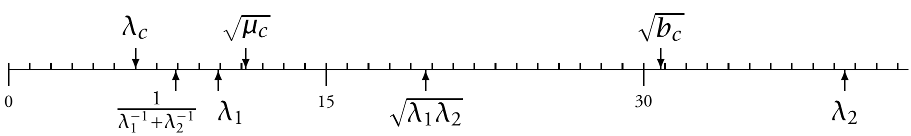
3. Explicit numbers

For a number of special domains some of these critical numbers are explicitly known. We list λ_1, λ_2, and the known critical constants for the unit interval and some special planar domains Ω with fixed area $|\Omega| = 1$. We start with one dimension.

Proposition 3.1. *The following explicit numbers can be given for I the unit interval $\langle 0, 1\rangle$:*

$$\begin{aligned}
\lambda_c(I) &= 6, \\
\lambda_1(I) &= \pi^2 && \approx 9.86960, \\
\lambda_2(I) &= 4\pi^2 && \approx 39.4784, \\
\mu_c(I) &= 4(\kappa_0)^4 && \approx 125.141, \\
b_c(I) &= 4(\kappa_1)^4 && \approx 950.884,
\end{aligned}$$

where κ_0 is the first positive zero of $\tan(x) + \tanh(x)$ *and where κ_1 is the first positive zero of* $\tan(x) - \tanh(x)$. *Scaled to homogeneity we may visualize:*



The derivation of these numbers is postponed to the last section. In Remark 6.1 in that section one may also find a physical interpretation of these numbers.

In two dimensions little is known. It is known by Griffin, McConnell, and Verchota in [8, Proposition 4.2] that

$$\lambda_c^*(D) := \left(\sup_{x\in D, y\in\partial D} \frac{G_2(x,y)}{G_1(x,y)} \right)^{-1} = \frac{\pi}{2\log 2 - 1}.$$

Some heuristic reasons seem to indicate that

$$\sup_{x\in\Omega, y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega) = \sup_{x,y\in\Omega} \mathbb{E}_x^y(\tau_\Omega), \tag{3.1}$$

and hence by (2.2) and (2.3) it would follow that $\lambda_c^*(D) = \lambda_c(D)$. No proof of (3.1) seems yet available.

Let us fix this result and compare the number with the first eigenvalues.

Proposition 3.2. *Let $D \subset \mathbb{R}^2$ be a disk with area equal to* 1. *If* (3.1) *holds, then*

$$\begin{aligned} \lambda_c(D) &= \frac{\pi}{2\log 2 - 1} &&\approx 8.13264, \\ \lambda_1(D) &= \pi j_{0,1}^2 &&\approx 18.1685 \\ \lambda_2(D) &= \pi j_{1,1}^2 &&\approx 46.1246, \end{aligned}$$

where $j_{0,1}$ and $j_{1,1}$ are the first positive zeros of the Bessel functions J_0, respectively J_1.

For the limit behaviour of a long rectangle we again refer to [8, Theorem 3.1].

Proposition 3.3. *If R_a is a rectangle with sides $a \geq 1$ and $1/a$, then*

$$\begin{aligned} \lambda_c(R_a) &\to \pi \quad \text{as } a \to \infty, \\ \lambda_1(R_a) &= \pi^2(a^2 + a^{-2}), \\ \lambda_2(R_a) &= \pi^2(a^2 + 4a^{-2}). \end{aligned}$$

Now that the conjecture of McKenna and Walter does not hold in its full generality, one might ask if such a type of result would hold when restricting the class of domains. For example one could ask if the disk maximizes $b_c(\Omega)$ among all convex domains of equal area. In fact, it has been conjectured in [8] that, among all convex planar domains of fixed area, the disk minimizes $\sup_{x\in\Omega, y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega)$, or in other words, maximizes $\lambda_c^*(\Omega)$. This is not true: recently in [11] it has been shown that for some convex sector S, with area 1, it holds that

$$\lambda_c^*(D) < \lambda_c^*(S) = \frac{8\pi}{3} \approx 8.37758.$$

It is expected that these expressions equal $\lambda_c(D)$ respectively $\lambda_c(S)$ which would imply that $\lambda_c(D) < \lambda_c(S)$. We expect similar results for μ_c and b_c, that is, the disk does not maximize these numbers in the class of convex domains. But an actual proof, by supplying a counterexample, seems to be very hard. Proceeding by an explicit Green function as in [11] is not very likely.

The results in this section are restricted to domains in dimensions 1 and 2. A tool that is strongly used in [8] and [11], which have a two-dimensional setting, is the Riemann mapping theorem. In higher dimensions no such tool exists. Even if for the 3-dimensional ball heuristic reasoning will lead to x and y in (2.3) lying as opposite poles on the ball, an analytical proof of such a fact is not obvious.

4. Identities involving inverse sums of eigenvalues

In a number of basically one-dimensional cases the following identity holds. It should be compared with (2.6).

$$\frac{1}{\lambda_c(\Omega)} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\Omega)}. \tag{4.1}$$

For the interval in $\mathbb{R}$ we refer to [19]. Caristi and Mitidieri in [2] proved (4.1) for the radially symmetric case on a ball in $\mathbb{R}^n$. Identity (4.1) has only been verified through tedious explicit computation of both sides.

In higher dimensions the growth rate of the eigenvalues is such that the right hand side of (4.1) does not converge, and hence such a formula cannot hold. However, another related identity holds. An alternating series of inverse eigenvalues does converge and it converges to the inverse of four times the 'anti'-eigenvalue.

Lemma 4.1. *For D the disk in 2 dimensions it holds true that*

$$\frac{1}{\lambda_c^*(D)} = 4 \sum_{\nu=0}^{\infty} (-1)^{\nu} \sum_{k=1}^{\infty} \frac{m_{\nu,k}}{\lambda_{\nu,k}(D)}. \tag{4.2}$$

Here the eigenvalues are counted including multiplicity $m_{\nu,k}$, with $k-1$ denoting the number of circular nodal lines of the eigenfunction inside the disk and with ν the number of radial nodal lines. One has $m_{\nu,k} = 1$ for $\nu = 0$, and $m_{\nu,k} = 2$ for $\nu \geq 1$.

Let us remark that the eigenfunctions above are symmetric for ν even, i.e., $\varphi(x) = \varphi(-x)$, while for ν odd $\varphi(x) = -\varphi(-x)$.

A similar formula again holds in one dimension with factor 2 instead of 4 (for the third identity see [1, Formula 23.2.26]):

$$\frac{1}{\lambda_c(I)} = \frac{1}{6} = \frac{2}{\pi^2}\frac{\pi^2}{12} = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\lambda_k(I)}. \tag{4.3}$$

At least this formula can be explained in a heuristic way. We know from (2.2) that

$$\lambda_c(I)^{-1} = \sup_{x,y\in I} \frac{G_2(x,y)}{G_1(x,y)} = \sup_{x,y\in I} \frac{\sum_{k=0}^{\infty} \frac{1}{\lambda_k^2}\varphi_k(x)\varphi_k(y)}{\sum_{k=0}^{\infty} \frac{1}{\lambda_k}\varphi_k(x)\varphi_k(y)} \tag{4.4}$$

$$= \sup_{x,y\in I} \frac{\sum_{k=0}^{\infty} \frac{1}{(\pi k)^4}\sqrt{2}\sin(k\pi x)\sqrt{2}\sin(k\pi y)}{\sum_{k=0}^{\infty} \frac{1}{(\pi k)^2}\sqrt{2}\sin(k\pi x)\sqrt{2}\sin(k\pi y)}.$$

By the symmetry in x and y we may assume that $x \le y$. Since the enumerator and denominator go to 0 when x or y move to the boundary, it seems appropriate to divide both by $k\pi x$ and $k\pi(1-y)$ to find

$$\lambda_c(I)^{-1} = \sup_{x,y\in I} \frac{\sum_{k=0}^{\infty} (-1)^{k-1} \frac{1}{(\pi k)^2} \frac{\sin(k\pi x)}{k\pi x} \frac{\sin(k\pi(1-y))}{k\pi(1-y)}}{\sum_{k=0}^{\infty} (-1)^{k-1} \frac{\sin(k\pi x)}{k\pi x} \frac{\sin(k\pi(1-y))}{k\pi(1-y)}} \tag{4.5}$$

By the Fourier-series for t^2 on $[-\pi,\pi]$, we find that the denominator equals $\frac{1}{2}$ for all $0 < x \le y < 1$:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(k\pi x)}{k\pi x} \frac{\sin(k\pi(1-y))}{k\pi(1-y)} \tag{4.6}$$

$$= \frac{4\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi(1+x-y))}{k^2} - 4\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi(1-x-y))}{k^2}}{8\pi^2 x(1-y)}$$

$$= \frac{\pi^2(1+x-y)^2 - \pi^2(1-x-y)^2}{8\pi^2 x(1-y)} = \frac{1}{2}.$$

Hence, convinced that $\sup_{x,y\in I}$ can be replaced by $\lim_{x\downarrow 0, y\uparrow 1}$, we continue by

$$\lambda_c(I)^{-1} = \lim_{x\downarrow 0, y\uparrow 1} 2\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(\pi k)^2} \frac{\sin(k\pi x)}{k\pi x} \frac{\sin(k\pi(1-y))}{k\pi(1-y)}$$

$$= 2\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(\pi k)^2} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\lambda_k}.$$

Could (4.1) *and* (4.2) *be just a coincidence or is there a similar heuristic explanation beyond computation?*

Proof of Lemma 4.1. On the disk $D = B(0, \pi^{-1/2})$ a complete set of eigenfunctions is given in radial coordinates (r, θ) by

$$\begin{aligned} \varphi_{e,\nu,i}(r,\theta) &= \cos(\nu\theta) J_\nu(j_{\nu,i}\sqrt{\pi}r) \quad \text{with } \nu \in \mathbb{N},\ i \in \mathbb{N}^+, \\ \varphi_{o,\nu,i}(r,\theta) &= \sin(\nu\theta) J_\nu(j_{\nu,i}\sqrt{\pi}r) \quad \text{with } \nu,\ i \in \mathbb{N}^+, \\ \lambda_{o,\nu,i} &= \lambda_{e,\nu,i} = \pi j_{\nu,i}^2. \end{aligned}$$

By Rayleigh ([16], see also [22, Section 15.51]) one gets for $\nu \in \mathbb{N}$

$$\sum_{i=1}^{\infty} \frac{1}{j_{\nu,i}^2} = \frac{1}{4(\nu+1)},$$

and since $\sum_{\nu=0}^{\infty} (-1)^\nu/(\nu+1) = \log 2$, it follows that

$$\begin{aligned} &\sum_{\nu=0}^{\infty} (-1)^\nu \sum_{\substack{i=1,\\ \sigma\in\{0,e\}}}^{\infty} \frac{1}{\lambda_{\sigma,\nu,i}(\Omega)} \\ &\quad = \sum_{i=1}^{\infty} \frac{1}{\pi j_{0,i}^2} + 2\sum_{\nu=1}^{\infty} (-1)^\nu \sum_{i=1}^{\infty} \frac{1}{\pi j_{\nu,i}^2} \\ &\quad = \frac{1}{4\pi}\left(1 + 2\sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu+1}\right) = \frac{1}{4\pi}(2\log 2 - 1). \end{aligned}$$

□

5. Proof of the Theorem

In the sequel we will denote the eigenfunctions for the Dirichlet Laplace operator by φ_i, that is

$$\begin{cases} -\Delta\varphi_i = \lambda_i(\Omega)\varphi_i & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

The Dirichlet Laplace operator can be considered as an unbounded operator in $L^2(\Omega)$ or $C(\bar{\Omega})$. Since we are considering bounded domains with Lipschitz boundary, it follows that its inverse, the solution operator $\mathcal{G}$, is compact in both spaces. Hence in both cases its spectrum consists of eigenvalues. By using regularity results, one obtains that the corresponding spectra and eigenfunctions coincide. For regularity in domains satisfying an exterior cone condition see [7, Theorem 8.30]; for compact imbeddings see [7, Theorem 7.22].

Proof of Theorem 2.1. The theorem states in particular that the sets of b, μ, and λ for which the respective solution operators are positivity preserving are intervals. It will be convenient to define b_c, μ_c, and λ_c as the infimum of nonadmissible positive b, μ, and λ; e.g.

$$b_c := \inf\{b > 0 \mid \exists f \geq 0 \text{ and } \exists v \not\geq 0, \text{ such that } v \text{ solves (1.1)}\}.$$

After recalling that the sets of admissible parameters contain a right neighborhood of zero, we prove that the operators are not positivity preserving for some large numbers b, μ, and λ. In the course of this we derive (2.4), (2.5), and (2.6). Then we show that the sets of admissible coefficients are connected. Finally, we prove the change of sign for solutions with large coefficients and appropriate f. For nonpositive numbers (1.2) falls in the class of cooperative elliptic systems and the bound $-\lambda_1(\Omega)^2$ follows from arguments using the classical maximum principle. See [14]. In the sequel we restrict ourselves to positive parameters.

Bounds from below for the critical number. From [14] it follows that in all three cases a small positive number ε exists such that, for $b, \lambda \in [0, \varepsilon]$, the solutions v and u, v of respectively (1.1), (1.2), and (1.3) are positive for positive f.

Existence of non-admissible positive coefficients b, λ, and μ. Explicit bounds from above for $\lambda_c(\Omega)$ and $\mu_c(\Omega)$ are stated in (2.6) and (2.5). These follow by choosing $f = \varphi_1 - \gamma\varphi_2$ with

$$\gamma := \left(\sup_{x\in\Omega} \frac{\varphi_2(x)}{\varphi_1(x)} \right)^{-1}. \tag{5.2}$$

Note that γ is the largest number such that f is nonnegative. Solving (1.3) with this f one finds

$$u = \left(\frac{1}{\lambda_1} - \frac{\lambda}{\lambda_1^2} \right) \varphi_1 - \gamma \left(\frac{1}{\lambda_2} - \frac{\lambda}{\lambda_2^2} \right) \varphi_2, \tag{5.3}$$

and consequently $u \geq 0$ if and only if

$$\left(\frac{1}{\lambda_1} - \frac{\lambda}{\lambda_1^2} \right) \geq \left(\frac{1}{\lambda_2} - \frac{\lambda}{\lambda_2^2} \right). \tag{5.4}$$

Since (5.4) can be rewritten as

$$\lambda \leq \left(\frac{1}{\lambda_1(\Omega)} + \frac{1}{\lambda_2(\Omega)} \right)^{-1},$$

the estimate in (2.6) follows from the definition of $\lambda_c(\Omega)$. Notice that (2.6) implies in particular $\lambda_c(\Omega) < \lambda_1(\Omega)$. This argument is found in [19].

To prove (2.5) we start with the same ansatz for f. By similar arguments as above we obtain then the bound for $\mu_c(\Omega)$.

Another argument is needed in order to show that $b_c(\Omega)$ is bounded. We will modify an example from [9, Lemma 6.3]. Set $v_0 = \mathcal{G}^2 1$ and assume that $B(0,3R) \subset \Omega$. By the strong maximum principle there exists $s > 0$ such that $v_0(x) \geq s$ for $x \in B(0,2R)$. Let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$ with $\chi(r) = 0$ for $r > R$ and $\chi(r) = 1$ for $r < \frac{1}{2}R$. Set

$$\tilde{v}(x) = \chi(|x|)(16|x|^2 - 1) + (1 - \chi(|x|))v_0(x). \tag{5.5}$$

The function $\tilde{v}$ is constructed in such a way that it changes sign. In fact $\tilde{v}(0) = -1$, and one may show that for some positive δ and M:

$$\begin{aligned}
\Delta^2\tilde{v} &= 0 && \text{and} \quad \tilde{v} < 0 && \text{in } B(0, \tfrac{1}{4}R),\\
\Delta^2\tilde{v} &= 0 && \text{and} \quad \tilde{v} \geq 0 && \text{in } B(0, \tfrac{1}{2}R) \setminus B(0, \tfrac{1}{4}R),\\
\Delta^2\tilde{v} &\geq -M && \text{and} \quad \tilde{v} \geq \delta && \text{in } B(0, R) \setminus B(0, \tfrac{1}{2}R),\\
\Delta^2\tilde{v} &= 1 && \text{and} \quad \tilde{v} \geq 0 && \text{in } \Omega \setminus B(0, R).
\end{aligned}$$

If we set $a(x) = M\delta^{-1}(1 - \chi(2|x|))$ and note that a vanishes in $B(0, \frac{1}{4}R)$, then

$$\Delta^2\tilde{v} + a(x)\tilde{v} =: g \geq 0 \quad \text{in } \Omega. \tag{5.6}$$

Let us define

$$b := \max\{a(x) \mid x \in \Omega\}, \tag{5.7}$$

and see that $\tilde{v}$ satisfies

$$\Delta^2 v + bv = g + (b - a(x))v \quad \text{in } \Omega. \tag{5.8}$$

Assuming that the inverse operator for $\Delta^2 + b$ with appropriate boundary conditions is positivity preserving, we will obtain a contradiction. Let us denote the solution operator by $\mathcal{B}_b$, that is, $w = \mathcal{B}_b f$ solves $\Delta^2 w + bw = f$ in Ω, and $w = \Delta w = 0$ on $\partial\Omega$. The eigenvalues of $\Delta^2 + bI$ under Navier boundary conditions are $\lambda_i(\Omega)^2 + b$, and the corresponding eigenfunctions (which coincide with eigenfunctions for the Dirichlet Laplacian) form a complete orthonormal system in $L^2(\Omega)$. Since the inverse operator $\mathcal{B}_b$ is compact, its spectrum consists of the inverted eigenvalues $(\lambda_i(\Omega)^2 + b)^{-1}$ and hence its spectral radius equals $(\lambda_1(\Omega)^2 + b)^{-1}$. Denoting the multiplication with $(b - a(x))$ by $\mathcal{M}_{b-a}$, the spectral radius of $\mathcal{B}_b\mathcal{M}_{b-a}$ is bounded by $b/(\lambda_1(\Omega)^2 + b) < 1$ and it follows that

$$\tilde{v} = (I - \mathcal{B}_b\mathcal{M}_{b-a})^{-1}\mathcal{B}_b g = \sum_{k=0}^{\infty} (\mathcal{B}_b\mathcal{M}_{b-a})^k \mathcal{B}_b g. \tag{5.9}$$

But $\mathcal{B}_b$ and $\mathcal{M}_{b-a}$ are positive operators. Therefore (5.9) and (5.6) imply that $\tilde{v}$ is positive, a contradiction. This proves that, for the positive b from (5.7), the solution operator to (1.1) is not positivity preserving.

Comparison estimates. The estimates (2.5) and (2.6) were already obtained as a byproduct in the second step of the proof. The proof of the right hand side estimate in (2.4) is a consequence of the maximum principle. To prove the left hand side estimate in (2.4), we use a Neumann series expansion for an operator containing the inverse Dirichlet Laplacian. With this $\mathcal{G}$, which is defined at the beginning of the section, (1.2) is reformulated as $(I + \mu\mathcal{G}^2)u = \mathcal{G}f$ or $u = (I + \mu\mathcal{G}^2)^{-1}\mathcal{G}f$, provided the inverse exists. Since the spectral radius of $\mathcal{G}$ is $1/\lambda_1(\Omega)$, the Neumann series

$$u = \sum_{k=0}^{\infty} (-\mu\mathcal{G}^2)^k \mathcal{G}f \tag{5.10}$$

does converge to $(I + \mu\mathcal{G}^2)^{-1}\mathcal{G}f$ when $\mu < \lambda_1(\Omega)^2$. But now elementary algebra leads to

$$u = \left(\sum_{k=0}^{\infty} (\mu^2\mathcal{G}^4)^k\right)(I + \sqrt{\mu}\mathcal{G})(I - \sqrt{\mu}\mathcal{G})\mathcal{G}f. \tag{5.11}$$

All but one of the factors in (5.11) are positive operators. The operator $(I - \sqrt{\mu}\mathcal{G})$ is never positive for $\mu > 0$, but for $\sqrt{\mu} \le \lambda_c(\Omega)$ the product $(I - \sqrt{\mu}\mathcal{G})\mathcal{G}$ is positive. This concludes the proof of (2.4).

Connected intervals. Due to the trivial coupling in (1.3) one has $u = \mathcal{G}f - \lambda\mathcal{G}^2 f$, and one directly finds that if for some $f \le 0$ and $\lambda > 0$ the solution u of (1.3) changes sign, then for any $\lambda^* > \lambda$ the corresponding solution u^* satisfies $u^* < u$ and is hence somewhere negative. Therefore the set of admissible λ's is an interval.

In order to show that the μ and b for which (1.1) respectively (1.2) are positivity preserving form an interval, one may proceed as follows. Since $\mathcal{G}$ is compact, $I + \mu\mathcal{G}^2$ is Fredholm and since the eigenvalues equal $1 + \mu(\lambda_i(\Omega))^{-2}$, the operator $(I + \mu\mathcal{G}^2)$ is invertible for all $\mu \ge 0$. Hence $u = (I + \mu\mathcal{G}^2)^{-1}\mathcal{G}f$ and one finds

$$\frac{d}{d\mu}(I + \mu\mathcal{G}^2)^{-1}\mathcal{G} = -(I + \mu\mathcal{G}^2)^{-2}\mathcal{G}^3,$$

which means that $\mu \mapsto (I + \mu\mathcal{G}^2)^{-1}\mathcal{G}$ is decreasing as long as $(I + \mu\mathcal{G}^2)^{-1}\mathcal{G}$ is positivity preserving. To make the argument precise, let us fix some $\mu > 0$ for which the system is positivity preserving and set $\mu_* \in (0, \mu)$. We may assume that $\mu \le \lambda_1(\Omega)\lambda_2(\Omega)$. Replacing μ by μ_* in (1.2) we may solve by

$$u_* = (I + \mu\mathcal{G}^2)^{-1}\mathcal{G}(\mu - \mu_*)\mathcal{G}u_* + (I + \mu\mathcal{G}^2)^{-1}\mathcal{G}f, \tag{5.12}$$

with $(I+\mu\mathcal{G}^2)^{-1}\mathcal{G}$ positive by assumption. Since for the spectral radius holds

$$\nu((I+\mu\mathcal{G}^2)^{-1}\mathcal{G}(\mu-\mu_*)\mathcal{G}) \le \frac{\mu}{\lambda_1(\Omega)^2+\mu} < 1, \tag{5.13}$$

we may solve by a Neumann series of positive operators as in (5.9) implying that u_* is positive:

$$u_* = \sum_{k=0}^{\infty} ((I+\mu\mathcal{G}^2)^{-1}\mathcal{G}(\mu-\mu_*)\mathcal{G})^k (I+\mu\mathcal{G}^2)^{-1}\mathcal{G}f.$$

In a similar way one proceeds for b, using $v = (I+\mu\mathcal{G}^2)^{-1}\mathcal{G}^2 f$. In fact an inspection of (5.8), with $a(x)$ replaced by $b_* \in (0,b)$, shows that it is still positivity preserving for b_*.

Sign change. We have shown for all three cases there is $f > 0$ such that, when the parameter becomes large, the corresponding solution cannot be positive. It remains to show that it cannot be purely negative either, but that it must change sign. Testing the corresponding solution with φ_1 one finds, respectively for (1.1), (1.2), and (1.3), that

$$\int_\Omega \varphi_1 v\,dx = \frac{1}{\lambda_1^2+b}\int_\Omega \varphi_1 f\,dx > 0,$$
$$\int_\Omega \varphi_1 u\,dx = \frac{\lambda_1}{\lambda_1^2+b}\int_\Omega \varphi_1 f\,dx > 0,$$
$$\int_\Omega \varphi_1 u\,dx = \frac{\lambda_1-\lambda}{\lambda_1^2}\int_\Omega \varphi_1 f\,dx.$$

The first two inequalities imply that u and v are somewhere positive, and hence sign-changing. For the solution of (1.3) this argument holds true only for $\lambda \in \langle \lambda_c(\Omega), \lambda_1(\Omega)\rangle$. If $\lambda \ge \lambda_1(\Omega)$, then u is somewhere negative for every positive f. In this situation it remains to show that there is a positive f such that u is somewhere positive. For this purpose one can take f sufficiently close to a Dirac δ-function at y, with y sufficiently close to the boundary for small dimensions, or even anywhere for higher dimensions. Then $u(\cdot)$ is close to a linear combination $G_1(\cdot,y) - \lambda G_2(\cdot,y)$ of Green functions. Therefore we can use explicit estimates for the Green function from [10, 20], [15, page 143].

We have to distinguish several cases according to the space dimension n.

If $n = 1$, $u(x) = (1-x)(1-\lambda x(2-x)/6)$ and $v(x) = 1-x$ satisfy (1.3) in the limit case $k \to \infty$ for $f_k = \frac{2}{3}k^2\mathcal{X}_{[1/k,2/k]}$, with $\mathcal{X}$ the characteristic function. Note that u is positive for x near zero for any $\lambda > 0$, and negative near 1 for $\lambda > 6 = \lambda_c(I)$. See also the proof of Proposition 3.1.

If $n = 2$, then the Green function G_1 has a logarithmic singularity $-\log|x-y|$, while G_2 is bounded. Therefore $G_1(\cdot, y) - \lambda G_2(\cdot, y)$ is positive near y.

If $n \geq 3$, then the singular part of $G(x,y)$ behaves like $|x-y|^{2-n}$, while G_2 has a singularity which grows like $|x-y|^{4-n}$ for $n > 4$. For $n = 4$ the singularity of G_2 is logarithmic, and for $n = 3$ the function G_2 is even bounded. Therefore in each case $G_1(\cdot, y) - \lambda G_2(\cdot, y)$ is positive near y. □

6. DERIVATION OF THE EXPLICIT NUMBERS IN ONE DIMENSION

Proof of Proposition 3.1.
◇ The number λ_c follows through direct computation of the Green function and the iterated Green function as has been done in [19]. Let us recall some elementary results. Direct computations yield

$$G_1(x,y) = \begin{cases} x(1-y) & \text{for } 0 \leq x \leq y \leq 1, \\ y(1-x) & \text{for } 0 \leq y < x \leq 1, \end{cases}$$

$$G_2(x,y) = \begin{cases} \frac{1}{6}x(1-y)(1-x^2-(1-y)^2) & \text{for } 0 \leq x \leq y \leq 1, \\ \frac{1}{6}y(1-x)(1-y^2-(1-x)^2) & \text{for } 0 \leq y < x \leq 1, \end{cases}$$

and $\frac{1}{6}$ as the maximum of the quotient $G_2(x,y)/G_1(x,y)$. Hence $\lambda_c(I) = 6$ follows from (2.2).

The solution v of (1.1) can be written in terms of a Green function:

$$v(x) = \int_\Omega B_\delta(x,y) f(y)\, dy, \tag{6.1}$$

with $\delta = \sqrt[4]{b/4}$ and where for $x, y \in \langle 0, 1\rangle$

$$\begin{aligned} &8\delta^3(\cosh(2\delta) - \cos(2\delta))B_\delta(x,y) \\ &\quad = \cosh((2-|x-y|)\delta)\sin(|x-y|\delta) + \cos((2-|x-y|)\delta)\sinh(|x-y|\delta) \\ &\qquad - \cosh((2-(x+y))\delta)\sin((x+y)\delta) - \cos((2-(x+y))\delta)\sinh((x+y)\delta) \\ &\qquad + \cosh((x-y)\delta)\sin((2-|x-y|)\delta) + \cos((x-y)\delta)\sinh((2-|x-y|)\delta) \\ &\qquad - \cosh((x+y)\delta)\sin((2-(x+y))\delta) - \cos((x+y)\delta)\sinh((2-(x+y))\delta). \end{aligned}$$

Such a formula has been derived by Ulm in [21]. The numbers μ_c and b_c appear as follows. We look for a parameter δ so that the one-dimensional function $x \mapsto B_\delta(x,y)$ loses its concavity for some y, while b_c appears when this mapping ceases to be positive. Schröder [18, page 101] suggests (and for b_c he proves) that both effects happen first when x and y are in opposite endpoints, e.g for $x = 0$ and $y = 1$. A direct approach for b_c can be found in [21].

Remark 6.1. A physical interpretation goes as follows. See also [12]. We interpret $B_\delta(x, y)$ as the deformation at point x of a hinged beam, embedded in an elastic ambient medium with elasticity constant b, and exposed to an upward pointing point load at y. The ambient medium exerts a force proportional to the vertical displacement which tries to drive the beam back into the trivial shape. For $b = \delta = 0$, and any y, the deformation is clearly concave in x, but for larger δ we expect its influence to destroy concavity or even positivity. In fact, obviously concavity gets destroyed before positivity, which is expressed in (2.4). Both effects are most likely to happen when x is as far away as possible from y.

◇ The number b_c can be traced in the the papers of Schröder, [17] and [18]. His elegant arguments can be applied in a rather general setting. Unaware of this earlier result, the number has recently been computed again by a brute force method in [21].

Schröder finds that this critical number b_c appears as an eigenvalue with a positive eigenfunction for the following problem:

$$\begin{cases} \psi^{iv} + b\psi = 0 & \text{in } \langle 0, 1\rangle, \\ \psi(0) = \psi'(0) = \psi''(0) = 0, \\ \psi(1) = 0. \end{cases} \tag{6.2}$$

The corresponding eigenfunction, with $4\delta_0^4 = b_c$, is as follows:

$$\psi(x) = \sin(\delta_0 x)\cosh(\delta_0 x) - \cos(\delta_0 x)\sinh(\delta_0 x). \tag{6.3}$$

The condition $\psi(1) = 0$ determines δ_0 as the first positive zero κ_1 of $\tan(\delta) - \tanh(\delta)$. The corresponding $b = 4(\kappa_1)^4$ is called b_c.

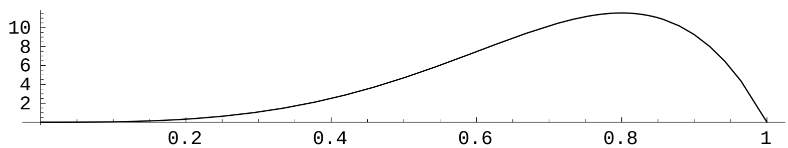


FIGURE 1. The eigenfunction for (6.2) with $\delta_0 = \kappa_1$.

Since the paper of Schröder [17] is not easily accessible, we take the liberty to give a rough sketch of his arguments.

Let us denote by B_δ the Green function for (1.1), where $4\delta^4 = b$ and $\Omega = \langle 0, 1\rangle$. For $\delta = 0$ the Green function is positive, and by continuously increasing δ one reaches a first $\delta_0 > 0$ such that a further increase leads to sign change. For the Green function B_{δ_0} there exists x_0 and y_0, with $0 \le x_0 \le y_0 \le 1$, such that

$$B_{\delta_0}(x_0, y_0) = \frac{\partial}{\partial x} B_{\delta_0}(x_0, y_0) = 0. \tag{6.4}$$

Note that the eigenvalues lie on the other side of 0. Let us assume that x_0 or y_0 lies in the interior. Using the continuity of $(\partial^k/\partial x^k)B_\delta(\cdot, y_0)$, $k = 0, 1, 2$, and the jump in the third derivative at y_0, one obtains two coupled eigenvalue problems for $x \mapsto B_\delta(x, y_0)$ on $(0, y_0)$ and $(y_0, 1)$. The 'smallest' δ that allows a nontrivial function satisfying (6.4), one gets for $x \downarrow 0$. In a similar way, considering $y \mapsto B_\delta(x_0, y)$, the smallest δ 'appears' for $y \uparrow 1$. The word appears is misleading since $\lim_{y\uparrow 1} B_\delta(x, y) \equiv 0$. Only after scaling to

$$\psi(x) := \lim_{y\uparrow 1} \frac{B_{\delta_0}(x, y)}{1 - y} \tag{6.5}$$

one obtains a nontrivial solution to (6.2). The jump-condition replaces

$$\frac{\partial^2}{\partial x^2} B_{\delta_0}(x, 1)_{x=1} = 0$$

by $(\partial/\partial x)B_{\delta_0}(x, 1)_{x=1} = 0$. The scaling in (6.5) replaces $(\partial/\partial x)B_{\delta_0}(x, 1)_{x=1} = B_{\delta_0}(x, 1)_{x=1} = 0$ by $\psi(1) = 0$; the condition in (6.4) adds $\psi'(0) = 0$. In other words, ψ is an eigenfunction to (6.2).

$\diamond$ The number μ_c is obtained by a similar approach, but will involve a nonlocal term. Although the methods of Schröder do give the result, we have not been able to trace an actual statement in the literature.

In terms of the inverted Dirichlet Laplace operator $\mathcal{G}$ we can rewrite (1.2) as

$$\begin{cases} -u'' + \mu \mathcal{G} u = f & \text{in } \langle 0, 1\rangle, \\ u(0) = u(1) = 0, \end{cases} \tag{6.6}$$

and we are interested in the critical μ for which the positivity preserving property of (6.6) breaks down. Let us denote D_δ the Green function for (6.6), with $4\delta^4 = \mu$. Again one increases δ until one encounters δ_0, after which a sign change will start. It will not be an eigenvalue, since these lie on the other side of 0. The remaining possibility is that there is (x_0, y_0) with

$$D_{\delta_0}(x_0, y_0) = 0. \tag{6.7}$$

Again we may assume that $0 \le x_0 \le y_0 \le 1$. Like for B_δ, one finds by comparison that the critical case occurs for $x_0 \downarrow 0$ and $y_0 \uparrow 1$. And again a nontrivial function is obtained after rescaling

$$\psi(x) := \lim_{y\uparrow 1} \frac{D_{\delta_0}(x, y)}{1 - y}. \tag{6.8}$$

This scaling removes the condition $\psi(1) = 0$ and by (6.7) an extra condition at 0, namely $\psi'(0) = 0$, appears. In other words, ψ is a positive eigenfunction to

$$\begin{cases} -\psi'' + 4\delta_0^4 \mathcal{G} \psi = 0 & \text{in } \langle 0, 1\rangle, \\ \psi(0) = \psi'(0) = 0, \end{cases} \tag{6.9}$$

which is, using the boundary conditions for $\mathcal{G}$, equivalent to

$$\begin{cases} \psi^{iv} + 4\delta_0^4\psi = 0 & \text{in } \langle 0, 1\rangle, \\ \psi(0) = \psi'(0) = \psi''(0) = 0, \\ \psi''(1) = 0. \end{cases} \tag{6.10}$$

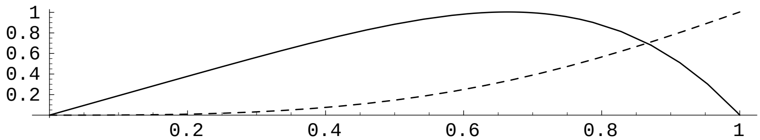


FIGURE 2. The eigenfunction ψ (dashed) and $u = \psi''$ (solid) for (6.10) with $\delta_0 = \kappa_0$, both ψ and u are normalized.

Hence ψ is as in (6.3), except that the condition at 1 now determines δ_0 as the first positive zero of

$$\psi''(x) = 2\delta_0^2(\cos(\delta_0 x)\sinh(\delta_0 x) + \sin(\delta_0 x)\cosh(\delta_0 x)), \tag{6.11}$$

that is, the first positive zero κ_0 of $\tan(\delta) + \tanh(\delta)$. The corresponding $\mu = 4(\kappa_0)^4$ we call μ_c. □

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