Convex sets of constant width

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A bounded convex set has constant width d iff any two parallel (and nonidentical) tangent planes to it have identical distance d from each other. Clearly balls have this property, but there are also other sets of constant width. This lecture was originally designed for a general audience as part of a series of lectures during the German "Year of Mathematics" 2008. It starts by presenting evidence from the Challenger desaster [11]. A lack of geometric insight was a serious contributing factor to this accident. Then the lecture treats two- (and later three-)dimensional sets of constant width and their occurence in daily life, for instance as shapes of coins [18]. These require in general less material than circular ones, because Barbier proved the following interesting isoperimetric property at the age of 21.

Theorem 1: (Barbier 1860) *All plane convex sets of constant width d have the same perimeter πd as the disc of diameter d.*

So by the classical isoperimetric inequality the disc has maximal area among all plane convex sets of given constant width. Another elegant and elementary proof that uses only the theorem of Pythagoras and polar coordinates was given by Littlewood in [19].

There are many plane convex sets of constant width. Their support function $p(\theta)$ necessarily satisfies the functional equation

$$p(\theta) + p(\theta + \pi) = d \qquad \text{on } [0, 2\pi], \tag{1}$$

and this equation has many solutions, for instance $p(\theta) = \frac{d}{2} + \varepsilon \sin(k\theta)$. It is only natural to ask for the shape of a coin that uses least material for a given width, and this question is answered by what is commonly called the Theorem of Blaschke and Lebesgue.

Theorem 2: *Among all plane convex sets of constant width d the Reuleaux-triangle minimizes area.*

The Reuleaux triangle is the intersection of three discs of radius d with centers at the corners of an equilateral triangle with sides of length d. Its beauty has inspired artists and architects as well as engineers. Very different proofs of this theorem were provided by Blaschke [4], Lebesgue [21, 22], Fujiwara [13], Eggleston [10], Besicovich [3], Ghandehari [14], Campi, Colesanti & Gronchi [7] and Harrell [15]. One can calculate the area A of a convex set in terms of its support function p and try to minimize A. This leads to the variational problem of minimizing the functional

$$A(p) := \int_0^{2\pi} \{p(\theta)^2 - p'(\theta)^2\}\, d\theta \qquad \text{among } 2\pi\text{-periodic functions } p \tag{2}$$

under the nonlocal constant width constraint (1) and the convexity constraint

$$p''(\theta) + p(\theta) \geq 0 \qquad \text{on } [0, 2\pi]. \tag{3}$$

Notice that (2) is a nonconvex minimization problem under nonstandard side constraints, and it cannot be attacked by direct methods in the calculus of variations.

Reuleaux triangles can be used to construct drills that drill square holes, see [23] for a video-clip or [27] for an instructive animation, or machines that transform a rotation into a sliding and stopping motion, see [26]. Reuleaux built and sold collections of small gears as instructional tools for students, and one of those collections has survived at Cornell University and was recently put on the web. Such movements were used in movie-projectors, see [28]. Reuleaux was an impressive scientist. I report on some of his achievements. He has only recently been compared to Leonardo da Vinci by Francis C. Moon, who discovered and saved the collection of Reuleaux's kinematic models at Cornell University [24, 26].

There are also threedimensional convex bodies of constant width. Photos of plaster models can be found in [16] or [12], other shapes in the website that comes with [6]. Incidentally, Stefan Cohn-Vossen was a postdoc of Courant, came to Cologne and gave his "Antrittsvorlesung"on convex surfaces on Feb 22, 1932. In April 1933 he was temporarily suspended from teaching because he was Jewish, in September 1933 he lost his job permanently. He emigrated to Moscow, where he died 1936 of pneumonia. It is truely admirable that Richard Courant, who was also driven out of this country, was later of instrumental help in supporting the Oberwolfach Institute. As in the twodimensional case, one ask if there is an analogue to Barbier's Theorem and one can try to maximize or minimize the volume of convex bodies of constant width.

Theorem: (Blaschke 1915) *Among all 3d convex bodies of given width d the ball maximizes volume and surface area, and the one that minimizes volume also minimizes surface area.*

The body of minimal volume or surface area is unknown, but there is a suspect.

Conjecture: (documented 1934 by Bonnesen & Fenchel) *The threedimensional convex bodies of constant width that minimize volume are exactly Meissner's bodies.*

Pictures of these bodies can be found at [25, 16, 12]. They are essentially constructed from modifications of a Reuleaux-tetrahedron, the intersection of four balls centered at the corners of a regular tetrahedron. There is, however an answer to the minimal volume problem if we look in the smaller class of rotational bodies,

Theorem: (Campi et al. 1996) *Among the class of rotational convex bodies of constant width d, the one that minimizes volume is the rotated Reuleaux triangle.*

Finally I mention two recent results that support the above conjecture. The first one shows how to construct an n-dimensional body of constant width from an $(n-1)$-dimensional one.

Theorem: (Lachand-Robert, Oudet 2006) *Suppose that $E_\pm$ denote the upper and lower half-plane in $\mathbb{R}^n$. Let $K_0 \subset E_+ \cap E_-$ be an $(n-1)$–dim. const. width body, and $Q \subset \mathbb{R}^n$ satisfy $K_0 \subset Q \subset E_- \bigcap_{x \in K_o} B(x,d)$. Set $K_+ := E_+ \cap \bigcap_{x \in Q} B(x,d)$ and $K_- := E_- \bigcap_{x \in K_+} B(x,d)$. Then $K := K_+ \cup K_-$ is an n-dimensional constant width body with K_0 as a cross section.*

If in this construction $n = 2$ and $Q = K_0 = (0,d)$, then K is the Reuleaux-triangle, and if $n = 3$ and $Q = K_0 =$ Reuleaux-triangle, then K is a Meissner body. Although this construction seems to be exhaustive only for $n = 2$, see [9], it can be used to randomly generate many threedimensional bodies of constant width. A student of mine, Martin Müller, has recently generated one million of those. None of them had smaller volume than the Meissner bodies.

And in a recent paper [2] Bayen, Lachand-Robert & Oudet derive a necessary condition that characterizes a 3d constant width body of (locally) minimal volume: If one squeezes such a body between two parallel planes, at one of the two points of tangency its surface is not smooth.

Clearly Meissner's bodies satisfy this condition, while a ball does not, and this supports the conjecture, but it does not prove it.

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[28] URL http://www.etudes.ru/ru/mov/mov001/index.php (animation of applications of the Reuleaux triangle in Wankel enginges and movie projectors)