Remarks on Quenching

Bernd Kawohl

Received: November 2 1995
Revised: April 29, 1996

Communicated by Bernold Fiedler

Abstract. Consider the parabolic problem

$$u_t - \text{div } (a(u, \nabla u)\nabla u) = -u^{-p} \tag{1}$$

for $t > 0$, $x \in \mathbb{R}^n$ under initial and boundary conditions $u = 1$, say. Since p is assumed positive, the right hand side becomes singular as $u \to 0$. When u reaches zero in finite or infinite time, one says that the solution quenches in finite or infinite time. This article gives a survey of results on this kind of problem and emphasizes those that have been obtained at the SFB 123 in Heidelberg. It is an updated version of an invited survey lecture at the International Congress of Nonlinear Analysts in Tampa, August 1992. To be specific, I shall cover existence and nonexistence of quenching points, asymptotic behaviour of the solutions in space and time near the quenching points, qualitative behaviour, application to mean curvature flow and phase transitions, reaction in porous medium flow etc..

The tools are variational methods and suitable maximum principles. Many of the results presented in this article were obtained with my coauthors Acker, Dziuk, Fila, Kersner and Levine, but related results will also be mentioned.

1991 Mathematics Subject Classification: 35K65, 35K57, 35K60, 35B05, 35B65

Model Problems

For the sake of simplicity I shall discuss four special cases of (1), namely:

$$u_t - \Delta u = -u^{-p}, \tag{A}$$

$$u_t - (\varphi(u_x))_x = -u^{-p}, \tag{B}$$

$$u_t - \frac{u_{xx}}{1 + u_x^2} = -\frac{1}{u}, \tag{B'}$$

$$u_t - (u^m)_{xx} = -u^{-p}. \tag{C}$$

Note that for $n = 1$ case (A) is a special case of both (B) and (C). Equation (B') is a special case of (B), which has a significant application in the mathematical description of mean curvature flow of rotationally symmetric twodimensional surfaces in $\mathbb{R}^3$. To see this imagine the x-axis to be the axis of a revolutionary body whose surface at time t is described by $u(t,x)$. Then (see Figure 1) its inward velocity v is given by

$$v = \frac{u_t}{\sqrt{1+u_x^2}},$$

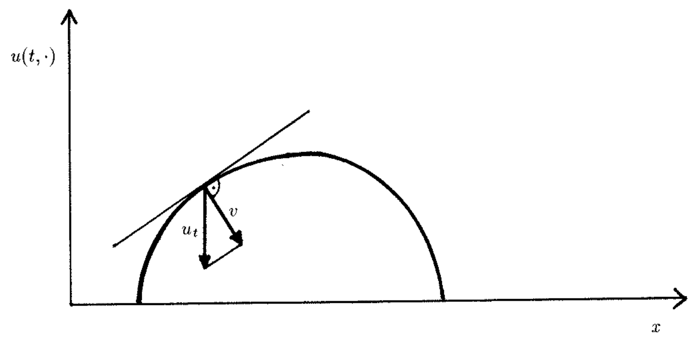

FIGURE 1: Derivation of (B')

while its principle curvatures are

$$\frac{u_{xx}}{\sqrt{1+u_x^2}^3} \qquad \text{in } x\text{−direction,}$$
$$-\frac{1}{u}\frac{1}{\sqrt{1+u_x^2}} \qquad \text{in } v\text{−direction.}$$

Therefore, after rescaling time by a factor of two, the mean curvature flow of our surface is described by

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{1}{u} = (\arctan u_x)_x - \frac{1}{u},$$

and, incidentally, this is how one can see that (B') is a particular case of (B).

Chronologically ordered, my coauthors and I wrote the following papers on Problems (A), (B) and (C). Problem (A) was dealt with in [AK, KP, FK1, FK2] and [K], Problem (B') was studied in [DK,K], Problem (B) was treated in [FKL] and Problem (C) in [KK].

Quenching Occurs

Let us first consider Problem (A):

$$\begin{aligned} u_t - \Delta u &= -u^{-p}, && x \in \Omega,\ t > 0, \\ u &= 1 && \text{on the parabolic boundary} \end{aligned} \tag{2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$. For $n = 1$ and $p = 1$ this problem was studied by Kawarada [Ka] and he stated the following result.

Theorem 1: *If there are no stationary solutions to* (2), *i.e. if Ω is too large, then*
i) u reaches zero in some point x_0 in finite time T.
ii) $u_t(t, x_0) \to -\infty$ as $t \nearrow T$.

Statement i) has been derived for numerous more general situations, e.g. by Acker and Walter, Levine and Montgomery or Lieberman to higher dimension, hyperbolic equations, nonlinear boundary conditions such as $(\partial u/\partial n) = -u^{-p}$ and so on. One can prove i) by energy methods or by comparison principles. The proof of ii) was wrong as stated by Kawarada. This was noted and corrected by Chan and Kwong, and by Acker and myself in 1987. Moreover, it was shown in [AK] that both statements of Theorem 1 hold for general $n \in \mathbb{N}$ and $p > 0$, provided Ω is a ball. In this case x_0 is uniquely determined and is the center of the ball.

In the same year I discovered why quenching and blow up problems have so much in common. In fact one can be transformed into the other, see [KP]. The fact that both classes of problems are amenable to similar techniques had puzzled people, e.g. Bandle and Stakgold [BS] or Friedman and Herrero [FH], who had studied equations like (A) with $p \in (-1, 0]$, a less singular case than ours. This observation was useful, because now one could try to mimic blow-up results like the ones of Friedman and McLeod for quenching problems. And in fact, using techniques from [FM], Deng and Levine were able to show in 1988 that both statements of Theorem 1 could be extended from balls to convex domains Ω. A year later Fila and I found the blow up rate of $|u_t|$.

Time Asymptotics

Let u be a solution of (2) and suppose that u quenches at $t = T,\ x = 0$. Then the following estimates are known.

There exists a constant $c \geq 0$ such that for $t < T$

$$c \leq \min_{x \in \Omega}\{u(t,x)\}(T-t)^{-1/(1+p)} \leq (1+p)^{1/(1+p)}. \tag{3}$$

Moreover, if Ω is convex, $c > 0$, see [FK1]. Relation (3) holds for general $n \in \mathbb{N}$, $p > 0$ and Ω.

For any positive constant C and for $t < T$, $|x| \leq C(T-t)^{1/2}$ we have

$$\lim_{t \to T} u(t,x)(T-t)^{-1/(1+p)} = (1+p)^{1/(1+p)} \quad . \tag{4}$$

This result was first established for $n = 1$ and $p \geq 3$ by Guo in 1988, and subsequently generalized to $n = 1$ and $p \geq 1$ by Fila and Hulshof, and to general n, nonnegative p and Ω a ball by Fila, Hulshof and Quittner.

Time asymptotics of this nature have been extended to equations of type (B) and (B'), see [FKL] or [SS]. It is also possible to extend such results to equation (C), see [KK].

How does one get the upper bound in (3)? This one is easy. In a spatial minimum we have $\Delta u \geq 0$, so there $u_t \leq -u^{-p}$, or equivalently

$$\frac{1}{p+1}(u^{p+1})_t = u^p \; u_t \leq -1. \tag{5}$$

An integration of (5) from t to T yields $0 - u^{p+1}(t) \leq -(p+1)(T-t)$, i.e the desired upper bound for u. To derive the lower bound in (3) one shows

$$u_t + \delta u^{-p} \leq 0$$

for some $\delta > 0$ and for (t, x) in some subcylinder of $(0, T) \times \Omega$. Here the idea of proof is essentially due to [FM].

SPACE ASYMPTOTICS AT $t = T$

Consider equation (A) and suppose that $\Omega \subset \mathbb{R}^n$ is a ball with center in the origin. For simplicity, suppose that $u(0, x) \equiv 1$. Then the following inequalities were derived in 1989 by M.Fila and myself, see [FK1].

$$u(T, r) \leq \left[\frac{(p+1)^2}{2(1-p)}\right]^{1/(1+p)} \left(r^2\right)^{1/(1+p)} \qquad \text{for } 0 < p < 1, \tag{6}$$

$$u(t, r) \quad \geq \quad C_\varepsilon \left(r^2\right)^{\varepsilon + 1/(1+p)} \qquad \text{for } 0 < p, \tag{7}$$

for $t < T$. These inequalities tell us, that for $p < 1$ the function $u(T, \cdot)$ is of class C^1 at the origin, while for $p > 1$ it has a cusp-singularity and is merely Hölder continuous in the origin, see Figure 2.

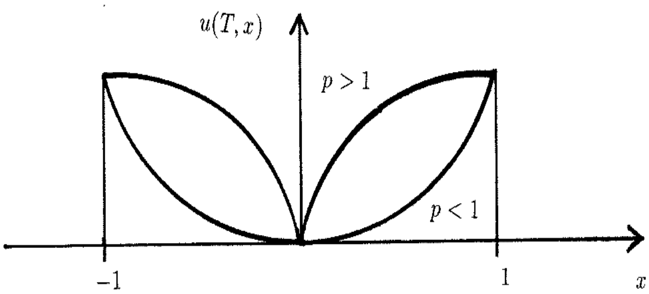

FIGURE 2: Shape of $u(\cdot, T)$

This distinction is consistent with the observation that $p < 1$ means less absorption than $p > 1$. Inequalities (6) and (7) are an ε apart. So the exact profile was still to be found. I had conjectured, but was unable to prove that for $p = 1$ the solution should develop a corner in the origin, like $u(r, T) \approx |r|$. I had been wrong, because in 1991 Filippas and Guo were able to find the exact asymptotics in the case $n = 1$ as follows

$$u(t,x) = \left[\frac{(p+1)^2}{8p}\right]^{1/(1+p)} \left(\frac{|x|^2}{|\ln|x||}\right)^{1/(1+p)} (1 + o(1)) \tag{8}$$

as $|x| \to 0$. This is definitely a sharper result. Again the method of proof relied on a corresponding blow up result, this time due to Herrero and Velazquez. In 1991 Fila, Levine and I generalized estimates (6) and (7) to equations (B) and (B'). In the context of equation (B') and for $p = 1$ one can interpret (7) as characterizing the rate at which the curvature of a rotational surface blows up. In fact, differential geometers like Huisken have found similar estimates for mean curvature flow in nonrotational settings as well.

Why is the assumption $p < 1$ made in (6)? To see this and to present another popular trick consider a solution u of the equation

$$u_t - \Delta u = -f(u)$$

and set

$$P(t,x) = \frac{1}{2}|\nabla u|^2 - F(u),$$

where $F'(u) = f(u)$. The letter P stands for L.Payne, who made this trick widely known, see [S]. A straightforward calculation shows that P satisfies the differential equation

$$P_t - \Delta P + b \cdot \nabla P \leq 0 \tag{9}$$

with $b = |\nabla u|^{-2}(2f(u)\nabla u - \nabla P)$ in $L^\infty_{loc}(\{(t,x) \mid 0 < t < T, |\nabla u(t,x)| \neq 0\})$. Now for $p < 1$ we have $P = -F(u) = -\frac{1}{1-p}u^{1-p} \leq 0$ in those points where $|\nabla u| = 0$. Thus, by the maximum principle, P must attain its maximum initially or on the lateral boundary of $[0,T] \times \Omega$. Since for convex Ω one can rule out that P attains its maximum on the boundary, and since $P(0,x) \leq 0$, we know $P(t,x) \leq 0$ or, in the case that Ω is a ball

$$u_r^2 \leq \frac{2}{1-p}u^{1-p}.$$

But now $u^{(p-1)/2}u_r \leq \sqrt{2/(1-p)}$ or

$$\frac{\partial}{\partial r}\left(u^{(p+1)/2}\right) = \left(\frac{1+p}{2}\right) u^{(p-1)/2}u_r \leq \left(\frac{1+p}{2}\right)\sqrt{2/(1-p)} = \frac{1+p}{\sqrt{2(1-p)}}.$$

An integration at $t = T$ yields

$$u^{(p+1)/2}(T,r) - u^{(p+1)/2}(T,0) \leq \left[\frac{1+p}{\sqrt{2(1-p)}}\right] r,$$

that is (6)

$$u(T,r) \le r^{2/(1+p)} \left[\frac{1+p}{\sqrt{2(1-p)}} \right]^{2/(1+p)} .$$

Location of Quenching Points

Can one predict the points where a solution will quench? This question is related to the prediction of blow up points, and one of the early results on blow up stated single point blow up, see Weissler [W]. For the case of equation (A) and Ω being a ball, and under restrictions on the initial data, in 1987 Acker and I derived the inequalities

$$\begin{aligned} u_t &\le 0 \\ ru_r = x \cdot \nabla u &\ge 0 \\ u_{rt} &\ge 0 \end{aligned}$$

in the parabolic time space cylinder $(0,T) \times \Omega$, and this implied that u quenches only in the origin, so one has a single point quenching result.

But more can sometimes be said for general initial data. In fact, for $n = 1$ and $p < 0$, Chen, Matano and Mimura have been able to derive finite point quenching results. They used lap-number type arguments to justify the occurrence of finitely many spatial oscillations of u after short time; and then they localized the above type of inequalities. This was nontrivial, because spatial minima of u can move in time.

Of course single point quenching results can also be shown for more general equations such as (B) or (B'), see [DK,FKL,AAG]. It is important to note though that in general it is necessary that $u_t \le 0$ when u gets small.

Life after Quenching

What happens after $t = T$ to a solution u of (1)? The answer depends on the notion of solution that we are willing to accept and on p. Suppose that the nonlinearity u^{-p} is regularized by the finite nonlinearity $u/(\varepsilon + u^{p+1})$. One can hope that then a classical global solution u_ε of (1) exists for every positive ε, that u_ε is decreasing in ε, and that it has a limit U as $\varepsilon \to 0$, which coincides with u for $t < T$. This hope has been replaced by a proof

a) in case of equation (A) and for $p < 1$ by D.Phillips [P], and
b) in case of equation (C) and for $p < m$, $m \ge 1$ in [KK].

In both cases there are regions in which $U = 0$, and in case a) the ω-limit set of U consists of equilibria or steady states, see [FLV,KK]. Moreover U is a global weak solution of

$$u_t - \Delta(u^m) = -u^{-p}\chi_{\{u>0\}},$$

for which uniqueness still appears to be open. So much for the case $0 < p < 1$.

If $p > 1$ and $n = 1$ I conjecture total quenching, that means I believe that $\lim_{\varepsilon\to 0} u_\varepsilon(x, T+\delta) = 0$ for every $x \in \Omega$ and every $\delta > 0$. The heuristic reason for this conjecture is the nonintegrability of u^{-p} as well as a corresponding total blow up result of Baras and Cohen. For $n \geq 2$ the situation is more complicated, see [FK1]. If Ω is convex, then

$$\int_\Omega u^{-q}\,dx \begin{cases} < \infty & \text{for } q < \frac{N}{2}(1+p) \\ = \infty & \text{for } q \geq \frac{N}{2}(1+p) \end{cases} \qquad \text{as } t \to T.$$

Another indication for complete quenching was kindly pointed out to the author by the referee. Using a transformation as in [KP], Galaktionov and Vazquez [GV1,GV2] converted the quenching problem to a blow-up problem. After deriving blow up results for the Cauchy problem and quasi-linear parabolic equations they were recently able to confirm my conjecture on total quenching for the Cauchy problem on $\mathbb{R}^n \times \mathbb{R}_+$ for equations of type (B) and (C).

If $p = 1$, little seems to be known for equations (A) and (C), but much is known for (B'). In fact, if $u(x,t)$ describes the radius of a compact rotational surface moving by mean curvature, then $u_x = \pm\infty$ on the boundary of the support of u. So near this boundary, the dependent and independent variable can be interchanged and the surface could also be described by a function $v(r)$, see Figure 3.

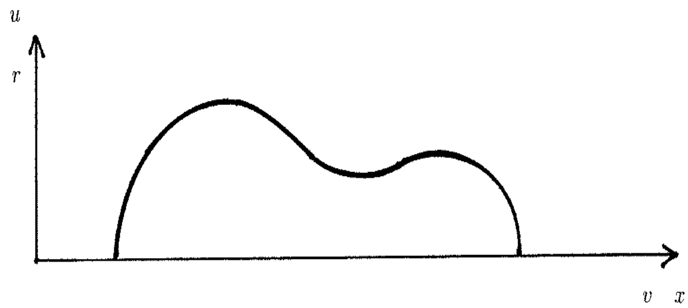

Figure 3: $u(t)$ and $v(r,t)$

The equation (B') for the horizontal graph

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{1}{u} \tag{B'}$$

is then transformed into almost the same equation for the vertical graph, see [AAG],

$$v_t = \frac{v_{rr}}{1+v_r^2} + \frac{1}{r}v_r \quad . \tag{10}$$

So we have a parabolic equation, i.e. (B'), whose solution exhibits a hyperbolic phenomenon: finite speed of propagation of the free boundaries. This is reminiscent of phenomena described in [BD]. Nevertheless, equation (10) enables one to continue the analysis until the surface completely collapses. Eventually, it collapses into isolated points but u can stay non-concave in x until the time of collapse, see [AAG].

If one tries to apply the same transformation trick to equation (A), the outcome looks as follows

$$v_t = \frac{1}{v_r^2} v_{rr} + \frac{v_r}{r^p}. \tag{11}$$

Now (11) is totally different in nature from (10), because on the free boundary $r = 0$ we have $v_r = 0$, and so (11) reflects a very degenerate situation with "infinite" diffusion, while $v_r = 0$ causes no problems in the coefficients of (10). Again infinite diffusion seems to support the idea of total quenching mentioned above.

It is interesting to note that equation (11) can be rewritten in divergence form as

$$v_t = -\left(\frac{v_r}{|v_r|^2}\right)_r + \frac{v_r}{r^p},$$

and this in turn is equivalent to

$$v_t = -\mathrm{div}(|\nabla v|^{q-2}\nabla v) + \frac{v_r}{r^p} \quad \text{with } q = 0 \quad . \tag{12}$$

Now (12) looks like a backward heat equation with the Laplacian replaced by Δ_q for $q = 0$. Forward equations with Laplacian replaced by Δ_q and $q > 1$ are somewhat understood, see [EV], but (12) is far away from this situation. Therefore studying the operator Δ_q for $q = 0$ appears to be worthwhile and not just another academic exercise.

REFERENCES

[AK] A.Acker & B.Kawohl: Remarks on quenching. Nonlinear Analysis, Theory, Methods and Applications, **13** (1989) p. 53–61

[AW] A.Acker & W.Walter: The quenching problem for nonlinear parabolic differential equations. In: *Ordinary and partial differential equations, Dundee 1976*, Eds. W.M.Everitt & B.D.Sleeman, Springer Lecture Notes in Math. **564** (1976) p. 1–12

[AAG] S.Altschuler, S.Angenent, Y.Giga: Mean curvature flow through singularities for surfaces of revolution. J. Geom. Anal. **5** (1995) p. 293–358

[BC] P.Baras & L.Cohen: Complete blow up after T_{max} for the solution of a semilinear heat equation. J. Funct. Anal. **71** (1987) p. 142–174

[BS] C.Bandle & I.Stakgold: The formation of the dead core in parabolic reaction diffusion problems. Trans. Amer. Math. Soc. **286** (1984) p. 275–293

[BD] M.Bertsch & R.Dal Passo: Hyperbolic phenomena in a strongly degenerate parabolic equation. Arch. Ration. Mech. Anal. **117** (1992) p. 349–387

[CK] C.Y.Chan & M.K.Kwong: Quenching phenomena for singular nonlinear parabolic equations. Nonlin. Anal. TMA **12** (1988) p. 1377–1383

[CMM] X.Chen, H.Matano & M.Mimura: Finite point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption. J. reine angew. Math. **459** (1995) p. 1–36

[DK] G.Dziuk & B.Kawohl: On rotationally symmetric mean curvature flow, J. Differ. Equations, **93** (1991) p. 142–149

[EV] J.R. Esteban & J.L.Vazquez: Homogeneous diffusion in $\mathbb{R}$ with power-like non-linear diffusivity. Arch. Ration. Mech. Anal. **103** (1988) p. 39–80

[FH] M.Fila & J.Hulshof: A note on the quenching rate. Proc. Amer. Math. Soc. **112** (1991) p. 473–477

[FHQ] M.Fila, J.Hulshof & P.Quittner: The quenching problem on the N-dimensional ball. In: *Nonlinear Diffusion Equations and their Equilibrium States, 3.* Eds.: N.G.Lloyd, W.M.Ni, J.Serrin, L.A.Peletier, Birkhäuser, Boston, (1992) p. 183–196

[FLV] M.Fila, H.A.Levine & J.L.Vazquez: Stabilization of solutions of weakly singular quenching problems. Proc. Amer. Math. Soc., **119** (1993) p. 555–559

[FK1] M.Fila & B.Kawohl: Asymptotic analysis of quenching problems, Rocky Mt. J. Math. **22** (1992) p. 563–577.

[FK2] M.Fila & B.Kawohl: Is quenching in infinite time possible? Quarterly of Applied Math. **48** (1990) p. 531–534

[FKL] M.Fila, B.Kawohl & H.Levine: Quenching for quasilinear parabolic equations. Comm. Partial Differ. Equations **17** (1992) p. 593–614.

[FG] S.Filippas, J.S.Guo: Quenching profiles for one-dimensional semilinear heat equations. Quarterly of Applied Math. **51** (1993) p. 713–729

[FH] A.Friedman & M.Herrero: Extinction properties of semilinear heat equations with strong absorption. J.Math. Anal. Appl. **124** (1987) p. 530–547

[FM] A.Friedman & J.B.McLeod: Blow up of positive solutions of semilinear heat equations. Indiana Univ. Math. J. **35** (1985) p. 425–447

[GV1] V. Galaktionov & J.L.Vazquez: Necessary and sufficient conditions of complete blow-up and extinction for one-dimensional quasilinear heat equations. Arch. Ration. Mech. Anal. **129** (1995) p.225–244.

[GV2] V. Galaktionov & J.L.Vazquez: Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, manuscript, 54 pp.

[G] J.Guo: On the quenching behaviour of the solution of a semilinear parabolic equation. J. Math. Anal. Appl. **151** (1990) p. 58–79.

[H] G.Huisken: Asymptotic behaviour for singularities of the mean curvature flow. J. Differ. Geometry **31** (1990) p. 285-299

[Ka] H.Kawarada: On solutions of initial boundary problem for $u_t = u_{xx} + 1/(1-u)$. Publ. Res. Inst. Math. Sci. **10** (1975) p. 729–736

[K] B.Kawohl: Remarks on quenching, blow up and dead cores. In: *Nonlinear Diffusion Equations and their Equilibrium States, 3.* Eds.: N.G.Lloyd, W.M.Ni, J.Serrin, L.A.Peletier, Birkhäuser, Boston, (1992) p. 275–286

[KK] B.Kawohl & R.Kersner: On degenerate diffusion with very strong absorption. Math. Methods Appl. Sci. **15** (1992) p. 469–477

[KP] B.Kawohl & L.Peletier: Observations on blow up and dead cores for nonlinear parabolic equations, Math. Zeitschr. **202** (1989) p. 207–217

[Le] H.Levine: Advances in quenching. in: *Nonlinear Diffusion Equations and their Equilibrium States, 3.* Eds.: N.G.Lloyd, W.M.Ni, J.Serrin, L.A.Peletier, Birkhäuser, Boston, (1992) p.319–346

[LM] H.A.Levine & J.T.Montgomery: Quenching of solutions of parabolic equations with nonlinear boundary conditions in several dimensions. J. reine angew. Math. **345** (1983) p. 23–28

[L] G.Lieberman: Quenching of solutions of evolution equations. Proc. Centre Math. Anal. Austral. Natl. Univ. **8** (1984), Miniconference on Nonlinear Analysis p. 151–157

[P] D.Phillips: Existence of solutions to a quenching problem. Appl. Anal. **24** (1987) p. 253-264

[SS] H.M.Soner & P.E.Souganidis: Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature. Comm. PDE **18** (1993) p. 859–894

[S] R.Sperb: *Maximum principles and their applications,* Academic Press, New York 1981

[W] F.B.Weissler: Single point blow up for a semilinear initial value problem. J. Differ. Equations **55** (1984) p. 204-224

Bernd Kawohl
Mathematisches Institut
Universität zu Köln
D 50923 Köln
Germany
kawohl@mi.uni-koeln.de