Simplicity of the principal eigenvalue for indefinite quasilinear problems

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Abstract

Given any domain $\Omega \subseteq \mathbb{R}^N$, $w \in L^1_{loc}(\Omega)$ and a differentiable function $A : \mathbb{R}^N \to [0, \infty)$ which is p-homogeneous and strictly convex, we consider the minimization problem

$$\inf \left\{ \frac{\int_\Omega A(\nabla u)}{\left(\int_\Omega w(x)|u|^q\right)^{\frac{p}{q}}} : \, u \in \mathcal{D}_0^{1,p}(\Omega), \quad 0 < \int_\Omega w(x)|u|^q < \infty \right\}.$$

If the infimum is achieved and $q = p > 1$, without additional regularity assumptions on Ω or the weight function w, we show that the minimizer is unique up to a constant factor. The same conclusion holds when A is allowed to depend on $x \in \Omega$ and satisfies natural growth assumptions. Some of our results also hold when $q < p$.

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1 Introduction

Given a bounded domain of $\mathbb{R}^N$, it is well known that the first eigenvalue of the Laplacian with homogeneous Dirichlet data is simple, in the sense that its associated eigenspace has dimension one. Furthermore the corresponding eigenfunctions are characterized by the fact that they do not vanish in the domain (see [9]). These classical results have been generalized in [23] by Manes-Micheletti to weighted elliptic eigenvalue problems in divergence form:

$$-\sum_{i,j=1}^{N} \partial_i(a^{ij}(x)\partial_j u) = \lambda w(x)u, \qquad u \in W_0^{1,2}(\Omega), \tag{1.1}$$

where w is allowed to change sign. In this case it was shown in [23] that the smallest positive eigenvalue exists and is simple whenever $w^+ \not\equiv 0$ and $w \in L^r(\Omega)$ with $r > \frac{N}{2}$. Related works for the Laplace operator were done by Brown *et al.* [10, 11], Allegretto [1]. A generalization to nonlinear eigenvalue problems involving the p-Laplacian

$$-\Delta_p u = \lambda w(x)|u|^{p-2}u, \qquad u \in W_0^{1,p}(\Omega), \tag{1.2}$$

has been obtained by Anane [3], Lindqvist [19], [20]. For unbounded domains and singular weights allowed to change sign we refer to Szulkin and Willem [29].

In both eigenvalue problems (1.1) and (1.2) the first eigenvalue is obtained by minimizing the associated Rayleigh quotient. Thus one may study more generally the structure of the set of minimizers for the problem:

$$\inf\left\{ \frac{\int_\Omega |\nabla u|^p}{\left(\int_\Omega w(x)|u|^q\right)^{\frac{p}{q}}} \;:\; u \in W_0^{1,p}(\Omega), u \not\equiv 0 \right\}. \tag{1.3}$$

For positive weight $w \in L^\infty(\Omega)$, in a bounded domain Ω, it has been shown by Kawohl [17] (for $w \equiv 1$), Nazarov [25] that the minimizers of (1.3) are unique up to a constant factor whenever $1 < q \leq p$.

A common feature in all the previous works concerning the simplicity of the principal eigenvalue is that they rely on Harnack's inequality in order to ensure the associated eigenfunctions to be continuous and strictly positive (or negative) in the domain. In [21] it was observed that a weaker version of the Strong Maximum Principle due to Ancona [4] and Brezis-Ponce [7] allows to prove simplicity of the principal eigenvalue for a larger class of linear problems. By exploiting furthermore some fine properties of the Sobolev functions, it is proved in [22] that the principal eigenvalue (positive or negative) of Problem (1.2) is simple even when

$w \in L^{N/p}(\Omega)$ ($w \not\equiv 0$), a class of weight functions for which Harnack's inequality may fail. The aim of the present paper is to study in a very general framework the set of minimizers of problems of the type (1.3). We shall work with any function $w \in L^1_{loc}(\Omega)$ allowed to change sign in the domain $\Omega \subseteq \mathbb{R}^N$ (which may be unbounded) and consider instead of the p-Dirichlet integral a more general functional.

More specifically, let $p \in (1, \infty)$, $\Omega \subseteq \mathbb{R}^N$ be a domain (open, connected open set) such that

$$\Omega \text{ is bounded if } p \geq N, \tag{1.4}$$

while Ω can be any domain if $p \in (1, N)$. Consider the "Beppo Levi space" $\mathcal{D}_0^{1,p}(\Omega)$ defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ (set of smooth functions having compact support) with respect to the norm $\|u\| := \left(\int_\Omega |\nabla u|^p\right)^{1/p}$. Under the restriction (1.4), it is known that $\mathcal{D}_0^{1,p}(\Omega)$ can be identified with a subspace of the space of distributions (see [13]). Given $w \in L^1_{loc}(\Omega)$ with $w^+ \not\equiv 0$ and $q \geq 1$, let us introduce the following sets:

$$\mathcal{W} := \{\varphi \in \mathcal{D}_0^{1,p}(\Omega) : \; w|\varphi|^q \in L^1(\Omega)\}, \tag{1.5}$$

$$\mathcal{W}^+ := \{\varphi \in \mathcal{W} : \; \int_\Omega w|\varphi|^q > 0\}. \tag{1.6}$$

Clearly $\mathcal{W}$ is a vector subspace of $\mathcal{D}_0^{1,p}(\Omega)$ and $C_0^\infty(\Omega)$ is contained in $\mathcal{W}$ (since $w \in L^1_{loc}(\Omega)$). Furthermore since $w^+ \not\equiv 0$, standard arguments show that $\mathcal{W}^+ \neq \emptyset$. It is then meaningful to consider the functional:

$$J : \mathcal{W}^+ \to \mathbb{R}, \quad J(u) = \frac{\int_\Omega A(x, \nabla u)}{\left(\int_\Omega w(x)|u|^q\right)^{\frac{p}{q}}}, \tag{1.7}$$

and to study the structure of the class of minimizers of J under the following assumptions:

(H1) condition (1.4) holds, $w \in L^1_{loc}(\Omega)$ with $w^+ \not\equiv 0$ and $A : \Omega \times \mathbb{R}^N \to [0, \infty)$ is a Carathéodory function satisfying:

$$0 < A(x, \eta) \leq C|\eta|^p, \qquad \text{a.e. } x \in \Omega, \, \forall \eta \in \mathbb{R}^N \setminus \{0\};$$

(H2) (a) for some $p > 1$, we have

$$A(x, t\eta) = |t|^p A(x, \eta), \qquad \text{a.e. } x \in \Omega, \, \forall (t, \eta) \in \mathbb{R} \times \mathbb{R}^N,$$

(b) $\eta \mapsto A(x, \eta)$ is convex for a.e. $x \in \Omega$;

(H3) there exists $\Phi \in \mathcal{W}^+$ solving the minimization problem

$$J(\Phi) = \Lambda := \inf\{J(u) : \ u \in \mathcal{W}^+\}. \tag{1.8}$$

The structure conditions (H1), (H2) on the function A are modeled on the example $A(\eta) = |\eta|^p$ with $p > 1$ or more generally

$$A(x, \eta) = \sum_{i=1}^{N} a_i(x)|\eta_i|^p \ \ (p > 1), \quad a_i \in L^\infty(\Omega), \quad \inf_{\Omega} a_i > 0.$$

The assumption that A is a Carathéodory function means:

$$\begin{cases} x \mapsto A(x, \eta) \text{ measurable } \forall \eta \in \mathbb{R}^N, \\ \eta \mapsto A(x, \eta) \text{ continuous a.e. } x \in \Omega, \end{cases}$$

and ensures the function $x \mapsto A(x, u(x))$ to be a Lebesgue measurable function whenever $u : \Omega \to \mathbb{R}$ has this property. Hence if (H1) holds, the functional J defined by (1.7) is well-defined. As stated above, (H3) holds for instance if $w \in L^{N/p}(\Omega)$ (see [29]), or for some Hardy-type weights (see [24], [28]), but otherwise it is a genuine assumption. Under the assumptions (H1)-(H3), one may ask if minimizers are unique up to a constant factor. When $1 \leq q < p$, a first answer to this question is given by the following result:

Theorem 1.1. *Let (H1) to (H3) be satisfied with $1 < q < p$, and consider two minimizers $\Phi_1, \Phi_2 \geq 0$ of (1.8). Then, there exists $t > 0$ such that*

$$\Phi_2 = t\,\Phi_1. \tag{1.9}$$

For $q = 1$, conclusion (1.9) holds if furthermore $\eta \mapsto A(x, \eta)$ is assumed strictly convex (a.e. $x \in \Omega$).

Let us emphasize that conclusion (1.9) holds for *non-negative* minimizers. Without additional assumptions the minimizers can either change sign or vanish on a set of positive measure. But if one realizes that $|\Phi|$ is a minimizer whenever Φ is, one consequence of Theorem 1.1 is that, up to a set of measure zero, the set of zeroes is the same for any two minimizers of Problem (1.8).

In order to derive a stronger conclusion than (1.9) and also a uniqueness result that holds for $q = p$, we must strengthen our hypotheses by requiring:

(H4) (a) For a.e. $x \in \Omega$, $\eta \mapsto A(x, \eta)$ is differentiable and $\exists C > 0$ such that:

$$C|\eta|^p \leq \langle \mathbf{a}(x, \eta), \eta \rangle, \qquad |\mathbf{a}(x, \eta)| \leq C|\eta|^{p-1} \quad \forall \eta \in \mathbb{R}^N, \tag{1.10}$$

where $\mathbf{a}(x, \eta) := \nabla_\eta A(x, \eta)$.

(b) If $q = 1$ or $q = p$, $\eta \mapsto A(x, \eta)$ is strictly convex a.e. $x \in \Omega$.

This hypothesis will be necessary to get more information on the set of zeroes of the minimizers. In the case $q \in [1, p)$, by applying Harnack's inequality we can derive:

Theorem 1.2. *Assume (H1) to (H4) hold with $1 \leq q < p$ and that the set $\{w > 0\}$ is an open connected set. Then the minimizer Φ given in (1.8) is unique up to a constant factor and does not change sign.*

For the case $q = p$, we will be able to get a uniqueness result without making any assumptions on the set $\{w > 0\}$. To do this, we will adapt to our framework a version of the strong maximum Principle found in [4] and [7] that does not rely on Harnack's inequality. With this tool in hand, we will prove the following:

Theorem 1.3. *Assume (H1) to (H4) hold and $1 < q = p$. Then the minimizer Φ given in (1.8) is unique up to a constant factor. It does not change sign and the set of zeroes of its precise representative has $W^{1,p}$-capacity zero.*

Let us remark that some of our assumptions are necessary in order to handle functions A which depend on the variable $x \in \Omega$. But if A is autonomous (i.e. $A(x, \eta) = A(\eta)$), it is enough in above proposition to assume A to be a differentiable non-negative function which is p-homogeneous and strictly convex.

We have organized the paper as follows. In Section 2, we recall several properties of the variational capacity. This tool is used in Section 3 to prove a strong maximum principle for quasilinear operator in the same spirit of what has been done in [7]. Section 4 provides more details on the set $\mathcal{W}^+$ defined by (1.6) and a discussion of the Gâteaux-differentiability of the functional J. The sign of minimizers of the functional J and their set of zeroes is discussed in Section 5. Finally in Section 6 we prove our results about the uniqueness of minimizers.

2 Preliminaries

One difficulty that arises in our framework is the possible lack of regularity of the minimizers given in (H3), which may not even be continuous. In the case $q = p > 1$, we shall overcome this problem by exploiting some fine properties of $W^{1,p}_{loc}$-functions that we recall throughout this section.

A main ingredient is the notion of "variational capacity", that in the modern form was introduced by Choquet [8]. For an extended discussion we also refer to the books of Evans-Gariepy [14] and Heinonen *et al* [16]. Given a compact set K contained in an open subset U of $\mathbb{R}^N$ and $p \geq 1$, the $W^{1,p}$-capacity of the pair (K, U) is defined as

$$\mathrm{Cap}_p(K, U) := \inf \left\{ \int_U |\nabla \varphi|^p \;:\; \varphi \in C_0^\infty(U),\, \varphi \geq 1 \text{ on } K \right\}.$$

If U' is an open subset of U, the corresponding $W^{1,p}$-capacity is defined as

$$\mathrm{Cap}_p(U', U) := \sup \left\{ \mathrm{Cap}_p(K, U) :\; K \subset U', K \text{ compact } \right\},$$

and the definition is extended to a general set $E \subset U$ as follows:

$$\mathrm{Cap}_p(E, U) := \inf \left\{ \mathrm{Cap}_p(U', U) :\; U' \text{ open}, E \subset U' \subset U \right\}.$$

A set $E \subset \mathbb{R}^N$ is said to be of $W^{1,p}$-capacity zero, and we write $\mathrm{Cap}_p(E) = 0$, if $\mathrm{Cap}_p(E \cap U, U) = 0$ for any open set $U \subset \mathbb{R}^N$. We say that $E \subset \mathbb{R}^N$ is $W^{1,p}$-*quasi open* (resp. $W^{1,p}$-*quasi closed*) if for every $\epsilon > 0$ there is an open set $U \subseteq \mathbb{R}^N$ (resp. closed set) such that $E \subseteq U$ and $\mathrm{Cap}_p(U \setminus E) < \epsilon$.

Given a function $f \in L^1_{loc}(\mathbb{R}^N)$, the *precise representative* of f is defined as:

$$f^*(x) := \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} f & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

A function $u : \Omega \to \mathbb{R}^N$ is $W^{1,p}$-*quasi continuous* if for each $\epsilon > 0$ there is an open set $U \subset \Omega$ such that $\mathrm{Cap}_p(U, \Omega) < \epsilon$ and $f|_{\Omega \setminus U}$ is continuous. It is known that the precise representative of a Sobolev function $u \in W^{1,p}_{loc}(\Omega)$ $(1 \leq p < n)$ is $W^{1,p}$-quasi continuous (see [14], p.160).

Some of our results would be be easier to prove if we knew that the set of zeroes of the minimizer Φ (given in (H3)) is a closed set. But with our assumptions we can only ensure the level sets of Φ (which we can choose to be a $W^{1,p}$-quasi continuous function) to be, up to a set of capacity zero, a countable union of closed sets:

Proposition 2.1. *Let $u : \Omega \to \mathbb{R}$ be a $W^{1,p}$-quasi continuous function and consider its level set $\mathcal{Z}_t := \{u = t\}$. Then there exist closed sets $F_n \subset \Omega$ $(n = 1, 2, \cdots)$ such that*

$$\bigcup_{n \in \mathbb{N}} F_n \subseteq \mathcal{Z}_t \quad \textit{and} \quad Cap_p(\mathcal{Z}_t \setminus \bigcup_{n \in \mathbb{N}} F_n) = 0. \tag{2.1}$$

Proof: Since the set $\mathcal{Z}_t$ is $W^{1,p}$-quasi closed, for each $n \in \mathbb{N}$ there exists a closed set $F_n \subset \Omega$ such that $F_n \subseteq \mathcal{Z}_t$ and $\mathrm{Cap}_p(\mathcal{Z}_t \setminus F_n, \Omega) \le \frac{1}{n}$. Now for any $n \in \mathbb{N}$ we have

$$\mathrm{Cap}_p(\mathcal{Z}_t \setminus \bigcup_{n\in\mathbb{N}} F_n, \Omega) \;\le\; \mathrm{Cap}_p(\mathcal{Z}_t \setminus F_n, \Omega) \;\le\; \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

and so (2.1) follows. □

In particular Proposition 2.1 shows that we may assume that a minimizer Φ given by (H3) is a $W^{1,p}$-quasi continuous function whose set of zeros is an F_δ-set (and so a Borel set).

3 A strong maximum principle for quasilinear operators

This section is of independent interest and introduces a version of the strong maximum principle adapted to our setting. Given $p, q \in (1, \infty)$, $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $V \in L^1_{loc}(\Omega)$, consider the differential inequality (in the sense of distributions):

$$-\,\mathrm{div}\,(\mathbf{a}(x, \nabla u)) + V(x)|u|^{q-2}u \;\ge\; 0, \quad u \in W^{1,p}_{loc}(\Omega). \tag{3.1}$$

The classical strong maximum principle asserts that under suitable assumptions on $\mathbf{a}$ and V, a non-negative function u satisfying (3.1) is either strictly positive or identically zero. For the model example $\mathbf{a}(x, \eta) = |\eta|^{p-2}\eta$, such an alternative can be obtained as a consequence of Harnack's inequality if $V \in L^s_{loc}(\Omega)$ with $s > \frac{N}{p}$. Even when the potential V is not so regular, it was pointed out in the works of Ancona [4] and Bénilan-Brezis [Appendix C, [6]] that it is still possible to state a weaker form of the strong maximum principle. When $\mathbf{a}(x, \eta) = \eta$ and $q = 2$, Brezis-Ponce [7] have proven under a very mild assumption on V, that the precise representative of a function u satisfying (3.1) is either identically zero or that its set of zeroes has C_2-capacity zero. Inspired by these works, we shall state in this section a similar result for quasilinear operators.

Throughout this section, we assume

$$V \in L^1_{loc}(\Omega), \quad V \ge 0, \quad q \ge p > 1, \tag{3.2}$$

and $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ to be a Carathéodory map satisfying for some constant

$\alpha > 0$ the following growth conditions:

$$\langle \mathbf{a}(x,\eta),\eta\rangle \;\geq\; \frac{1}{\alpha}|\eta|^p \quad \text{a.e. } x\in\Omega,\ \forall \eta\in\mathbb{R}^N, \tag{3.3}$$

$$|\mathbf{a}(x,\eta)| \;\leq\; \alpha|\eta|^{p-1} \quad \text{a.e. } x\in\Omega,\ \forall \eta\in\mathbb{R}^N. \tag{3.4}$$

To avoid any misunderstandings, the meaning of inequality (3.1) is the following:

Definition 3.1. *Let (3.2) to (3.4) be satisfied. We say that $u \in W^{1,p}_{loc}(\Omega)$ satisfies (3.1) if $V|u|^q \in L^1_{loc}(\Omega)$ and if the following inequality holds:*

$$\textstyle\int_\Omega \{\langle \mathbf{a}(x,\nabla u),\nabla\xi\rangle \,+\, V|u|^{q-2}u\xi\} \,\geq\, 0, \quad \forall \xi\in C_0^\infty(\Omega), \xi\geq 0. \tag{3.5}$$

Under the general assumptions (3.2) to (3.4) one cannot prove that the set of zeroes of a non-negative function $u \not\equiv 0$ satisfying (3.1) is empty. But the following result shows that it must be very small:

Proposition 3.2. *Assume that (3.2) to (3.4) hold. Given a non-negative and $W^{1,p}$-quasi continuous solution $u \in W^{1,p}_{loc}(\Omega)$ of (3.1), consider its set of zeroes*

$$\mathcal{Z} := \{x\in\Omega :\ u(x) = 0\}. \tag{3.6}$$

Then either $Cap_p(\mathcal{Z}) = 0$ or $u \equiv 0$.

Proof: Assume $\mathrm{Cap}_p(\mathcal{Z})$ is not zero. Then we need to show that $u \equiv 0$.

Let $\delta > 0$ and $\xi \in C_0^\infty(\Omega)$ with $0 \leq \xi \leq 1$ (to be chosen later). As in [7], the main idea is to prove the existence of a constant $C_0 := C_0(u,\xi) > 0$ such that

$$\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p \leq C_0, \quad \forall \delta > 0. \tag{3.7}$$

To derive (3.7), we proceed as follows.

$$\begin{aligned}
&\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p = \int_\Omega |\nabla u|^p\,(u+\delta)^{-p}\,\xi^p \\
&\quad\leq\; \alpha\int_\Omega \langle \mathbf{a}(x,\nabla u),\nabla u\rangle\,(u+\delta)^{-p}\,\xi^p \qquad \text{(by (3.3))}\\
&\quad=\; -\frac{\alpha}{p-1}\int_\Omega \left\langle \mathbf{a}(x,\nabla u),\nabla(u+\delta)^{1-p}\right\rangle\,\xi^p\\
&\quad=\; -\frac{\alpha}{p-1}\int_\Omega \left\langle \mathbf{a}(x,\nabla u),\nabla\left(\frac{\xi^p}{(u+\delta)^{p-1}}\right) - \frac{\nabla\xi^p}{(u+\delta)^{p-1}}\right\rangle.
\end{aligned}$$

Note that the differential inequality (3.5) and a density argument show that

$$-\int_\Omega \left\langle \mathbf{a}(x,\nabla u), \nabla\left(\frac{\xi^p}{(u+\delta)^{p-1}}\right)\right\rangle \le \int_\Omega V(x)|u|^{q-1}\,\frac{\xi^p}{(u+\delta)^{p-1}}. \tag{3.8}$$

From (3.8) and the assumption (3.4), we then obtain:

$$\begin{aligned}
&\frac{p-1}{\alpha}\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p \\
&\quad\le \int_\Omega V(x)|u|^{q-1}\,\frac{\xi^p}{(u+\delta)^{p-1}} \;+\; \alpha\int_\Omega \left(\frac{|\nabla u|}{u+\delta}\right)^{p-1} |\nabla \xi^p| \\
&\quad= \underbrace{\int_\Omega V(x)|u|^{q-1}\,\frac{\xi^p}{(u+\delta)^{p-1}}}_{I_1} \;+\; \underbrace{\alpha p\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^{p-1} \xi^{p-1}\,|\nabla\xi|}_{I_2}.
\end{aligned} \tag{3.9}$$

To estimate I_1, we note that assumptions (3.2), $q \ge p$ and $u, V, \xi \ge 0$ lead to:

$$\begin{aligned}
I_1 &\le \int_{\{0\le u<1\}} V(x)\,\frac{|u|^{p-1}}{(u+\delta)^{p-1}}\xi^p \;+\; \int_{\{u\ge 1\}} V(x)|u|^q\xi^p \\
&\le \int_\Omega V(x)\xi^p \;+\; \int_\Omega V(x)|u|^q\xi^p,
\end{aligned} \tag{3.10}$$

the last inequality following from the fact that $0 \le \frac{u}{u+\delta}, \xi \le 1$.

Let us now estimate I_2. By using the inequality $ab \le \epsilon^r \frac{a^r}{r} + \frac{b^s}{s\epsilon^s}$ $(\frac{1}{r}+\frac{1}{s}=1)$, with $r = \frac{p}{p-1}$, we can find a constant $C = C(u,\xi)$ such that

$$I_2 \le \frac{p-1}{2\alpha}\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p + C. \tag{3.11}$$

Hence, by plugging estimates (3.10) and (3.11) in (3.9), we get

$$\frac{p-1}{2\alpha}\int_\Omega \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p \;\le\; \int_\Omega V(x)(1+|u|^q)\xi^p + C.$$

Therefore, given $\xi \in C_0^\infty(\Omega)$, we can find a constant $C_0 := C_0(u,\xi)$ such that (3.7) holds.

To proceed with the proof of Prop. 3.2, let us emphasize that

(a) by modifying u on a set of $W^{1,p}$-capacity zero, we may assume that $\mathcal{Z}$ is a Borel set by Prop. 2.1;

(b) the set of zeroes of the function $x \mapsto \log\left(1+\frac{u(x)}{\delta}\right)$ coincides with $\mathcal{Z}$ for any $\delta > 0$.

Now if $\mathcal{Z}$ has positive $W^{1,p}$-capacity, we may find an open set $\omega_0 \subset\subset \Omega$ such that $Cap_p(\mathcal{Z} \cap \omega_0, \omega_0) > 0$. Consider then an arbitrary but fixed open set ω such that $\omega_0 \subset \omega \subset\subset \Omega$ and define

$$\gamma := Cap_p(\mathcal{Z} \cap \omega, \omega) > 0.$$

Choose $\xi \in C_0^\infty(\Omega)$, $0 \leq \xi \leq 1$ such that $\xi \equiv 1$ on ω. Then, by applying the Poincaré type inequality as stated in [Cor. 4.5.2, [32]] to the functions $\log\left(1 + \frac{u}{\delta}\right)$ and by using (3.7), we find a constant $C := C(\omega, \gamma)$ such that

$$\int_\omega \left|\log\left(1 + \frac{u}{\delta}\right)\right|^p \leq C, \quad \forall \delta > 0.$$

Since δ is arbitrary, the above uniform bound implies that $u = 0$ $\mathcal{H}^N$-a.e. in ω. Since ω can be chosen arbitrarily, we deduce that $u = 0$ $\mathcal{H}^N$-a.e. in Ω, as claimed. $\square$

Corollary 3.3. *Assume that (3.2) to (3.4) hold and let $u \geq 0$ satisfy (3.1). Then, if $u \not\equiv 0$, the set of zeroes $\mathcal{Z}$ of u satisfies*

(a) $\mathcal{H}^s(\mathcal{Z}) = 0$ for each $s > N - p$ if $p < N$;

(b) If $p \geq N$, then the Hausdorff dimension of $\mathcal{Z}$ is zero. In particular, $\mathcal{Z}$ is a discrete set.

Proof: For the proof of statement (a), we refer to [[14], p. 156] and (b) follows from (a). $\square$

Remark 3.4. *Proposition 3.2 may fail if $q < p$. Consider for example in the domain $A = \{x \in \mathbb{R}^N : 1/2 < |x| < 2\}$ the C^∞-function defined by*

$$u(x) = ||x| - 1|^\gamma, \quad \textit{with } \gamma \geq 1. \tag{3.12}$$

By defining $V := \frac{\Delta_p u}{|u|^{q-1}}$, an explicit calculation shows that $V = O\left(||x| - 1|^{(p-q)\gamma - p}\right)$ and in particular

$$V \in L^1(A) \quad \Longleftrightarrow \quad q < 1 + \frac{(\gamma - 1)(p - 1)}{\gamma}. \tag{3.13}$$

Hence, for each $q < p$, we can find a non-negative function of the type (3.12) which solves the problem:

$$-\Delta_p u + V(x)|u|^{q-1}u = 0, \quad u \geq 0, \tag{3.14}$$

with a non-negative $V \in L^1(A)$. But the set of zeroes of the function (3.12) is given by the sphere $|x| = 1$, which is not a set of zero capacity, because its complement is not connected (see [Lemma 2.46, [16]]).

If more regularity is assumed on the function V, then the unique continuation property holds (see [31]).

4 Functional framework

Recalling the restriction (1.4), let us emphasize that the elements of $\mathcal{D}_0^{1,p}(\Omega)$ are distributions and the following embeddings hold (see [13]):

$$\begin{cases} \text{for } p \in (1,N): \ \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \ \text{with } p^* = \frac{Np}{N-p}, \\ \text{for } p = N: \ \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \forall r \in [1,\infty), \\ \text{for } p \in (N,\infty): \ \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega). \end{cases} \tag{4.1}$$

The set $\mathcal{W}$ defined by (1.5) contains $C_0^\infty(\Omega)$ (since $w \in L^1_{loc}(\Omega)$) and is a subspace of $\mathcal{D}_0^{1,p}(\Omega)$. Our next proposition shows that the space $\mathcal{W}$ differs from $\mathcal{D}_0^{1,p}(\Omega)$ only when w^- is very singular.

Proposition 4.1. *Assume (H1) holds and that w^- satisfies one of the following hypotheses:*

(a) $p > N$, $w^- \in L^1(\Omega)$,

(b) $p = N$, $w^- \in L^r(\Omega)$ for some $r \in (1,\infty]$,

(c) $p \in (1,N)$, $q \in [1,p^*)$ and $w^- \in L^r(\Omega)$ with $r = \frac{Np}{N[p-q]+pq}$.

Then $\mathcal{W} \neq \mathcal{D}_0^{1,p}(\Omega)$ implies $\inf\limits_{u \in \mathcal{W}^+} J(u) = 0$ (and so (H3) cannot be satisfied).

Proof: Choose $u_0 \in \mathcal{D}_0^{1,p}(\Omega) \setminus \mathcal{W}$. Let then $\varphi_n \in C_0^\infty(\Omega)$ be a sequence such that φ_n converges strongly to u_0 in $\mathcal{D}_0^{1,p}(\Omega)$. Then by (H1) we readily get

$$\liminf_{n\to\infty} \int_\Omega A(x,\nabla\varphi_n) \;=\; \int_\Omega A(x,\nabla u_0). \tag{4.2}$$

By using the embeddings (4.1) and our hypotheses, we easily check that

$$\int_\Omega w^-|u_0|^q \;<\; \infty, \qquad \int_\Omega w^+|u_0|^q \;=\; \infty, \quad \sup_{n\in\mathbb{N}} \left\{ \int_\Omega w^-|\varphi_n|^q \right\} \;<\; \infty. \tag{4.3}$$

Furthermore, since $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$, up to a subsequence φ_n converges pointwise to u_0 a.e. in Ω. So by Fatou's Lemma we obtain

$$\liminf_{n\to\infty} \int_\Omega w^+|\varphi_n|^q \;\geq\; \int_\Omega w^+|u_0|^q \;=\; \infty. \tag{4.4}$$

Therefore from (4.3) and (4.4) we deduce $\varphi_n \in \mathcal{W}^+$ (for n sufficiently large) and $\liminf\limits_{n\to\infty} \int_\Omega w|\varphi_n|^q = +\infty$. Therefore $\liminf\limits_{n\to\infty} J(\varphi_n) = 0$, which concludes the proof. $\square$

Our discussion above shows that the space $\mathcal{W}$ is of real interest when $p \leq N$ and when the weight w is a singular sign-changing function with a "strong" negative part. Let us illustrate with an example why the spaces $\mathcal{W}$ and $\mathcal{D}_0^{1,p}(\Omega)$ do not always coincide. Consider the function $\varphi := e^{-|x|^2/2}$ defined on $\mathbb{R}^N$. We check easily that φ satisfies

$$-\Delta\varphi = w\varphi, \quad \varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \quad \varphi > 0,$$

with

$$w(x) = N - |x|^2, \quad x \in \mathbb{R}^N. \tag{4.5}$$

Consider then the functional J defined by (1.7) with the weight (4.5). By applying [Thm. 2.2, [29]], we note on one hand that J admits a minimizer. On the other hand φ belongs to $\mathcal{W}^+$ and is *strictly positive*. Therefore, it must be a minimizer of the functional J (see Corollary 5.6 for more details). But we find easily a function ξ which is such that

$$\xi \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} w^- \xi^2 = \infty,$$

and therefore $\mathcal{D}_0^{1,2} \setminus \mathcal{W} \neq \emptyset$ in this case. Hence by working in the space $\mathcal{W}$ we also admit weight functions of the type (4.5).

Let us now turn our attention to the subset $\mathcal{W}^+$ of $\mathcal{W}$ defined in (1.6). The restriction to this set is clearly required only when w changes sign, and the next proposition shows that it is never empty.

Proposition 4.2. *Let $w \in L^1_{loc}(\Omega)$ with $w^+ \not\equiv 0$ and $q \geq 1$. Then $\mathcal{W}^+ \neq \emptyset$.*

Proof: By applying the Lebesgue-Besicovitch Theorem (see [14]) and using the assumptions that $w \in L^1_{loc}(\Omega)$ with $w^+ \not\equiv 0$, we can find a point $a \in \Omega$ such that

$$\lim_{\epsilon \to 0} \frac{1}{|B(a,\epsilon)|} \int_{B(a,\epsilon)} w = w(a) > 0.$$

Hence there exist two balls $B(a,r) \subset\subset B(a,R) \subset \Omega$ such that

$$\int_{B(a,r)} w > 0 \quad \text{and} \quad \int_{B(a,R)\setminus B(a,r)} |w| < \frac{1}{2} \int_{B(a,r)} w. \tag{4.6}$$

Choose a function φ such that

$$\varphi \in C_0^\infty(B(a,R)), \quad 0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \text{ on } B(a,r). \tag{4.7}$$

Then, (4.6) and (4.7) imply

$$\int_\Omega w\varphi^q = \int_{B(a,r)} w\varphi^q + \int_{B(a,R)\setminus B(a,r)} w\varphi^q \frac{1}{2} \int_{B(a,r)} w > 0.$$

Therefore, $\varphi \in \mathcal{W}^+$. $\square$

We need to make some comments on the differentiability of the mappings:

$$\begin{array}{rcl}\widetilde{A}:\mathcal{D}_0^{1,p}(\Omega) & \to & \mathbb{R}\\ u & \mapsto & \int_\Omega A(x,\nabla u),\end{array} \qquad \begin{array}{rcl}\widetilde{W}:\mathcal{W} & \to & \mathbb{R}\\ u & \mapsto & \int_\Omega w|u|^q.\end{array} \tag{4.8}$$

Under the assumptions (H1) and (H4), it is known that $\eta \mapsto A(x,\eta)$ is of class $C^1(\mathbb{R}^N)$ (see [Cor. 25.5.1, [27]]). As a consequence, we deduce that the mapping $\widetilde{A}$ is of class C^1 on $\mathcal{D}_0^{1,p}(\Omega)$, and that its Fréchet derivative at each $u_0 \in \mathcal{D}_0^{1,p}(\Omega)$ equals

$$D\widetilde{A}_{u_0}(\xi) = \int_\Omega \langle \mathbf{a}(x,\nabla u_0), \nabla\xi\rangle, \qquad \text{where } \mathbf{a}(x,\eta) := \nabla_\eta A(x,\eta). \tag{4.9}$$

But the situation is quite different for $\widetilde{W}$ which in our setting may not even be continuous. Nevertheless we have

Proposition 4.3. *Let $w \in L^1_{loc}(\Omega)$ and $u_0 \in \mathcal{W}$. If $q > 1$, the Gâteaux-derivative of the mapping $\widetilde{W}$ at u_0 exists and is given by:*

$$D\widetilde{W}_{u_0}(\xi) = q\int_\Omega w|u_0|^{q-2}u_0\xi, \quad \forall \xi \in \mathcal{W}. \tag{4.10}$$

If $q = 1$, we have

$$\lim_{t\downarrow 0} \frac{\widetilde{W}(u_0+t\xi) - \widetilde{W}(u_0)}{t} = \int_{\{u_0\neq 0\}} w\frac{u_0}{|u_0|}\xi + \int_{\{u_0=0\}} w|\xi| \qquad \forall \xi \in C_0^\infty(\Omega), \tag{4.11}$$

and if furthermore $u_0 > 0$ $\mathcal{H}^N$-a.e. in an open subset Ω' of Ω, one has

$$\lim_{t\to 0} \frac{\widetilde{W}(u_0+t\xi) - \widetilde{W}(u_0)}{t} = \int_\Omega w\xi, \qquad \forall \xi \in C_0^\infty(\Omega'). \tag{4.12}$$

Proof: Let $q > 1$. Given $u_0, \xi \in \mathcal{W}$ and $t \in \mathbb{R}$, let us set

$$f_t(x) := \left(\frac{w|u_0+t\xi|^q - w|u_0|^q}{t}\right)(x), \quad x \in \Omega.$$

We first note that

$$\lim_{t\to 0} f_t(x) = qw(x)|u_0(x)|^{q-2}u_0(x)\xi(x), \quad a.e.\, x \in \Omega.$$

Furthermore,

$$\begin{aligned} f_t(x) &= \frac{1}{t}\int_0^1 \frac{d}{ds}\left[w(x)\,|u_0+st\xi|^q\right] ds \\ &= q\int_0^1 \left\{w(x)\,|u_0+st\xi|^{q-2}\,(u_0+st\xi)\xi\right\} ds, \end{aligned}$$

which implies

$$|f_t| \le C_0 \left\{ |w||u_0|^{q-1}|\xi| + |t||w||\xi|^q \right\}. \tag{4.13}$$

We check easily that the right hand-side of (4.13) is in $L^1(\Omega)$ for any $u_0, \xi \in \mathcal{W}$. The conclusion follows by applying Lebesgue's dominated convergence Theorem.

In the case $q = 1$, we simply note that

$$\begin{aligned}
\lim_{t\to 0} \frac{|u_0 + t\xi| - |u_0|}{t}(x) &= \tfrac{u_0(x)}{|u_0(x)|}\xi(x) \quad \text{if } u_0(x) \ne 0,\\
\lim_{t\downarrow 0} \frac{|u_0 + t\xi| - |u_0|}{t}(x) &= |\xi|(x) \qquad\quad \text{if } u_0(x) = 0.
\end{aligned}$$

Again using Lebesgue's dominated convergence Theorem, (4.11) and (4.12) follow. □

Proposition 4.4. *Let (H1) to (H4)(a) be satisfied and (Λ, Φ) be a solution of the minimization problem (1.8). If $q > 1$ one has*

$$\int_\Omega \langle \mathbf{a}(x, \nabla\Phi), \nabla\xi\rangle = p\frac{\Lambda}{\left(\int_\Omega w(x)|\Phi|^q\right)^{1-\frac{p}{q}}} \int_\Omega w(x)|\Phi|^{q-2}\Phi\xi, \quad \forall \xi \in \mathcal{W}. \tag{4.14}$$

If $q = 1$ and $\Phi \ge 0$ $\mathcal{H}^N$-a.e, then for any $\xi \in C_0^\infty(\Omega)$ with $\xi \ge 0$

$$\int_\Omega \langle \mathbf{a}(x, \nabla\Phi), \nabla\xi\rangle \ge p\Lambda \left(\int_\Omega w|\Phi|\right)^{1-p} \left(\int_\Omega w\xi\right), \tag{4.15}$$

furthermore if $\Phi > 0$ $\mathcal{H}^N$-a.e. in an open subset Ω' of Ω, we have

$$\int_\Omega \langle \mathbf{a}(x, \nabla\Phi), \nabla\xi\rangle = p\Lambda \left(\int_\Omega w|\Phi|\right)^{1-p} \left(\int_\Omega w\xi\right), \quad \forall \xi \in C_0^\infty(\Omega'). \tag{4.16}$$

Proof: Note that for each $(u_0, \xi) \in \mathcal{W}^+ \times \mathcal{W}$, the dominated convergence theorem ensures the existence of $\epsilon > 0$ such that

$$u_0 + t\xi \in \mathcal{W}^+, \quad \forall t \in (-\epsilon, \epsilon). \tag{4.17}$$

Assume first $q > 1$. Given a minimizer Φ of $J : \mathcal{W}^+ \to \mathbb{R}$, the Gâteaux derivative at Φ in any direction $\xi \in \mathcal{W}$ is well-defined, because of (4.17), (4.9) and (4.10). Since Φ is a minimizer of J, we have that the Gâteaux derivative $DJ_\Phi = 0$. Hence for any $\xi \in \mathcal{W}$ we get

$$\int_\Omega \langle \mathbf{a}(x, \nabla\Phi), \nabla\xi\rangle \ = \ p\frac{\int_\Omega A(x, \nabla\Phi)}{\int_\Omega w(x)|\Phi|^q} \int_\Omega w(x)|\Phi|^{q-2}\Phi\xi,$$

and so (4.14) follows.

In the case $q = 1$, the functional $\widetilde{W}$ defined by (4.8) fails to be Gâteaux differentiable at some points. But since Φ is a minimizer, we always have:

$$\liminf_{t \downarrow 0} \frac{J(\Phi + t\xi) - J(\Phi)}{t} \geq 0. \tag{4.18}$$

One the other hand, we note that (4.11) applied at Φ (non-negative a.e. by assumption) gives:

$$\lim_{t \downarrow 0} \frac{\widetilde{W}(\Phi + t\xi) - \widetilde{W}(\Phi)}{t} = \int_\Omega w\xi, \quad \forall \xi \in C_0^\infty(\Omega),\ \xi \geq 0. \tag{4.19}$$

By applying (4.18) together with (4.9) and (4.19) we derive (4.15).

If we make the stronger assumption that $\Phi > 0$ $\mathcal{H}^N$-a.e. in an open subset Ω' of Ω, then the functional $\widetilde{W}$ is Gâteaux differentiable in any direction $\xi \in C_0^\infty(\Omega')$ by (4.12). So in this case (4.16) will follow by applying (4.9) and (4.12). $\square$

5 Sign of the minimizers

The study of the sign of minimizers is a main point in the derivation of the uniqueness results in Theorems 1.2 and 1.3. We start with the following Lemma:

Lemma 5.1. *Assume (H1), (H3) hold and let Φ be a minimizer of Problem (1.8). Then $\int_\Omega w|\Phi^\pm|^q \geq 0$ and*

$$\int_\Omega w|\Phi^\pm|^q = 0 \quad \Longleftrightarrow \quad \Phi^\pm \equiv 0. \tag{5.1}$$

Proof: We show this for Φ^-, since the same proof applies to Φ^+. Note first that $\int_\Omega w|\Phi|^q > 0$ (by (H3)). If $\int_\Omega w|\Phi^-|^q < 0$, we get $0 < \int_\Omega w|\Phi|^q < \int_\Omega w|\Phi^+|^q$. Hence $\Phi^+ \in \mathcal{W}^+$ and

$$\frac{\int_\Omega A(x, \nabla\Phi^+)}{\left(\int_\Omega w|\Phi^+|^q\right)^{\frac{p}{q}}} < \frac{\int_\Omega A(x, \nabla\Phi)}{\left(\int_\Omega w|\Phi|^q\right)^{\frac{p}{q}}} = \Lambda, \tag{5.2}$$

contradicting the definition of Λ. Thus $\int_\Omega w|\Phi^-|^q \geq 0$.

Assume now $\int_\Omega w|\Phi^-|^q = 0$. In such a case, we have $\Phi^+ \in \mathcal{W}^+$ and

$$\Lambda = \frac{\int_\Omega A(x, \nabla\Phi^+) + \int_\Omega A(x, \nabla\Phi^-)}{\left(\int_\Omega w|\Phi^+|^q\right)^{\frac{p}{q}}} \geq \frac{\int_\Omega A(x, \nabla\Phi^+)}{\left(\int_\Omega w|\Phi^+|^q\right)^{\frac{p}{q}}} \geq \Lambda. \tag{5.3}$$

Therefore $A(x, \nabla\Phi^-) \equiv 0$ and (H1) yields $\Phi^- \equiv 0$. $\square$

Before going further, we emphasize that under the assumptions (H1), (H2) and (H4)(a) the mapping $\mathbf{a}(x,\eta) := \nabla_\eta A(x,\eta)$ fulfills the conditions (3.3) and (3.4). Indeed since A is differentiable and convex in the second variable, it is well-known that $A(x,\cdot) \in C^1(\mathbb{R}^N)$ a.e. $x \in \Omega$ (see [27]). In particular, by using the fact that $x \mapsto A(x,\eta)$ is measurable for any $\eta \in \mathbb{R}^N$, we deduce that $\nabla_\eta A(x,\eta)$ is also a Carathéodory mapping.

Proposition 5.2. *Assume (H1) to (H4) hold and the set $\{w>0\}$ is open. Let Φ be a minimizer of J.*

(a) Then there exists at least one connected component $\widetilde{\Omega}$ of $\{w>0\}$ such that $|\Phi|>0$ a.e. in $\widetilde{\Omega}$.

(b) If $\{w>0\}$ is also connected then Φ does not change sign, i.e. either $\Phi^- \equiv 0$ or $\Phi^+ \equiv 0$.

Proof: Since $A(x,\eta) = A(x,-\eta)$ (by (H2)), we deduce that $|\Phi|$ is also a minimizer.

If $\Phi \equiv 0$ in $\{w>0\}$, we would have $\int_\Omega w|\Phi|^q \leq 0$. This is not possible because $\Phi \in \mathcal{W}^+$ by assumption. Choose then a connected component $\widetilde{\Omega}$ of $\{w>0\}$ in which $\Phi \not\equiv 0$. It follows from (4.14) and (4.15) that $|\Phi|$ satisfies in the sense of distributions:

$$-\mathrm{div}\,(\mathbf{a}(x,\nabla|\Phi|)) \geq 0 \ \text{ in } \widetilde{\Omega}, \quad |\Phi| \not\equiv 0 \ \text{ in } \widetilde{\Omega}.$$

Now the classical strong maximum principle ([16]) implies

$$\mathrm{essinf}_\omega |\Phi| > 0, \quad \forall \omega \subset\subset \widetilde{\Omega}. \tag{5.4}$$

So claim (a) of the proposition follows.

Let us prove the second statement. Assume $\Phi^+ \not\equiv 0$. Since $\{w>0\}$ is connected, part (a) implies

$$\Phi > 0 \ \mathcal{H}^N - a.e. \text{ in } \{w>0\}. \tag{5.5}$$

Consider then Φ^-. By (5.5), we have

$$\int_\Omega w|\Phi^-|^q = \int_{\{w\leq 0\}} w|\Phi^-|^q \leq 0.$$

Therefore Lemma 5.1 implies $\Phi^- \equiv 0$. $\square$

Proposition 5.2 holds for any $p > 1$, $q \geq 1$ and does *not* exclude the possibility of a minimizer Φ being identically zero on an open set. But when $q \geq p$, without making any assumption on the set $\{w > 0\}$, we can actually prove that the set of zeroes of a minimizer has capacity zero.

Proposition 5.3. *Let (H1) to (H4) be satisfied and $q \geq p$. Then exactly one of the following alternative holds:*

$$\Phi > 0 \quad \mathcal{H}^N\!-a.e. \qquad or \qquad \Phi < 0 \quad \mathcal{H}^N\!-a.e. \tag{5.6}$$

and the set of zeroes of Φ satisfies

$$Cap_p(\{\Phi = 0\}) = 0. \tag{5.7}$$

Proof: Assume that $\Phi^+ \not\equiv 0$ and let us show then that $\Phi^- \equiv 0$.

Note that for any $\eta \in \mathbb{R}^N$ we have $\langle \mathbf{a}(x,\eta), \eta \rangle = pA(x,\eta)$ (Euler differentiation formula for a homogeneous mapping). So by using Φ^+ and Φ^- as test functions in (4.14), we deduce that

$$\int_\Omega A(x, \nabla\Phi^+) \quad = \quad \frac{\Lambda}{\left(\int_\Omega w(x)|\Phi|^q\right)^{1-\frac{p}{q}}} \int_\Omega w(x)|\Phi^+|^q. \tag{5.8}$$

Hence,

$$\Lambda \leq \frac{\int_\Omega A(x,\nabla\Phi^+)}{\left(\int_\Omega w(x)|\Phi^+|^q\right)^{\frac{p}{q}}} \quad = \quad \Lambda \left(\frac{\int_\Omega w(x)|\Phi^+|^q}{\int_\Omega w(x)|\Phi|^q} \right)^{1-\frac{p}{q}}. \tag{5.9}$$

If $q > p$, we get from (5.9) that $\int_\Omega w(x)|\Phi^-|^q = 0$. Thus, $\Phi^- \equiv 0$ by Lemma 5.1.

If $q = p$, relation (5.9) shows that Φ^+ is a minimizer of Problem (1.8). It satisfies therefore equation (4.14) and we get

$$\begin{cases} -\operatorname{div}(\mathbf{a}(x,\nabla\Phi^+)) + \Lambda w^-|\Phi^+|^{p-1} = \Lambda w^+|\Phi^+|^{p-1}, \\ \quad \Phi^+ \geq 0, \quad \Phi^+ \not\equiv 0. \end{cases}$$

By the strong maximum principle as stated in Proposition 3.2, we deduce that $\Phi^- \equiv 0$ and $\mathrm{Cap}_p(\{\Phi = 0\}) = 0$. □

Proposition 5.3 may fail when $q \in [1, p)$. Let us give one example in the simplest case $q = 1$.

Example 5.4. *For $q = 1$, the set of zeroes of the minimizers of the functional J can have positive measure. Consider the one-dimensional situation:*

$$\Omega = (-1, 1), \quad w(x) = -x, \quad J(u) = \frac{\int_{-1}^{1} |u'(x)|^2 dx}{\int_{-1}^{1} (-x)|u(x)| dx} \quad (u \in \mathcal{W}^+).$$

Since w is bounded, we have $\mathcal{W} = W_0^{1,2}((-1, 1))$, and classical arguments show that the functional J has a minimizer $\Phi \geq 0$ which is continuous. Let us show that $\{\Phi = 0\}$ must be a non-empty interval $(z_0, 1) \subset (0, 1)$. Indeed by Prop. 5.2, we know that $\Phi > 0$ in the interval $(-1, 0)$ (namely in the interval where w is positive). Let us set

$$z_0 := \inf\{z \in (-1, 1) \colon \Phi(z) = 0\},$$

and note by previous discussion that $z_0 \in [0, 1]$. Now in the interval $(-1, z_0)$, we know by Prop. 4.4 that Φ satisfies:

$$-\Phi'' = \lambda(-x) \text{ in } (-1, z_0), \quad \Phi \in C^\infty(-1, z_0), \quad \Phi > 0 \text{ in } (-1, z_0), \tag{5.10}$$

where $\lambda := 2\Lambda \left(\int_{-1}^{1} w\Phi \right)^{-1} > 0$. If $z_0 = 1$, since the right hand-side of the ODE in (5.10) has average zero, we deduce that $\Phi'(-1) = \Phi'(1)$. Since $\Phi \geq 0$ we deduce easily that

$$\Phi'(-1) = \Phi'(1) = 0,$$

which is not possible because $\Phi'(-1) > 0$ (by Hopf's Lemma). Therefore $z_0 \in [0, 1)$, which already shows that the set of zeroes of Φ is non-empty. We claim that actually Φ is identically zero in $[z_0, 1]$. If this would not be the case, by continuity of Φ we can find an interval $[a, b] \subset [z_0, 1]$ such that:

$$\Phi > 0 \text{ in } (a, b), \quad \Phi(a) = \Phi(b) = 0.$$

By applying again Prop. 4.4, we get

$$-\Phi'' = \lambda(-x) \text{ in } (a, b), \quad \Phi \in H_0^1((a, b)), \quad \Phi > 0 \text{ in } (a, b), \tag{5.11}$$

with λ defined as in before. From (5.11) we derive:

$$\int_a^b |\Phi'(t)|^2 = \lambda \int_a^b (-x)\Phi \leq 0,$$

and therefore $\Phi \equiv 0$ in $[a, b]$. So the set of zeroes of Φ is the full interval $[z_0, 1]$.

We conclude this section by showing that the minimizers of J are completely characterized by their sign when $q = p$. More precisely:

Proposition 5.5. *Assume (H1) to (H4) hold and $q = p$. Let (λ, φ) be a solution of*

$$-\mathrm{div}(\mathbf{a}(x, \nabla\varphi)) \geq \lambda w \varphi^{p-1}, \quad \varphi \geq 0 \text{ in } \Omega, \quad \varphi \not\equiv 0.$$

Then $\lambda \leq \Lambda$.

Proof: Let $\Phi \in \mathcal{W}^+$ be a minimizer of J, and choose it such that $\Phi \geq 0$. Hence, we have

$$\begin{aligned} -\mathrm{div}(\mathbf{a}(x, \nabla\Phi)) &= \Lambda w \Phi^{p-1}, \quad \Phi \geq 0 \text{ in } \Omega, && (5.12)\\ -\mathrm{div}(\mathbf{a}(x, \nabla\varphi)) &\geq \lambda w \varphi^{p-1}, \quad \varphi \geq 0 \text{ in } \Omega. && (5.13)\end{aligned}$$

For each $k \geq 0$, let us truncate Φ as follows:

$$\Phi_k(x) := \begin{cases} k & \text{if } \Phi(x) \geq k,\\ \Phi(x) & \text{if } \Phi(x) \in [0, k),\end{cases}$$

and for each $\epsilon > 0$ consider the function $\frac{\Phi_k^p}{(\varphi+\epsilon)^{p-1}}$. Note that

$$\Phi_k, \frac{\Phi_k^p}{(\varphi+\epsilon)^{p-1}} \in \mathcal{W} \cap L^\infty(\Omega),$$

and so they can be used as test functions in (5.12) and (5.13). By doing so, we get

$$\begin{aligned} &\int_\Omega \left\{ \langle \mathbf{a}(x, \nabla\Phi_k), \nabla\Phi_k\rangle - \left\langle \mathbf{a}(x, \nabla\varphi), \nabla\left(\frac{\Phi_k^p}{(\varphi+\epsilon)^{p-1}}\right)\right\rangle \right\}\\ &\quad \leq \int_\Omega \left\{ \Lambda w \Phi^{p-1}\Phi_k - \lambda w \varphi^{p-1} \frac{\Phi_k^p}{(\varphi+\epsilon)^{p-1}} \right\}. \qquad (5.14)\end{aligned}$$

We claim that the integrand in the left hand-side of (5.14) is non-negative. Indeed, the assumption that A is p-homogeneous in the second variable implies:

$$\begin{aligned} &\langle \mathbf{a}(x, \nabla\Phi_k), \nabla\Phi_k\rangle - \left\langle \mathbf{a}(x, \nabla\varphi), \nabla\left(\frac{\Phi_k^p}{(\varphi+\epsilon)^{p-1}}\right)\right\rangle\\ &= pA(x, \nabla\Phi_k) + p(p-1)\left(\frac{\Phi_k}{\varphi+\epsilon}\right)^p A(x, \nabla\varphi)\\ &\quad -p\left(\frac{\Phi_k}{\varphi+\epsilon}\right)^{p-1} \langle \mathbf{a}(x, \nabla\varphi), \nabla\Phi_k\rangle\\ &= p\left\{ A(x, \nabla\Phi_k) + (p-1)A(x, \frac{\Phi_k}{\varphi+\epsilon}\nabla\varphi) - \left\langle \mathbf{a}(x, \frac{\Phi_k}{\varphi+\epsilon}\nabla\varphi), \nabla\Phi_k\right\rangle \right\}. \quad (5.15)\end{aligned}$$

By using the property that $\eta \mapsto A(x, \eta)$ is convex, we now deduce easily that (5.15) is non-negative. Therefore, coming back to (5.14), we get

$$\int_\Omega \left\{ \Lambda w \Phi^{p-1} \Phi_k - \lambda w \left(\frac{\varphi}{\varphi + \epsilon} \right)^{p-1} \Phi_k^p \right\} \geq 0. \tag{5.16}$$

Since by Theorem 3.2 the set $\{\varphi = 0\}$ is of measure zero, (5.16) is equivalent to

$$\int_{\{\varphi > 0\}} \left\{ \Lambda w \Phi^{p-1} \Phi_k - \lambda w \left(\frac{\varphi}{\varphi + \epsilon} \right)^{p-1} \Phi_k^p \right\} \geq 0. \tag{5.17}$$

Now, letting $\epsilon \to 0$ and $k \to \infty$ in (5.17), we get

$$(\Lambda - \lambda) \int_\Omega w \Phi^p \geq 0. \tag{5.18}$$

Since, $\int_\Omega w \Phi^p > 0$, (5.18) implies $\lambda \leq \Lambda$. $\square$

Corollary 5.6. *Assume (H1) to (H4) hold. Let $(\lambda, \varphi) \in (0, \infty) \times \mathcal{D}_0^{1,p}(\Omega)$ be such that:*

$$-\text{div } (\mathbf{a}(x, \nabla \varphi)) = \lambda w \varphi^{p-1}, \quad \varphi \geq 0, \quad \varphi \not\equiv 0.$$

Then $\lambda = \Lambda$ and φ is a minimizer of the functional J.

Remark 5.7. *The fact that when $q = p$ any positive critical point of the functional J must be a minimizer has been known under several regularity assumptions either on the domain Ω or on the weight w (see [26], [12]). In [18] we can find a proof that holds for any bounded domain, when the weight function $w \equiv 1$.*

The proof of Prop. 5.5 refines some arguments found in the work of Cuesta [12] and holds on any domain, without making any assumptions on the sign or regularity of the weight function w.

6 Uniqueness of the minimizer

This last section provides the proofs of the propositions stated in the introduction. Consider two minimizers Φ_1, Φ_2 of J which, thanks to (H2)(a), can be normalized as follows:

$$\int_\Omega w|\Phi_1|^q = \int_\Omega w|\Phi_2|^q = 1. \tag{6.1}$$

Note that (for $i = 1, 2$):

$$J(|\Phi_i|) = J(\Phi_i) = \Lambda, \quad |\Phi_i| \in \mathcal{W}^+, \tag{6.2}$$

i.e. $|\Phi_i|$ are also minimizers of J. To derive our uniqueness results we define

$$\eta := \left(\frac{|\Phi_1|^q + |\Phi_2|^q}{2}\right)^{1/q}. \tag{6.3}$$

As in [Prop. 1.1, [25]] and [5] the main idea is to show that η is again a minimizer of J. For later purposes let us emphasize that

$$\eta \in \mathcal{W}^+ \quad \text{with} \quad \int_\Omega w|\eta|^q = \int_\Omega w\eta^q = 1, \tag{6.4}$$

and also

$$\nabla\eta = \begin{cases} 0 & \text{in } \{\Phi_1 = \Phi_2 = 0\}, \\ \left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{i=1}^{2} \left(\frac{|\Phi_i|^q}{|\Phi_1|^q + |\Phi_2|^q}\right)^{\frac{q-1}{q}} \nabla|\Phi_i| & \text{elsewhere.} \end{cases} \tag{6.5}$$

The proof that η is a minimizer of J will rely on the following proposition which is of independent interest.

Proposition 6.1. *Let $p > 1$ and $H : \mathbb{R}^N \to [0, \infty)$ be a convex, p-homogeneous function, i.e. $H(t\eta) = t^p H(\eta)$ for any $t > 0$. Then for any $q \in [1, p]$, $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1^q + \alpha_2^q = 1$ we have:*

$$H(\alpha_1^{q-1}\eta_1 + \alpha_2^{q-1}\eta_2) \le 2^{\frac{p}{q}-1}\{H(\eta_1) + H(\eta_2)\}, \quad \forall \eta_1, \eta_2 \in \mathbb{R}^N. \tag{6.6}$$

Furthermore, when $1 < q < p$, the inequality in (6.6) is strict if

$$\alpha_1 \ne \alpha_2 \quad \textit{and} \quad H(\eta_1) + H(\eta_2) \ne 0. \tag{6.7}$$

For $q = 1$ or $q = p$, the inequality in (6.6) is strict if H is strictly convex and

$$\eta_1 \ne \eta_2 \quad \textit{for } q = 1, \qquad \alpha_2\eta_1 \ne \alpha_1\eta_2 \quad \textit{for } q = p. \tag{6.8}$$

Proof: When $q = 1$, the proof of (6.6) is obvious since

$$H(\eta_1 + \eta_2) = 2^p H(\frac{1}{2}\eta_1 + \frac{1}{2}\eta_2) \le 2^{p-1}\{H(\eta_1) + H(\eta_2)\}. \tag{6.9}$$

Furthermore, under the assumption that H is strictly convex, we deduce that strict inequality holds in (6.9) whenever $\eta_1 \ne \eta_2$ (which is condition (6.8) when $q = 1$).

Let us now deal with the case $1 < q \leq p$. If $\alpha_1 = 0$, we have $\alpha_2 = 1$. The inequality (6.6) follows since $H \geq 0$ and $q \leq p$. Furthermore the inequality is obviously strict if (6.7) holds.

To handle the case $q > 1$ and $\alpha_1, \alpha_2 > 0$, we chose $\gamma > 0$ satisfying

$$\gamma + p(q - 1 - \gamma) = 0. \tag{6.10}$$

The choice of γ will become clear later, and for now we just note that

$$\gamma = p\frac{q-1}{p-1} \quad \text{and} \quad \frac{q}{\gamma} \geq 1 \text{ (since } 1 < q \leq p). \tag{6.11}$$

Write now

$$\sum_{i=1}^{2} \alpha_i^{q-1}\eta_i = \sum_{i=1}^{2} \underbrace{2^{\frac{\gamma}{q}-1}\alpha_i^{\gamma}}_{t_i} \underbrace{\alpha_i^{q-1-\gamma}2^{1-\frac{\gamma}{q}}}_{s_i} \eta_i,$$

and note that $t_1 + t_2 \neq 0$. Then by using the convexity of H with the coefficients t_i and homogeneity with s_i, we get

$$\begin{aligned} H\left(\sum_{i=1}^{2} t_i s_i \eta_i\right) &= (t_1+t_2)^p \; H\left(\sum_{i=1}^{2} \frac{t_i}{t_1+t_2} s_i\eta_i\right) \\ &\leq (t_1+t_2)^{p-1} \sum_{i=1}^{2} t_i s_i^p H\left(\eta_i\right). \end{aligned} \tag{6.12}$$

Recall now that for each $r \geq 1$, the following inequality holds

$$(a+b)^r \leq 2^{r-1}(a^r + b^r), \quad \forall a, b \geq 0. \tag{6.13}$$

By applying (6.13) with $a = \alpha_1^{\gamma}$, $b = \alpha_2^{\gamma}$ and $r = \frac{q}{\gamma} \geq 1$ (see (6.11)), we derive

$$t_1 + t_2 = 2^{\frac{\gamma}{q}-1}\left\{\alpha_1^{\gamma} + \alpha_2^{\gamma}\right\} \leq 1. \tag{6.14}$$

Therefore (6.12) and (6.14) yield

$$\begin{aligned} H\Big(\sum_{i=1}^{2} \alpha_i^{q-1}\eta_i\Big) &\leq \sum_{i=1}^{2} 2^{\frac{\gamma}{q}-1}\alpha_i^{\gamma} H(\alpha_i^{q-1-\gamma}2^{1-\frac{\gamma}{q}}\eta_i) \qquad (6.15) \\ &= \sum_{i=1}^{2} 2^{\frac{\gamma}{q}-1}\alpha_i^{\gamma}\alpha_i^{p(q-1-\gamma)}2^{p(1-\frac{\gamma}{q})} H(\eta_i) \\ &= 2^{(\frac{p}{q}-1)} \sum_{i=1}^{2} H(\eta_i), \qquad (6.16) \end{aligned}$$

where the equality (6.16) follows from the choice of γ defined in (6.10). This completes the proof of inequality (6.6).

Let us discuss now when the strict inequality holds. For $1 < q < p$, we note that inequality (6.13) is strict when $r > 1$ and $a \neq b$. Hence strict inequality holds in (6.14) whenever

$$1 < q < p \quad \text{and} \quad \alpha_1 \neq \alpha_2. \tag{6.17}$$

Therefore under the assumption (6.7), we deduce that (6.15), and as a consequence (6.6), are strict inequalities.

When $q = p$, notice that (6.13) turns out to be always an equality since $r = 1$ in this case. Hence we need to argue differently. When $\alpha_1 = 0$ or $\alpha_2 = 0$, one gets easily from assumption (6.8) that inequality (6.6) is strict. If $\alpha_1, \alpha_2 \neq 0$, inequality (6.15) becomes simply

$$H(\sum_{i=1}^{2} \alpha_i^{p-1}\eta_i) \leq \sum_{i=1}^{2} \alpha_i^p H(\alpha_i^{-1}\eta_i) \quad (\text{with } \alpha_1^p + \alpha_2^p = 1),$$

and strict inequality holds whenever $\alpha_1^{-1}\eta_1 \neq \alpha_2^{-1}\eta_2$ (since H is assumed to be strictly convex). $\square$

6.1 The Case $1 \leq q < p$

In this case, the fact that a *non-negative* minimizer is unique (up to a multiplicative factor) does not rely on the results obtained in previous sections.

Proof of Theorem 1.1: By normalizing $\Phi_1, \Phi_2 \geq 0$ as in (6.1), we are led to prove that $\Phi_1 = \Phi_2$. Consider the function η defined by (6.3) and introduce the following subsets of Ω:

$$\begin{aligned} E_1 &:= \{\Phi_1 = \Phi_2\}, \\ E_2 &:= \{\Phi_1 \neq \Phi_2\} \cap \{\nabla\Phi_1 = \nabla\Phi_2 = 0\}, \\ E_3 &:= \{\Phi_1 \neq \Phi_2\} \cap \big(\{\nabla\Phi_1 \neq 0\} \cup \{\nabla\Phi_2 \neq 0\}\big). \end{aligned}$$

We note that the sets E_i are pairwise disjoint and their union equals Ω. On E_1, we have $\nabla\eta = \frac{1}{2}\{\nabla\Phi_1 + \nabla\Phi_2\}$ (see (6.5)) and also (see [Cor. 1.21, [16]])

$$\nabla\Phi_1 = \nabla\Phi_2 \quad \text{a.e. in } E_1. \tag{6.18}$$

Hence

$$A(x, \nabla\eta) = \frac{1}{2}\{A(x, \nabla\Phi_1) + A(x, \nabla\Phi_2)\} \quad \textit{a.e.} \text{ in } E_1. \tag{6.19}$$

In the set E_2 we have

$$A(x,\nabla\eta)=0 \quad \text{ a.e. in } E_2 \tag{6.20}$$

In E_3, we apply Proposition 6.1 with $(i=1,2)$:

$$\alpha_i=\left(\frac{\Phi_i^q}{\Phi_1^q+\Phi_2^q}\right)^{\frac{1}{q}} \quad \text{ and } \quad \eta_i=\nabla\Phi_i.$$

Since at a.e. x in E_3 either $\nabla\Phi_1\neq 0$ or $\nabla\Phi_2\neq 0$, and $\alpha_1\neq\alpha_2$, we get from (6.7):

$$A(x,\nabla\eta)<\frac{1}{2}\{A(x,\nabla\Phi_1)+A(x,\nabla\Phi_2)\} \quad \textit{a.e.} \text{ in } E_3. \tag{6.21}$$

Assume then $\mathcal{H}^N(E_3)\neq 0$. Because of (6.19), (6.20), (6.21), we obtain:

$$J(\eta)<\frac{1}{2}\int_{E_1\cup E_3}\{A(x,\nabla\Phi_1)+A(x,\nabla\Phi_2)\}=J(\Phi_1).$$

This is in contradiction to the fact that Φ_1 is a minimizer of J. Therefore $\mathcal{H}^N(E_3)=0$. But from (6.18) and the definition of E_2 we deduce that:

$$\nabla\Phi_1=\nabla\Phi_2 \quad \text{ a.e. in } \Omega.$$

Since Ω is connected, we obtain $\Phi_1=\Phi_2$ a.e. in Ω.

If $q=1$ and A is strictly convex in the second variable, we argue as follows.

$$\begin{aligned} A(x,\nabla\eta) &= \frac{1}{2}\left\{A(x,\nabla\Phi_1)+A(x,\nabla\Phi_2)\right\}, \quad \text{ in } \{\nabla\Phi_1=\nabla\Phi_2\}, & (6.22)\\ A(x,\nabla\eta) &< \frac{1}{2}\left\{A(x,\nabla\Phi_1)+A(x,\nabla\Phi_2)\right\}, \quad \text{ in } \{\nabla\Phi_1\neq\nabla\Phi_2\}. & (6.23) \end{aligned}$$

Therefore (6.22) and (6.23) imply

$$J(\eta)<\Lambda \quad \text{ if } \mathcal{H}^N(\{\nabla\Phi_1\neq\nabla\Phi_2\})>0.$$

Hence from the definition of Λ we deduce $\mathcal{H}^N(\{\nabla\Phi_1\neq\nabla\Phi_2\})=0$, which implies $\Phi_1=\Phi_2$ a.e. in Ω. □

Corollary 6.2. *Assume (H1) to (H3) hold and let Φ_1, Φ_2 be two minimizers of J. Then, up to a set of zero measure, $\{\Phi_1=0\}=\{\Phi_2=0\}$.*

Proof: By (6.2) and Theorem 1.1, we deduce that $|\Phi_2|=t|\Phi_1|$ for some $t>0$, and so the conclusion follows. □

When $\{w > 0\}$ is open and connected, we get a result which holds for arbitrary minimizers.

Proof of Theorem 1.2: By Prop. 5.2 we have $\pm\Phi_i \geq 0$ a.e. in Ω $(i = 1, 2)$. Hence, the conclusion follows from (6.2) and by applying Thm. 1.1. $\square$

If $\{w > 0\}$ has more than one connected component, Theorem 1.2 may fail.

Example 6.3. *For $q = 1$ and $p = 2$, the functional J may have a changing sign minimizer and minimizers are not unique (up to a constant factor). Consider*

$$\Omega = (-1, 1), \quad w(x) = 2|x| - 1, \quad J(u) = \frac{\int_{-1}^{1} |u'(t)|^2 dt}{\int_{-1}^{1} w(t)|u(t)|dt} \quad (u \in \mathcal{W}^+).$$

Classical arguments show that J has a minimizer Φ. Let us define the two functions $\Phi_1(x) := |\Phi(x)|$ and $\Phi_2(x) = |\Phi(-x)|$. Using the property that w is even, we easily check that Φ_1, Φ_2 are two non-negative minimizers of J, and $\|\nabla\Phi_1\| = \|\nabla\Phi_2\|$. Therefore by applying Theorem 1.1 we get $\Phi_1 \equiv \Phi_2$, i.e. $|\Phi|$ is even. Arguing as in Example 5.4, we deduce that there exists a $z_0 \in (0, \frac{1}{2})$ such that

$$\Phi(x) > 0 \quad \text{for } x \in (-1, -z_0), \quad \Phi(x) = 0 \quad \text{for } x \in (-z_0, 0).$$

Due to the symmetry of Φ, we deduce that Φ vanishes in the interval $[-z_0, z_0]$ and is strictly positive elsewhere. Consider now the function defined by

$$\widetilde{\Phi}(x) := \begin{cases} \Phi(x) & x \in (-1, 0), \\ -\Phi(x) & x \in (0, 1). \end{cases} \tag{6.24}$$

Then $\widetilde{\Phi}$ is a minimizer of J that changes sign, and clearly Φ and $\widetilde{\Phi}$ are not multiples of each other.

6.2 The Case $q = p > 1$

Proposition 6.4. *Assume (H1) to (H4) hold with $q = p > 1$. For any two minimizers Φ_1, Φ_2 of J, we have*

$$\Phi_2 \nabla\Phi_1 = \Phi_1 \nabla\Phi_2 \quad \mathcal{H}^N - a.e \text{ in } \Omega. \tag{6.25}$$

Proof: Thanks to Prop. 5.3 we may assume $\Phi_1, \Phi_2 \geq 0$, and normalize Φ_1, Φ_2 as in (6.1). Therefore

$$A(x, \nabla\eta) = \left(\frac{1}{2}\right)^{\frac{p}{q}} A\left(x, \sum_{i=1}^{2} \left(\frac{\Phi_i}{(\Phi_1^q + \Phi_2^q)^{\frac{1}{q}}}\right)^{q-1} \nabla\Phi_i\right).$$

By applying Prop. 6.1 with $\alpha_i = \frac{\Phi_i}{(\Phi_1^q+\Phi_2^q)^{\frac{1}{q}}}$ and $\eta_i = \nabla\Phi_i$ $(i = 1, 2)$, we get

$$A(x, \nabla\eta) \leq \frac{1}{2}\{A(x, \nabla\Phi_1) + A(x, \nabla\Phi_2)\}. \tag{6.26}$$

Therefore, from (6.4) and (6.26) we deduce that $J(\eta) \leq J(\Phi_1)$. Thus η is a minimizer, and therefore equality must hold in (6.26), which by Lemma 6.1 implies (6.25). □

To prove uniqueness of the minimizer in the case $q = p$, we can now follow what has been done in [22].

Proof of Theorem 1.3: We sketch the main ideas, and refer to [22] for more details. Let Φ_1, Φ_2 be two minimizers of J (quasicontinuous) and define:

$$\mathcal{P} := \{x \in \Omega \,:\, (6.25) \text{ does not hold at } x\}. \tag{6.27}$$

Denote by $L_i(x)$ the line parallel to the $i-th$ coordinate axis containing the point $x \in \mathbb{R}^N$, and by $\Phi_j|_{L_i(x)}$ the restriction of Φ_j to L_i:

$$L_i(x) := \{x + te_i \,:\, t \in \mathbb{R}\},$$
$$\Phi_j|_{L_i(x)} : L_i(x) \cap \Omega \to \mathbb{R}, \quad t \mapsto \Phi_j(x + te_i),$$

and introduce for each $i \in \{1, \ldots, N\}$ the sets

$$\begin{aligned}
\mathcal{A}_i &:= \{x \in \Omega \,:\, \Phi_1|_{L_i(x)}, \Phi_2|_{L_i(x)} \text{ are not absolutely continuous }\},\\
\mathcal{Z}_i &:= \{x \in \Omega \,:\, L_i(x) \cap \Phi_1^{-1}(\{0\}) \neq \emptyset \text{ or } L_i(x) \cap \Phi_2^{-1}(\{0\}) \neq \emptyset\},\\
\mathcal{P}_i &:= \{x \in \Omega \,:\, \mathcal{H}^1(L_i(x) \cap \mathcal{P}) > 0\},\\
E_i &:= \mathcal{Z}_i \cup \mathcal{P}_i \cup \mathcal{A}_i.
\end{aligned}$$

By exploiting the property that $\mathrm{Cap}_p(\Phi_j^{-1}(\{0\})) = 0$ derived in Proposition 5.3, we can show as in Lemma 3.2 of [22] that

$$\mathcal{H}^{N-1}(\pi_i(E_i)) = 0, \quad i = 1, \ldots, N,$$

where π_i denotes the projection

$$\pi_i : \mathbb{R}^N \to \mathbb{R}^N, \qquad (x_1, ..., x_N) \mapsto (x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_N).$$

In the sequel, by a "coordinate path" in Ω, we mean a continuous map $\gamma : [0, 1] \to \Omega$ whose image runs only in cartesian directions, i.e.

$$\begin{gathered}\gamma([0, 1]) = \Gamma_1 \cup ... \cup \Gamma_N,\\ \text{where } \Gamma_i \subset \Omega,\ \pi_i(\Gamma_i) \text{ is a finite set } (i = 1, \ldots, N).\end{gathered} \tag{6.28}$$

Then, fix a cube $Q \subset \Omega$. From Proposition 3.4 in [22] we may construct a set $\widetilde{Q} \subset Q$ such that $\mathcal{H}^N(Q \setminus \widetilde{Q}) = 0$ and any two points $x, y \in \widetilde{Q}$ can be joined by a coordinate path $\gamma : [0, 1] \to \widetilde{Q}$ of the form (6.28) with $\Gamma_i \cap E_i = \emptyset$. Hence, from the definition of E_i, we see that on each compact connected segment $I \subset\subset \Gamma_i$ $(i = 1, \ldots, N)$, we have

$$(\Phi_1 \, \partial_i \Phi - \Phi_2 \, \partial_i \Phi_1)(x) = 0, \quad \mathcal{H}^1 - \text{a.e. } x \in I,$$
$$\Phi_1(x), \Phi_2(x) > 0, \quad \forall x \in I.$$

Hence for $i = 1, \ldots, N$, we deduce $\partial_i(\log \Phi_1 - \log \Phi_2) = 0$ $\mathcal{H}^1$-a.e. in I. This shows that Φ_1/Φ_2 is constant on Γ_i for all $i \in \{1, \ldots, n\}$. Therefore $\Phi_1/\Phi_2 \equiv C_Q$ for some constant C_Q on Q. Since $Q \subset \Omega$ is arbitrary and Ω is connected, we get that Φ_1/Φ_2 is constant $\mathcal{H}^N$ a.e. in Ω □

6.3 The Case $q > p > 1$

When $q > p$, Proposition 6.4 still holds, and so we know that the minimizers of J do not change sign. But we cannot conclude as in the case $q = p$ that the minimizers are unique up to a constant factor, because Proposition 6.1 is not anymore true. Actually this uniqueness may fail when $q > p$. Consider for example the situation when

$$\Omega = \mathbb{R}^N (N \geq 3), \quad w \equiv 1, \quad p = 2, \quad q = 2^* = \frac{2N}{N-2}.$$

In such a case assumptions (H1)-(H4) are satisfied and the functional space $\mathcal{W}$ coincides with $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$. Then, it is well-known by the work of [30] that the minimum of the Rayleigh quotient $J(u) := \frac{\|\nabla u\|_2}{\|u\|_{2^*}}$ is achieved by a family of functions parametrized by $(a, \mu) \in \mathbb{R}^N \times (0, \infty)$, explicitly given by:

$$\Phi_{a,\mu} = \left(\frac{2\mu}{1 + \mu^2 |x - a|^2} \right)^{\frac{N-2}{2}}. \tag{6.29}$$

On bounded domain, uniqueness is not either expected when qp. Indeed, in [17], [25] it has been shown that when the domain is a ball or an annulus, the minimizers of (1.8) are in general non-radial and so they cannot be multiple of each others. The proof rests on the observation that the second variation of J in a supposedly radial minimizer can become negative in direction of non-radial admissible functions.

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