An isoperimetric inequality related to a Bernoulli problem*

Daniel Daners Bernd Kawohl

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Abstract

Given a bounded domain Ω we look at the minimal parameter $\Lambda(\Omega)$ for which a Bernoulli free boundary value problem for the p-Laplacian has a solution minimising an energy functional. We show that amongst all domains of equal volume $\Lambda(\Omega)$ is minimal for the ball. Moreover, we show that the inequality is sharp with essentially only the ball minimising $\Lambda(\Omega)$. This resolves a problem related to a question asked in [Flucher et al., J. Reine Angew. Math. **486** (1997), 165–204.].

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1 Motivation and Result

For given $\lambda > 0$ consider the following Bernoulli type free boundary problem

$$\begin{aligned} \Delta v &= 0 && \text{in } \Omega \setminus D, \\ v &= 0 && \text{on } \partial\Omega, \\ v &\equiv 1 && \text{on } D, \\ |\nabla v| &= \lambda && \text{on } \partial D, \end{aligned} \tag{1.1}$$

on a given bounded open set $\Omega \subset \mathbb{R}^N$, where $D \subset \Omega$ is an unknown closed set. Such free boundary value problems originally arose from two dimensional

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flows (see [2, 7]), but also have applications to heat flows or electro-chemical machining (see the references in [4]).

It was shown in [1, Section 1.3] that some solutions to (1.1) can be obtained as non-trivial minimisers of the the functional

$$J_\lambda(u) := \int_\Omega |\nabla u(x)|^2\,dx + \lambda^2|\{u<1\}| \tag{1.2}$$

over all $u \in H_0^1(\Omega)$ (Replace u by $1-u$ in [1]), where $\{u<1\} := \{x \in \Omega\colon u(x) < 1\}$ and is $|\{u<1\}|$ its Lebesgue measure. One can interpret the second term in J_λ as penalising the support of $(1-u)^+$. By reducing λ we expect the support of $(1-u)^+$ to grow or D to shrink. When we look at (1.1), we also expect $|\nabla u|$ to decrease as D shrinks. Hence the minimal λ for which a solution exists should occur when the distance between ∂D and $\partial\Omega$ becomes maximal. Therefore we expect an optimal configuration to maximise this distance, and a ball is very likely to do so. We set

$$\Lambda_2(\Omega) := \inf\{\lambda > 0\colon J_\lambda \text{ has a non-trivial minimiser}\}.$$

and prove that $\Lambda_2(\Omega) \geq \Lambda_2(\Omega^*)$, where Ω^* denotes the ball of same volume as Ω. We also prove that equality holds if and only if Ω is a ball.

We will look at a more general problem. In [4] it is shown that for $1 < p < \infty$ non-trivial minimisers of the functional

$$J_{\lambda,p}(u) := \int_\Omega |\nabla u|^p\,dx + (p-1)\lambda^p|\{u<1\}| \tag{1.3}$$

on $W_0^{1,p}(\Omega)$ solve the over-determined free boundary problem

$$\begin{aligned} \Delta_p v &= 0 && \text{in } \Omega \setminus D, \\ v &= 0 && \text{on } \partial\Omega, \\ v &\equiv 1 && \text{on } D, \\ |\nabla v| &= \lambda && \text{on } \partial D. \end{aligned} \tag{1.4}$$

Similarly as before we set

$$\Lambda_p(\Omega) := \inf\{\lambda > 0\colon J_{\lambda,p} \text{ has a non-trivial minimiser}\}. \tag{1.5}$$

First we establish the following existence result.

Theorem 1.1. *The functional $J_{\lambda,p}$ has a non-trivial minimiser if and only if $\lambda \geq \Lambda_p(\Omega)$. Moreover,* $\min J_{\lambda,p} = J_{\lambda,p}(0)$ *if and only if $\lambda \leq \Lambda_p(\Omega)$.*

As zero is the only minimiser of $J_{0,p}(u) = \|\nabla u\|_p^p$ the above theorem implies that $\Lambda_p(\Omega) > 0$. Our main result is the following isoperimetric inequality. The proof of the sharpness of that inequality relies in an essential way on the fact from Theorem 1.1 that zero and a nontrivial $u \in W_0^{1,p}(\Omega)$ both minimize $J_{\Lambda_p(\Omega),p}$.

Theorem 1.2. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$ and Ω^* a ball of same volume as Ω. Then*

$$\Lambda_p(\Omega) \geq \Lambda_p(\Omega^*), \tag{1.6}$$

with equality if and only if Ω is a ball up to a set of p-capacity zero. Moreover, if Ω^* has radius r then

$$\Lambda_p(\Omega^*) = \frac{p}{p-1}\Big(\frac{p}{N}\Big)^{(N-1)/(p-N)}\frac{1}{r}$$

if $N \neq p$ and

$$\Lambda_N(\Omega^*) = \frac{N}{N-1}e^{(1-1/N)}\frac{1}{r}.$$

if $N = p$.

Note that $\Lambda_p(\Omega^*)$ is a continuous function of $p \in (1, \infty)$. Also, if $p > N$, then points have positive p-capacity. Hence, if $\Lambda_p(\Omega) = \Lambda_p(\Omega^*)$ and $p > N$, then Ω is a ball.

Remark 1.3. If the integral $\int_\Omega |\nabla u|^p dx$ in $J_{\lambda,p}(u)$ is replaced by $\int_\Omega G(|\nabla u|)dx$, with suitable assumptions on G, including convexity of G, one can consider a more general quasi-linear equation for functions in the appropriate Orlicz space. Details of this can be found in [12].

A conjecture related to the above theorem appears in Flucher and Rumpf [5, page 202]. The difference is that we only look at solutions of (1.4) which minimise the energy functional $J_{\lambda,p}$, whereas [5] look at all solutions, that is,

$$\lambda_p(\Omega) := \inf\{\lambda > 0 \colon (1.4) \text{ has a non-trivial solution}\}.$$

A comparison of the optimal constants on the ball as computed in Section 4 reveals that $\lambda_p(B) < \Lambda_p(B)$ if $\Omega = B$ is a ball. The new result in Theorem 1.1 is that there exists a non-trivial minimiser for $\lambda = \Lambda_p(\Omega)$. A similar result is proved in [9, Theorem 3.1] for $\lambda = \lambda_p(\Omega)$ and for convex Ω, but with completely different techniques to the ones we use. Also, [11] claim to prove the conjecture by Flucher and Rumpf.

Since the energy minimising solutions have attracted quite some interest with the work in [1] in case $p = 2$ and [4] for general $p \in (1, \infty)$, our result should still be of interest. We give a proof of (1.6) in Section 3 and compute the optimal values in Section 4. Theorem 1.1 is proved in Section 2.

2 Existence of minimisers

In this section we establish the existence results for minimisers stated in Theorem 1.1. We throughout assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set.

Proposition 2.1. *Let $J_{\lambda,p}$ and $\Lambda_p(\Omega)$ be defined as in the previous section.*

(i) *If there exists $w \in W_0^{1,p}(\Omega)$ such that $J_{\lambda,p}(w) < J_{\lambda,p}(0) = (p-1)\lambda^p|\Omega|$, then $J_{\lambda,p}$ has a non-trivial minimiser.*

(ii) *For $\lambda > 0$ large enough $J_{\lambda,p}$ has a non-trivial minimiser.*

(iii) *Suppose $\mu > 0$ is such that $J_{\mu,p}$ has a nontrivial minimiser $u \in W_0^{1,p}(\Omega)$. Then $J_{\lambda,p}$ has a non-trivial minimiser for all $\lambda > \mu$.*

(iv) *We have $\min J_{\lambda,p} = J_{\lambda,p}(0)$ if and only if $\lambda \leq \Lambda_p(\Omega)$.*

Proof. (i) Since $J_{\lambda,p}(u) \geq 0$ for all $u \in W_0^{1,p}(\Omega)$ we can choose a minimising sequence $u_n \in W_0^{1,p}(\Omega)$ with $J_{\lambda,p}(u_n) \to \inf_{u \in W_0^{1,p}(\Omega)} J_{\lambda,p}(u)$. By definition of $J_{\lambda,p}$ the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$ and therefore has a subsequence (u_{n_k}) converging weakly in $W_0^{1,p}(\Omega)$ and pointwise almost everywhere in Ω to some function u. Hence

$$\|\nabla u\|_p^p \leq \liminf_{k\to\infty} \|\nabla u_{n_k}\|_p^p$$

and by Fatou's Lemma

$$\int_\Omega \chi_{\{u<1\}}\,dx \leq \liminf_{k\to\infty} \int_\Omega \chi_{\{u_{n_k}<1\}}\,dx,$$

where χ_A is the indicator function of a set $A \subseteq \mathbb{R}^N$ given by $\chi_A(x) = 1$ if $x \in A$ and zero otherwise. By definition of $J_{\lambda,p}$ and the choice of (u_n)

$$J_{\lambda,p}(u) \leq \liminf_{k\to\infty} J_{\lambda,p}(u_{n_k}) = \inf_{v\in W_0^{1,p}(\Omega)} J_{\mu,p}(v).$$

Thus, u is a minimiser. It is non-trivial since by assumption $J_{\lambda,p}(u) \leq J_{\lambda,p}(w) < J_{\lambda,p}(0)$.

(ii) Let $\varphi \in C_c^\infty(\Omega)$ such that $|\{\varphi \geq 1\}| > 0$. Then note that

$$J_{\lambda,p}(\varphi) - J_{\lambda,p}(\varphi)(0) = \|\nabla\varphi\|_p^p - (p-1)\lambda^p|\{\varphi \geq 1\}| < 0$$

for $\lambda > 0$ large enough. Now apply (i).

(iii) Clearly, if u is a non-trivial minimiser of $J_{\mu,p}$, then $J_{\lambda,p}(u) \leq J_{\lambda,p}(0)$. Also, $|\{u < 1\}| < |\Omega|$ since otherwise $J_{\lambda,p}(0) < J_{\lambda,p}(u)$ and u is not a minimiser. Hence from the definition of $J_{\mu,p}$ we have

$$\begin{aligned} J_{\lambda,p}(u) &= \|\nabla u\|_p^p + (p-1)\lambda^p|\{u<1\}| \\ &= J_{\mu,p}(u) + (p-1)(\lambda^p - \mu^p)|\{u<1\}| \\ &\leq (p-1)\mu^p|\Omega| + (p-1)(\lambda^p-\mu^p)|\{u<1\}| \\ &= (p-1)\lambda^p|\Omega| - (p-1)(\lambda^p-\mu^p)\big(|\Omega| - |\{u<1\}|\big). \end{aligned} \tag{2.1}$$

Since $|\{u < 1\}| < |\Omega|$ we conclude that $J_{\lambda,p}(u) < (p-1)\lambda^p|\Omega| = J_{\lambda,p}(0)$ for all $\lambda > \mu$. By (i) $J_{\lambda,p}$ has a non-trivial minimiser for all $\lambda > \mu$.

(iv) If $\lambda < \Lambda_p(\Omega)$, then clearly $\min_{u\in W_0^{1,p}(\Omega)} J_{\lambda,p}(u) = J_{\lambda,p}(0)$, so assume that $\mu := \Lambda_p(\Omega)$. Assume that u is a minimiser of $J_{\mu,p}$ and suppose that strict inequality holds in (2.1). Then clearly $J_{\lambda,p}(u) < J_{\lambda,p}(0) = (p-1)\lambda^p|\Omega|$ if $\lambda < \mu$ is close enough to μ. However, this contradicts the definition of $\mu = \Lambda_p(\Omega)$ since otherwise (i) implies the existence of a minimiser for some $\lambda < \mu$. □

To prove that $J_{\lambda,p}$ also has a non-trivial minimiser for $\lambda = \Lambda_p(\Omega)$ we need to compare $\|\nabla u\|_p$ with the measure of $\{u \geq 1\}$. In the following lemma we get such an estimate. It is motivated by the estimate of the measure of a set in terms of its capacity (see e.g. [6, page 5]), but does not rely on capacity.

Lemma 2.2. *Let $1 < p \leq N$. Then there exist $q > p$ and $C > 0$ only depending on N, p and $|\Omega|$ such that $|\{u \geq 1\}| \leq C\|\nabla u\|_p^q$ for all $u \in W_0^{1,p}(\Omega)$.*

Proof. If $1 < p < N$, by the Sobolev inequality there exists a constant $C > 0$ only depending on N and p such that

$$|\{u \geq 1\}| \leq \int_\Omega |u|^{Np/(N-p)}\,dx \leq C\|\nabla u\|_p^{Np/(N-p)}$$

for all $u \in W_0^{1,p}(\Omega)$. Hence we can set $q := Np(N-p) > p$. If $p = N$ choose $p_0 \in (N/2, N)$ and apply the above inequality and Hölder's inequality to get

$$|\{u \geq 1\}| \leq C\|\nabla u\|_{p_0}^{Np_0/(N-p_0)} \leq C|\Omega|^\theta\|\nabla u\|_p^{Np_0/(N-p_0)}$$

for all $u \in W_0^{1,p}(\Omega)$, where θ is a constant depending only on p_0 and N. Hence we can set $q := Np_0/(N-p_0)$. Clearly $q > N$ since $p_0 > N/2$. □

Since by definition of $\Lambda_p(\Omega)$ the functional $J_{\lambda,p}$ has no non-trivial minimiser for $\lambda < \Lambda_p(\Omega)$ the following proposition will conclude the proof of Theorem 1.1. It is also the most original and new part of the proof.

Proposition 2.3. *If $\mu = \Lambda_p(\Omega)$, then $J_{\mu,p}$ has a non-trivial minimiser.*

Proof. By definition of $\Lambda_p(\Omega)$ either there exists a non-trivial minimiser or there is a sequence (λ_n) such that $\lambda_n > \mu$ for all $n \in \mathbb{N}$, $\lambda_n \to \mu$ and $J_{\lambda_n,p}$ has a non-trivial minimiser $u_n \in W_0^{1,p}(\Omega)$ for every $n \in \mathbb{N}$. Then

$$J_{\lambda,p}(u_n) = \|\nabla u_n\|_p^p + (p-1)\lambda_n^p|\{u_n < 1\}| \leq (p-1)\lambda_n^p|\Omega|$$

for all $n \in \mathbb{N}$. Since (λ_n) is a convergent sequence, (u_n) is bounded in $W_0^{1,p}(\Omega)$. It therefore has a convergent subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and pointwise almost everywhere. Fix $v \in W_0^{1,p}(\Omega)$. As in the proof of Proposition 2.1(i)

$$J_{\mu,p}(u) \leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(u_{n_k}) \leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(v) = J_{\mu,p}(v) \tag{2.2}$$

where in the second inequality we use that u_{n_k} are minimisers for $J_{\lambda_{n_k},p}$. Hence $u \in W_0^{1,p}(\Omega)$ is a minimiser of $J_{\mu,p}$.

To conclude the proof we need to show that $u \neq 0$. If $u = 0$ and $p > N$, then $u_{n_k} \to 0$ uniformly as $k \to \infty$ since $W_0^{1,p}(\Omega)$ is compactly embedded into $C(\bar{\Omega})$. Therefore there exists $m \in \mathbb{N}$ such that $\|u_m\|_\infty < 1$ and so

$$J_{\lambda_m,p}(u_m) = \|\nabla u_m\|_p^p + (p-1)\lambda_m^p|\Omega| > (p-1)\lambda_m^p|\Omega| = J_{\lambda_m,p}(0)$$

since by assumption $u_m \neq 0$. As u_m was a non-trivial minimiser this is a contradiction, and so $u \neq 0$.

We next look at the case $1 < p \leq N$. Again assume that $u = 0$. Then by Rellich's theorem we have $u_{n_k} \to 0$ in $L_p(\Omega)$ and so

$$|\{u_{n_k} \geq 1\}| \leq \int_{\{u_{n_k}\geq 1\}} |u_{n_k}|^p \, dx \leq \|u_{n_k}\|_p^p \to 0$$

Hence (2.2) with $u = v = 0$ implies that

$$\begin{aligned}\mu^p(p-1)|\Omega| = J_{\mu,p}(0) &\leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(u_{n_k}) \\ &\leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(0) = J_{\mu,p}(0) = \mu^p(p-1)|\Omega|,\end{aligned}$$

and therefore $\|\nabla u_{n_k}\|_p \to 0$. As $J_{\lambda_n,p}(0) = (p-1)\lambda_n^p|\Omega|$, Lemma 2.2 implies the existence of constants $C > 0$ and $q > p$ such that

$$\begin{aligned}J_{\lambda_n,p}(u_n) &= J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p - (p-1)\lambda_n^p|\{u_n \geq 1\}| \\ &\geq J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p - C(p-1)\lambda_n^p\|\nabla u_n\|_p^q \\ &= J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p\Big(1 - C(p-1)\lambda_n^p\|\nabla u_n\|_p^{q-p}\Big)\end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Since $\|\nabla u_{n_k}\|_p \to 0$, $\lambda_n \to \mu$ and $q > p$ there exists $m \in \mathbb{N}$ with

$$1 - C(p-1)\lambda_m^p \|\nabla u_m\|_p^{q-p} > 0$$

and hence by (2.3) we get $J_{\lambda_m,p}(u_m) > J_{\lambda_m,p}(0)$. This is a contradiction since u_m was assumed to be a minimiser for $J_{\lambda_m,p}$, and so $u \neq 0$ as claimed. □

3 Proof of the isoperimetric inequality

This whole section is devoted to the proof of the first part of Theorem 1.2. We let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\Omega^* \subset \mathbb{R}^N$ be an open ball with the same volume as Ω. For $v \in W_0^{1,p}(\Omega^*)$ set

$$J_{\lambda,p}^*(v) = \int_{\Omega^*} \big(|\nabla v|^p \, dx + (p-1)\lambda^p |\{v < 1\}|$$

and recall that minimisers are solutions of (1.4) with Ω replaced by Ω^*. Let $\lambda \geq \Lambda_p(\Omega)$. By Theorem 1.1 $J_{\lambda,p}$ has a non-trivial minimiser $u \in W_0^{1,p}(\Omega)$. Consider its Schwarz symmetrisation u^* (see [10] for a definition and properties). By well known properties of Schwarz symmetrisation $u^* \in W_0^{1,p}(\Omega^*)$, $\|\nabla u^*\|_p \leq \|\nabla u\|_p$ and $|\{u^* < 1\}| = |\{u < 1\}|$. Also u^* is non-zero and

$$\begin{aligned} J_{\lambda,p}^*(u^*) &= \|\nabla u^*\|_p^p + (p-1)\lambda^p |\{u^* < 1\}| \\ &\leq \|\nabla u\|_p^p + (p-1)\lambda^p |\{u < 1\}| = J_{\lambda,p}(u). \end{aligned} \tag{3.1}$$

In particular $J_{\lambda,p}^*(u^*) \leq J_{\lambda,p}(u) \leq J_{\lambda,p}(0) = J_{\lambda,p}^*(0)$. If $J_{\lambda,p}^*(u^*) < J_{\lambda,p}^*(0)$, then by Proposition 2.1(i) $J_{\lambda,p}^*$ has a non-trivial minimiser. If $J_{\lambda,p}^*(u^*) = J_{\lambda,p}^*(0)$, then either u^* is a non-trivial minimiser, or $\inf J_{\lambda,p}^* < (p-1)\lambda^p|\Omega^*|$ and Proposition 2.1(i) implies the existence of a non-trivial minimiser. In any case, if $J_{\lambda,p}$ has a non-trivial minimiser, so does $J_{\lambda,p}^*$. Hence by definition of $\Lambda_p(\Omega)$ and $\Lambda_p(\Omega^*)$ the inequality (1.6) follows.

It remains to prove the sharpness of (1.6). We assume that $\Lambda_p(\Omega) = \Lambda_p(\Omega^*)$. The aim is to show that Ω is a ball up to a set of capacity zero. To simplify notation we denote the common value of $\Lambda_p(\Omega)$ and $\Lambda_p(\Omega^*)$ by Λ and let r be the radius of the ball Ω^*. By Theorem 1.1 zero is a minimiser for the problem on Ω and also on Ω^*. Hence, using (3.1)

$$(p-1)\Lambda^p|\Omega| = J_{\Lambda,p}^*(0) \leq J_{\Lambda,p}^*(u^*) \leq J_{\Lambda,p}(u) = J_{\Lambda,p}(0) = (p-1)\Lambda^p|\Omega|.$$

We conclude that $J_{\Lambda,p}^*(u^*) = J_{\Lambda,p}(u)$. In particular, u^* is a minimiser of $J_{\Lambda,p}^*$. Since there is a unique radially symmetric minimiser on Ω^* (see the argument at the start of Section 4) we conclude that u^* coincides with (4.1) if $p \neq N$ and (4.2) if $p = N$ with ρ given by (4.6) and (4.8), respectively. In particular, $\nabla u^*(x) = \nabla u_\rho(x) \neq 0$ whenever $0 < u_\rho(x) < 1 = \max u_\rho$. Therefore, [3, Theorem 1.1] applies and so, up to translation, $u = u^* = u_\rho$ almost everywhere. Extending u, u^* by zero outside Ω and Ω^*, respectively we can assume that $u, u^* \in W^{1,p}(\mathbb{R}^N)$. We can then replace u and u^* by a quasi-continuous representative as defined in [8, Theorem 4.5]. Since u_ρ is continuous and $u^* = u_\rho$ almost everywhere, u_ρ is the quasi-continuous representative of u^*. Hence $u_\rho = u$ quasi everywhere, that is, except possibly on a set of p-capacity zero. Also, as $u \in W_0^{1,p}(\Omega)$ we know from [8, Theorem 4.5] that $u = 0$ quasi everywhere on Ω^c. Combining the two facts we get $u = u_\rho = 0$ quasi-everywhere on $C := \Omega^* \setminus \Omega$. Since $u_\rho > 0$ on Ω^* we conclude that C must have p-capacity zero. Hence $\Omega = \Omega^*$ is a ball except possibly for a set of p-capacity zero.

4 The optimal constants

In this section we look at (1.4) in case $\Omega = B_r$ is a ball of radius $r > 0$ centred at the origin. We want to compute the value of $\Lambda_p(B_r)$. To do so we assume that $\lambda \geq \Lambda_p(B_r)$ and that $u \in W_0^{1,p}(B_r)$ is a minimiser of $J_{\lambda,p}$. Let $u^* \in W_0^{1,p}(B_r)$ be its Schwarz symmetrisation. According to (3.1) we have $J_{\lambda,p}(u^*) \leq J_{\lambda,p}(u)$. Hence there is a radially symmetric minimiser u_ρ and we can assume without loss of generality that $u_\rho = u_\rho^*$. Let $\rho > 0$ be the radius of the ball $\{u \geq 1\}$. By [4, Theorem 2.1] (or [1, Lemma 2.4] in case $p = 2$) the minimiser is p-harmonic on $B_r \setminus \bar{B}_\rho$ with $u = 0$ on ∂B_r and $u = 1$ on ∂B_ρ. As there is precisely one such p-harmonic function (see [8, Lemma 8.5])

$$u_\rho(x) = \begin{cases} \dfrac{|x|^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}}{\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}} & \text{if } \rho \leq |x| \leq r \\ 1 & \text{if } 0 \leq |x| \leq \rho \end{cases} \tag{4.1}$$

if $p \neq N$ and

$$u_\rho(x) = \begin{cases} \dfrac{\log|x| - \log r}{\log\rho - \log r} & \text{if } \rho \leq |x| \leq r \\ 1 & \text{if } 0 \leq |x| \leq \rho \end{cases} \tag{4.2}$$

if $p = N$ (see [5, 9]). Given $\rho \in (0, r)$ one can compute $\lambda = |\nabla u_\rho|$ for $|x| = \rho$, and then minimise λ. This yields the smallest possible value of λ such that (1.4) has a non-trivial solution. These optimal values have been calculated in [5] for $p = 2$ and in [9] for general $p \in (1, \infty)$. They are

$$\lambda_p(B_r) = \frac{\left|\frac{p-N}{p-1}\right|}{r\left|\left(\frac{p-1}{N-1}\right)^{(N-1)/(N-p)} - \left(\frac{p-1}{N-1}\right)^{(p-1)/(N-p)}\right|}$$

if $p \neq N$ and

$$\lambda_p(B_r) = \frac{e}{r}$$

if $p = N$. Unfortunately, the corresponding solution does *not* minimise $J_{\lambda_p,p}$. In case $p = 2$ this is pointed out in [5, Section 5.3], but also follows from the calculations below. To obtain $\Lambda_p(B_r)$ we start by computing $J_{\lambda,p}(u_\rho)$. We first consider the case $p \neq N$. An elementary calculation yields

$$|\nabla u_\rho(x)| = \Big|\frac{p-N}{p-1}\Big| \frac{|x|^{(1-N)/(p-1)}}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|}$$

for $\rho \leq |x| \leq r$ and zero elsewhere. Because

$$\begin{aligned}\int_\rho^r s^{p(1-N)/(p-1)} s^{N-1}\, ds &= \int_\rho^r s^{(p-N)/(p-1)-1}\, ds \\ &= \frac{p-1}{p-N}\big(r^{(p-N)/(p-1)} - \rho^{(p-N)/(p-1)}\big)\end{aligned}$$

we get

$$\int_{B_r} |\nabla u_\rho(x)|^p\, dx = \Big|\frac{p-N}{p-1}\Big|^{p-1} \frac{\omega_N}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|^{p-1}}, \tag{4.3}$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. According to Theorem 1.1 we have to find the smallest possible $\lambda > 0$ such that

$$J_{\lambda,p}(u_\rho) = J_{\lambda,p}(0) = (p-1)\lambda^p|B_r| = (p-1)\frac{\omega_N}{N} r^N \lambda^p.$$

Using the definition of $J_{\lambda,p}$ and u_ρ we therefore require that

$$\begin{aligned}\Big|\frac{p-N}{p-1}\Big|^{p-1} &\frac{\omega_N}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|^{p-1}} \\ &+ (p-1)\lambda^p \frac{\omega_N}{N}(r^N - \rho^N) = (p-1)\frac{\omega_N}{N} r^N \lambda^p \end{aligned}\tag{4.4}$$

or equivalently

$$N\Big|\frac{p-N}{p-1}\Big|^{p-1} = (p-1)\lambda^p \rho^N \Big|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}\Big|^{p-1}. \tag{4.5}$$

Clearly we get the smallest value of λ if we pick $\rho \in (0, r)$ such that

$$\rho^N \Big|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}\Big|^{p-1}$$

is maximal, and then compute the corresponding value of λ from (4.5). An elementary calculation shows that this is the case for

$$\rho = \left(\frac{N}{p}\right)^{(p-1)/(p-N)} r, \tag{4.6}$$

and hence, if we substitute that value of ρ into (4.5), then

$$\Lambda_p(B_r) = \frac{p}{p-1}\left(\frac{p}{N}\right)^{(N-1)/(p-N)} \frac{1}{r}. \tag{4.7}$$

We could confirm the above by computing $|\nabla u_\rho|$ for the above value of ρ. If $p = N$ we proceed in exactly the same way to get

$$\rho = e^{-(1-1/N)} r \tag{4.8}$$

and

$$\Lambda_N(B_r) = \frac{N}{N-1} e^{(1-1/N)} \frac{1}{r}. \tag{4.9}$$

It is also evident that

$$\left(\frac{p}{N}\right)^{(N-1)/(p-N)} = \left(1 + \frac{\frac{1}{N}}{\frac{1}{p-N}}\right)^{(N-1)/(p-N)} \to e^{(N-1)/N}$$

as $p \to N$, so $\Lambda_p(B_r) \to \lambda_N(B_r)$ as $p \to N$. Also note that $\lambda_p(B_r) < \Lambda_p(B_r)$ for all $p \in (1, \infty)$. In particular, this proves the second part of Theorem 1.2.

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Author's addresses:

Daniel Daners
School of of Mathematics and Statistics
The University of Sydney, NSW 2006, Australia
`daniel.daners@sydney.edu.au`

and

Bernd Kawohl
Mathematisches Institut
Universität zu Köln
D-50923 Köln, Germany
`kawohl@mi.uni-koeln.de`