A symmetry problem related to Wirtinger's and Poincaré's inequality

by M.Belloni (Parma) and B.Kawohl (Cologne)

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1. Introduction

In [1] Dacorogna, Gangbo and Subia investigate minimizers of

$$J(u;-1,1) = \frac{||u'||_p}{||u||_q}$$

among periodic functions in $W^{1,p}(-1,1)$ with mean value zero, and for $p, q \in (1,\infty)$. The minimizers are scaled in such a way that their maximum is $+1$ and their minimum is $-m \in (0,-1]$. They raise the question if $m = 1$, since this implies symmetries for the minimizers. Their answer is positive for $q \leq 2p$ and negative for $q >> 1$. Practically the same question is raised in [2] and [3]. If $m = 1$ minimizers must be odd (or symmetric in the sense that $u(x) = -u(-x)$). It was shown in [3] that for $p = q = 2$ the minimizer is symmetric, and that for $p = 2$, $q = \infty$ it is not symmetric. Moreover in [2] one can find that the minimizer is symmetric for $q \leq 2p$ and nonsymmetric for for $q > 4p - 1$. The nonsymmetry result was derived by Egorov in the following way. Restricted to the following set A

$$A := \{v \in W^{1,p}(-1,1) \mid \int_{-1}^{1} v\ dx = 0,\ \exists b \in (-1,1) \text{ s. th. } supp\ v_+ = (-1,b)$$
$$\text{and } v_+ \text{ resp. } v_- \text{ minimize } J(u;(-1,b) \text{ resp.} J(u;b,1)\ \}$$

the minimum of the functional $J(u;-1,1)$ could be rewritten as a function $f^{1/p}(b)$ of b only, and this function constitutes an upper bound for J, which is sharp when $b = 1/2$. Egorov showed that for $q > 4p - 1$ the second variation at $b = 1/2$ is negative.

The main results of the present paper are:

Theorem 1: *The minimizer is symmetric for $q \leq 2p + 1$.*

Theorem 2: *If the class of admissible functions is further restricted to A then the minimizer is symmetric for $q \leq 4p - 1$.*

The proof of Theorem 2 makes use of Theorem 1 when (18) is derived.

Remark 1: The sufficient upper bound $q \leq 2p+1$ for the existence of symmetric minimizers and Egorov's sufficient lower bound $q > 4p-1$ for nonsymmetric minimizers meet at $p = 1$. In this respect Theorem 1 is optimal.

Remark 2: Theorem 1 is optimal in another way as well. For $q > 2p+1$ the function $f(m, u)$ defined in (3) is no longer nonnegative. This can be seen as follows. Since $f(m, 1) = 0$ it suffices to show that

$$\frac{\partial f}{\partial u}(m, 1) = q(m^{2p} - m^q) - r(m)(m + m^{2p}) > 0$$

for some $m \in (0, 1)$. Choosing $m = 1 - \varepsilon$ as suggested by B.Dacorogna and developing f_u in powers of ε yields

$$\frac{\partial f}{\partial u}(1-\varepsilon, 1) = q(q - (2p+1))\varepsilon + o(\varepsilon) > 0,$$

as desired.

Remark 3: In [2] $J(u)$ was minimized among functions in $W_0^{1,p}$ with mean value zero. In [3] $J(u)$ was minimized among (not necessarily periodic) functions in $W^{1,p}(-1, 1)$ with mean value zero. In this case the minimizers tell us the best constant in Poincaré's inequality. If these are evenly reflected across 1 and then periodically continued, we arrive at the problem in [1] on the interval $(-2, 2)$.

Remark 4: Numerical experiments seem to suggest that the symmetry breaking occurs precisely at or slightly above $q = 3p$.

2. Proof of the results.

Proof of Theorem 1: We follow the same reasoning as in [1, Lemma 3.7]. We intend to show that the expression

$$F(m, p, q) :=$$
$$\int_0^1 \left\{ \frac{u}{[1 - u^q - r(m)(1-u)]^{1/p}} - \frac{m^2 u}{[1 - (mu)^q - r(m)(1+mu)]^{1/p}} \right\} du$$

is positive for $q \in (2p, 2p+1]$, $p \in (1, \infty)$ and $m \in [0, 1)$. Here $r(m) = (1 - m^q)/(1+m)$. We will achieve this by showing that the integrand is positive. After multiplying the integrand with the common denominator and dividing it by u we want to show

$$[1 - (mu)^q - r(m)(1+mu)]^{1/p} \geq m^2[1 - u^q - r(m)(1-u)]^{1/p}, \tag{1}$$

or equivalently

$$1-(mu)^q - r(m)(1+mu) \geq m^{2p} - m^{2p}u^q - r(m)m^{2p}(1-u). \tag{2}$$

For fixed p and q we can define $f(m,u)$ as the difference of the left and right hand side of (2). After sorting powers of u we want to show

$$f(m,u) = u^q\ (m^{2p}-m^q)\ - u\ r(m)(m+m^{2p})\ + (1-r(m))(1-m^{2p}) \geq 0. \tag{3}$$

It is easy to check that $f(m,0)=(1-r(m))(1-m^{2p})>0$ and that $f(m,1)=0$. For $q\leq 2p$ one realizes that f is monotone decreasing in u, and this is how the result in [1] was derived. We have $q>2p$ in mind, and therefore the map $u\mapsto f(m,u)$ might have a negative minimum at some $\tilde{u}(m)\in(0,1)$. Let us rule this out by contradiction. If f has such a negative minimum, then

$$\frac{\partial f}{\partial u}(m,\tilde{u}) = 0 = q\tilde{u}^{q-1}(m^{2p}-m^q) - r(m)(m+m^{2p}),$$

or

$$\tilde{u}^{q-1} = \frac{1}{q}\ \frac{r(m)(m+m^{2p})}{m^{2p}-m^q}. \tag{4}$$

Note that the denominator in (4) is positive. If we evaluate $f(m,\tilde{u})$ using (4) and $\tilde{u}\in(0,1)$, we obtain

$$\begin{aligned} f(m,\tilde{u}) &= \tilde{u}\left[(\frac{1}{q}-1)r(m)(m+m^{2p})\right] + (1-r(m))(1-m^{2p}) \\ &> \frac{(1-q)}{q}r(m)(m+m^{2p}) + (1-r(m))(1-m^{2p}) \\ &= \frac{1}{1+m}\left[\frac{(1-q)}{q}(1-m^q)(m+m^{2p}) + (m+m^q)(1-m^{2p})\right], \end{aligned} \tag{5}$$

and we want to show that the last square bracket is positive provided q is not too large. But

$$q(m+m^q)(1-m^{2p}) \geq (q-1)(1-m^q)(m+m^{2p}) \tag{6}$$

holds for $q\leq 2p+1$, since it can be written as

$$[-m^{2p+q}+(q-1)m^{q+1}] + q[m^q - m^{2p+1}] + [qm-(q-1)m^{2p}] \geq 0. \tag{7}$$

In fact, the first and last square bracket are nonnegative because $p,>1$ and $q>2p$, and the central square bracket is nonnegative when we assume $q\leq 2p+1$. This is the only place where the assumption $q\leq 2p+1$ enters the proof, and this completes the proof.

Proof of Theorem 2: In [2] Egorov minimized $J(v)$ among functions in $W_0^{1,p}(0,1)$ with mean value zero. In order to avoid confusion, we consider this minimization problem on the set $W_0^{1,p}(-1,1)$ with mean value zero. This is equivalent to the problem in [1]. He argued that a minimizer u had to have a zero at some $b \in (-1,1)$, and in fact using rearrangement methods one can show that there is only one zero. This motivates the choice of A. Functions that minimize $J(u;-1,1)$ in A have the property, that restricted to each of the subintervals $(-1,b)$ and $(b,1)$, they are the ground state associated to the corresponding subinterval, and by scaling one obtains (see [2, p.507] or the Appendix below),

$$(J(u))^p = c\ (1+b)^{p-p/q}\ (1-b)^{p-p/q}\ \frac{(1+b)^{1-2p} + (1-b)^{1-2p}}{[(1-b)^{q-1} + (1+b)^{q-1}]^{p/q}} := c\ f(b), \quad (8)$$

where c is a positive constant. Notice that f is an even function of b. Now minimizing J has become a one-dimensional problem and amounts to minimizing $f(b)$ on $[-1,0]$. A little computation shows that

$$f^q(b) = \{(1+b)^{1-2p} + (1-b)^{1-2p}\}^q\ [(1+b)^{1-q} + (1-b)^{1-q}]^{-p}, \quad (9)$$

so that

$$\begin{aligned}(f^q)'(b) = q\ &\{\ldots\}^{q-1}\ (1-2p)\ ((1+b)^{-2p} - (1-b)^{-2p})\ [\ldots]^{-p} \\ &- p\ \{\ldots\}^q\ (1-q)\ ((1+b)^{-q} - (1-b)^{-q})\ [\ldots]^{-p-1}\end{aligned} \quad (10)$$

or

$$\begin{aligned}\{\ldots\}^{1-q}\ [\ldots]^{p+1}\ (f^q)'(b) = q\ (1-2p)\ &[\ldots]\ ((1+b)^{-2p} - (1-b)^{-2p}) \\ &- p\ (1-q)\ \{\ldots\}\ ((1+b)^{-q} - (1-b)^{-q}).\end{aligned} \quad (11)$$

Clearly $f(b)$ has a critical point at $b=0$, and as Egorov has noted, for $q > 4p-1$ this critical point is a local maximum. Therefore u cannot be symmetric for $q > 4p-1$. Are there other values of $b \in (-1,0)$ for which the right hand side of (11) can become zero? After multiplication with $(1+b)^{2p+q-1}(1-b)^{2p+q-1}$ this is equivalent to showing

$$\begin{aligned}(2p-1)q\ &[(1-b)^{q-1} + (1+b)^{q-1}]\ ((1-b)^{2p} - (1+b)^{2p}) \\ &= (q-1)p\ \{(1-b)^{2p-1} + (1+b)^{2p-1}\}\ ((1-b)^q - (1+b)^q).\end{aligned} \quad (12)$$

If we set

$$\begin{aligned}g(b) :=&(2p-1)q\ [(1-b)^{q-1} + (1+b)^{q-1}]\ ((1-b)^{2p} - (1+b)^{2p}) \\ &- (q-1)p\ \{(1-b)^{2p-1} + (1+b)^{2p-1}\}\ ((1-b)^q - (1+b)^q),\end{aligned} \quad (13)$$

we ask for zeroes of g in $(-1,0)$. Clearly, at $b=-1$, since $p \geq 1 > q/(q+1)$, we have $p(q+1)-q = (2p-1)q - p(q-1) = 2^{1-2p-q} g(-1) > 0$ and $g(0)=0$. We intend to show that $g'(b) < 0$ on $(-1,0)$, provided $q \leq 4p-1$, because this will prove symmetry. A simple calculation gives

$$\begin{aligned} g'(b) =& (2p-1)\ q\ (q-1)\ [(1+b)^{q-2} - (1-b)^{q-2}]((1-b)^{2p} - (1+b)^{2p}) \\ &- (2p-1)\ q\ 2p\ [(1-b)^{q-1} + (1+b)^{q-1}]((1+b)^{2p-1} + (1-b)^{2p-1}) \\ &- p(q-1)(2p-1)\{(1+b)^{2p-2} - (1-b)^{2p-2}\}((1-b)^{q} - (1+b)^{q}) \\ &+ p\ (q-1)\ q\ \{(1-b)^{2p-1} + (1+b)^{2p-1}\}((1+b)^{q-1} + (1-b)^{q-1}). \end{aligned} \tag{14}$$

Note that the second and fourth line of (14) add up to

$$q\ p\ (q-(4p-1))\ [(1-b)^{q-1} + (1+b)^{q-1}]((1+b)^{2p-1} + (1-b)^{2p-1}) < 0, \tag{15}$$

if we use the assumption $q \leq 4p-1$. Therefore showing that $g'(b) < 0$ on $(-1,0)$ amounts to adding the first and third line of (14), dividing by (2p-1)(q-1), and establishing

$$\begin{aligned} &-q\ [(1-b)^{q-2} - (1+b)^{q-2}]\ [(1-b)^{2p} - (1+b)^{2p}] + \\ &+\ p\ [(1-b)^{q} - (1+b)^{q}]\ [(1-b)^{2p-2} - (1+b)^{2p-2}]\ < 0. \end{aligned} \tag{16}$$

A little calculation, using $(1+b)(1-b)^{-1} \in (0,1)$, gives

$$\begin{aligned} &[(1-b)^{t} - (1+b)^{t}]\ [(1-b)^{s-1} - (1+b)^{s-1}] \geq \\ &\geq [(1-b)^{t-1} - (1+b)^{t-1}]\ [(1-b)^{s} - (1+b)^{s}] \end{aligned} \tag{17}$$

for $s \geq t \geq 1$ so that an application of (17) to the first term in (16) gives

$$\begin{aligned} &[(1-b)^{q-2} - (1+b)^{q-2}]\ [(1-b)^{2p} - (1+b)^{2p}] \geq \\ &\geq [(1-b)^{q-1} - (1+b)^{q-1}]\ [(1-b)^{2p-1} - (1+b)^{2p-1}]. \end{aligned} \tag{18}$$

In fact we have set $t=2p$, $s=q-1$ and used $s \geq t$ or $q \geq 2p+1$ to arrive at (18). Next we set $t=2p-1$, $s=q$ and check $s \geq t$ to reach

$$\begin{aligned} &[(1-b)^{q-1} - (1+b)^{q-1}]\ [(1-b)^{2p-1} - (1+b)^{2p-1}] \geq \\ &\geq [(1-b)^{q} - (1+b)^{q}]\ [(1-b)^{2p-2} - (1+b)^{2p-2}]. \end{aligned} \tag{19}$$

Now a combination of (18), (19) gives

$$\begin{aligned} &[(1-b)^{q-2} - (1+b)^{q-2}]\ [(1-b)^{2p} - (1+b)^{2p}] \geq \\ &\geq [(1-b)^{q} - (1+b)^{q}]\ [(1-b)^{2p-2} - (1+b)^{2p-2}], \end{aligned} \tag{20}$$

and $p < q$ establishes (16) and completes the proof of our Theorem.

3. Appendix.

It is the purpose of this appendix to give a derivation of (8).

Lemma: *If u is a minimizer of $J(u,-1,1)$ on A then the following holds*

$$J(u) = c\ \frac{\left[(1-b)^p\ (1+b)^{1-p}\ +\ (1+b)^p\ (1-b)^{1-b}\right]^{1/p}}{\left[(1+b)\ (1-b)^q\ +\ (1-b)\ (1+b)^q\right]^{1/q}}, \tag{21}$$

where c is independent of b, see (24).

Proof of the Lemma: Let u_0 be a positive function minimizing $\int_{-1}^{1}\ |v'|^p\ dx$ on $\{v \in W_0^{1,p}(-1,1)\ |\ \int_{-1}^{1}\ v^q(x)\ dx = 1\ \}$.

Set $z_+(y) = u_0((2y+(1-b))/(1+b))$ and $y = (1+b)x/2 - (1-b)/2$. Then

$$\int_{-1}^{b}\ |z_+(y)|^q\ dy = \frac{(1+b)}{2} \quad \text{and} \quad \int_{-1}^{b}\ z_+(y)\ dy = \frac{(1+b)}{2}\int_{-1}^{1}\ u_0(x)\ dx. \tag{22}$$

Correspondingly set $z_-(y) = -u_0((2y-(1+b))/(1-b))$ and $y = (1-b)x/2+(1+b)/2$. Then

$$\int_{b}^{1}\ |z_-(y)|^q\ dy = \frac{(1-b)}{2} \quad \text{and} \quad \int_{b}^{1}\ z_-(y)\ dy = -\frac{(1-b)}{2}\int_{-1}^{1}\ u_0(x)\ dx. \tag{23}$$

Finally, continue z_+ and z_- outside their support by zero and set (we call those extension $\tilde{z}_+(y)$ and $\tilde{z}_-(y)$)

$$w(y) := \frac{(1-b)}{2}\tilde{z}_+(y) + \frac{(1+b)}{2}\tilde{z}_-(y) \qquad \text{for } y \in [-1,1].$$

It is easily seen that $w \in \{v \in W_0^{1,p}(-1,1)\ |\ \int_{-1}^{1}\ v(x)\ dx = 0\ \}$ is admissible for J. To calculate the Rayleigh quotient we observe that $\tilde{z}_+(y)$ and $\tilde{z}_-(y)$ have disjoint support. Thus, using (22) and (23)

$$\begin{aligned}\int_{-1}^{1} |w(y)|^q dy &= \left(\frac{1-b}{2}\right)^q \int_{-1}^{b} |z_+(y)|^q dy + \left(\frac{1+b}{2}\right)^q \int_{b}^{1} |z_-(y)|^q\ dy \\ &= \left(\frac{1-b}{2}\right)^q \left(\frac{1+b}{2}\right) + \left(\frac{1-b}{2}\right)\left(\frac{1+b}{2}\right)^q,\end{aligned}$$

so that $||w||_q = 2^{-(1+1/q)}\{((1-b)^q(1+b)+(1+b)^q(1-b)\}^{1/q}$. To calculate $||w'||_p$ we note (let us remember the definition of $\tilde{z}_+(y)$ and $\tilde{z}_-(y)$)

$$w'(y) = \frac{1-b}{1+b}\, u_0' \left(\frac{2y+(1-b)}{1+b}\right) - \frac{1+b}{1-b}\, u_0' \left(\frac{2y-(1+b)}{1-b}\right)$$

and find out

$$\begin{aligned} ||w'||_p^p = & \left(\frac{1-b}{1+b}\right)^p \int_{-1}^{b} \left| u_0' \left(\frac{2y+(1-b)}{1+b}\right)\right|^p \, dy \; + \\ & + \left(\frac{1+b}{1-b}\right)^p \int_{b}^{1} \left| u_0' \left(\frac{2y-(1+b)}{1-b}\right)\right|^p \, dy \\ = & \left(\int_{-1}^{1} |u_0'(x)|^p \, dx\right) \frac{1}{2} \{ \, (1-b)^p \, (1+b)^{1-p} \; + \; (1+b)^p \, (1-b)^{1-p} \, \} . \end{aligned}$$

Finally

$$J(w) = 2^{1+\frac{1}{q}-\frac{1}{p}} \; ||u_0'||_p \; \frac{\left[(1-b)^p \, (1+b)^{1-p} \; + \; (1+b)^p \, (1-b)^{1-p}\right]^{1/p}}{\left[(1+b) \, (1-b)^q \; + \; (1-b) \, (1+b)^q\right]^{1/q}},$$

and the proof is complete.

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