Newton's Aerodynamic Problem in the Presence of Friction

Dirk Horstmann, Bernd Kawohl
& Piero Villaggio

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Abstract: *This paper delivers an extension to a celebrated solution of Newton on the shape of the profile of an axisymmetric rigid body which, under prescribed length and caliber, has the least aerodynamic resistance. Newton assumed that the drag exerted by the surrounding gas on the body is exclusively due to the smooth impact of small gaseous particles against its surface. However, a more realistic description of the phenomenon must account for the friction between particles and body. This problem can be solved explicitly by essentially repeating Newton's argument and its subsequent improvements. The result is that the presence of friction has a noticeable effect both on the shape and resistance of the optimal body, even when the friction factor is small.*

1. Introduction.

In section VII of the second book of his *Principia* [7], Newton formulated and solved the problem of optimizing the shape of a solid of revolution, which, moving in a fluid along its axis, undergoes the least resistance, at parity of length and caliber. The solution obviously depends on how resistance is defined, and Newton assumed the fluid to be constituted by independent small particles hitting against the surface of the body and then reflected by this according to the law of elastic impact. As it is known, this hypothesis is not acceptable for projectiles travelling in the ordinary air, at relatively low altitude, but proved itself to be applicable some 250 years later to the flight of artificial satellites in rarefied atmosphere, as explained in Miele's book [6]. Its physical soundness notwithstanding, Newton's solution constitutes a landmark for the subsequent devlopment of the Calculus of Variations for having proposed in advance a weak solution to the problem because he found that the optimum profile is not necessarily a smooth surface, but is made by the combination of a flat circular front of radius r_0 and a convex surface of revolution connecting the boundary of the frontal circle with that of the rear, which has radius R (see Fig. 1). This quantity R, called the "caliber", and the "length" M are prescribed, whereas r_0 must be determined and so must the arc of meridian $\widehat{GD}$.

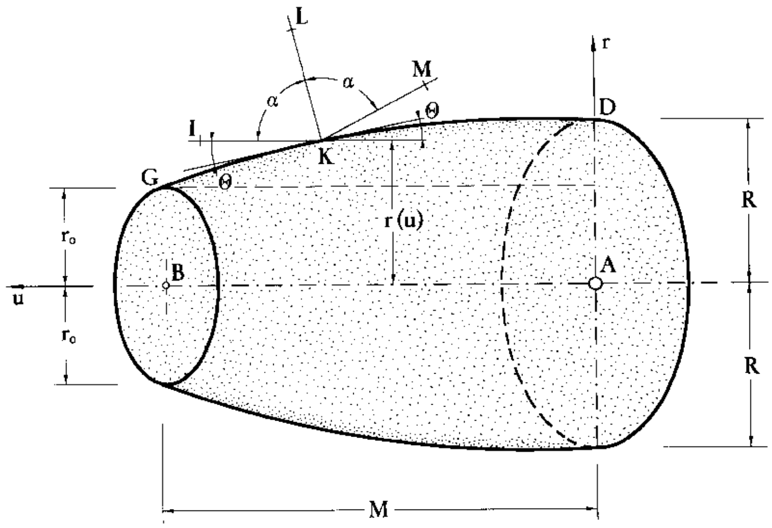

Figure 1: Illustration for Newton's body of minimal resistance

In Newton's problem the body with meridian B, G, D, A advances with velocity c_0 in a medium constituted by infinite small frozen drops which, after having hit the surface, rebound from it satisfying the classical law of reflection, according to which the angles $I\hat{K}L$ and $L\hat{K}M$ have the same value, say α. If we denote the slope of tangent to the curve $\overset{\frown}{GD}$ at a generic point K by Θ, we see from Fig. 1 that $\alpha = \pi/2 - \Theta$, because the straight line LK is normal to the meridian at K. The normal pressure exerted by the steady impact of particles on the surface is $\frac{1}{2}\varrho\left(c_0 \sin\Theta\right)^2$, where ϱ is the density of the medium. It follows that the horizontal component of this pressure is $\frac{1}{2}\varrho\left(c_0 \sin\Theta\right)^2 \sin\Theta$ and the total drag experienced by the body during its motion is

$$R_u = \int\int_S \frac{1}{2}\varrho c_0^2 \sin^3\Theta ds \quad , \tag{1.1}$$

where S is the surface of revolution formed by rotating the curve B, G, D about the longitudinal u-axis. If we take the origin of this axis at A and orient it from right to left, we see that our solid is contained between the two planes $u = 0$ and $u = M$ of Fig. 1. In addition, if we call $r(u)$ the radius of the generic cross-section passing through K, we can write

$$\sin\Theta = \frac{\tan\Theta}{\sqrt{1+\tan^2\Theta}} = \frac{r'(u)}{\sqrt{1+r'^2}} \tag{1.2}$$

in (1.1), and hence, recalling that in a surface of revolution $ds = 2\pi r(u)\sqrt{1+r'^2}du$, give (1.1) the form

$$R_u = \frac{1}{2}\varrho c_0^2 2\pi \int_0^M \frac{r(u)r'(u)}{(1+r'^2)} du \quad . \tag{1.3}$$

Expression (1.3) can be further simplified if we restrict the search for a minimum of R_u, subject to the constraint $r(u) \leq R$, to the class of strictly decreasing functions, a choice suggested by physical and mathematical arguments, as Buttazzo and Kawohl have explained in an expository paper [3]. In this case indeed we can introduce function $u = u(r)$ and transform (1.3) into the functional

$$R_u = \frac{1}{2}\varrho c_0^2 2\pi \int_0^R \frac{r}{(1+u'^2(r))} dr \quad . \tag{1.4}$$

Minimization of (1.4) by modern methods of Calculus of Variation (see Buttazzo, Ferone, Kawohl [2]) is just to look for a solution to Newton's problem.

Let us now assume that the impact of the solid particles against the surface S of our body is not perfectly smooth so that the normal pressure $\frac{1}{2}\varrho(c_0 \sin\Theta)^2$ considered above is associated to a tangential pressure $\frac{\mu_0}{2}\varrho(c_0 \sin\Theta)^2$, where μ_0 $(0 \leq \mu_0 \leq 1)$ is the friction factor. Then its horizontal component is $\frac{\mu_0}{2}(c_0 \sin\Theta)^2 \cos\Theta$ and the total contrasting force is

$$S_u = \int\int_S \frac{\mu_0}{2}\varrho c_0^2 \sin^2\Theta \cos\Theta dS \quad , \tag{1.5}$$

and this, by repeating the change of variable as before, can be reduced to the form

$$S_u = -\frac{1}{2}\varrho c_0^2 2\pi \int_0^R \frac{r\mu_0 u'(r)}{(1+u'^2(r))} dr \quad . \tag{1.6}$$

Thus the resulting drag is given by the sum $R_u + S_u$, and hence, by dropping the unessential constants, we are reduced to minimize the functional

$$F(u) = \int_0^R \frac{(1-\mu_0 u'(r))r}{(1+u'^2(r))} dr \tag{1.7}$$

in the class of strictly decreasing functions satisfying the constraint $0 \leq u \leq M$. The influence of friction on the resistance exerted on a body moving in a Newton's modelled fluid has been already studied by other authors like [1] and [4], but with models for the effect of friction which are not compatible with Newton's corpuscle theory.

2. Existence and properties of minimizers of (1.7).

Before studying the properties of minimizers of $F(u)$ it is first necessary to establish their existence. We follow essentially the same reasoning here that was recently introduced in the treatment of Newton's original problem, that is when μ_0 vanishes (*cf.* Buttazzo, Ferone, Kawohl [2]). Let draw the graph of the function

$$f(s) = \frac{1+\mu_0 s}{1+s^2} \qquad (0 \leq s < \infty) \quad , \tag{2.1}$$

which is also called the "indicatrix". Since we know that $0 \le \mu_0 \le 1$, the qualitative form of this curve is represented in Fig. 2. Notice that f is neither convex nor coercive.

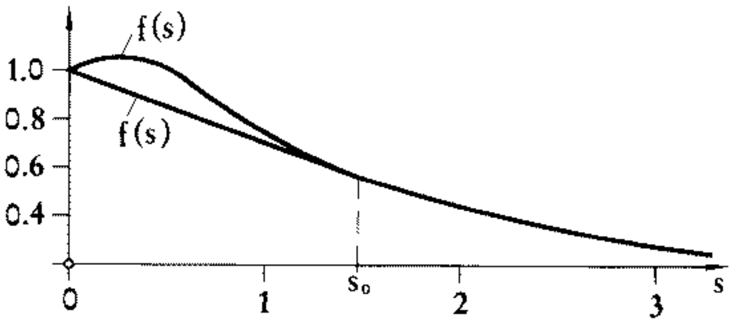

Figure 2: The graphs of f and $\tilde{f}$

The infimum of F will be the minimum of a relaxed variant $\tilde{F}$ of F, which we define as follows. Then let us trace from the point $O' = (0,1)$ the tangent $0'S$ to the graph of $f(s)$, which is the straight line of equation $1 - bs$, satisfying the conditions

$$\begin{array}{rclcl} f(s_0) & = & 1 - bs_0 & = & \frac{1+\mu_0 s_0}{1+s_0^2} \quad , \\ f'(s_0) & = & -b & = & \frac{\mu_0}{1+s_0^2} - \frac{2(1+\mu_0 s_0)s_0}{(1+s_0^2)^2} \quad . \end{array}$$

An easy calculation yields the values

$$s_0 = \mu_0 + \sqrt{1+\mu_0^2} \ge 1 \quad , \quad f(s_0) = \frac{1}{2} \quad . \tag{2.2}$$

Now, if we define the convex function

$$\tilde{f}(s) = \begin{cases} 1 - bs & \text{for } 0 \le s \le s_0 \quad , \\ \frac{1+\mu_0 s}{1+s^2} & \text{for } s > s_0 \quad , \end{cases} \tag{2.3}$$

and the functional

$$\tilde{F}(u) = \int_0^R r\tilde{f}(|u'|)dr \quad , \tag{2.4}$$

we easily see that the relation $\tilde{F}(u) \le F(u)$ holds, and hence that the minimum of $\tilde{F}(u)$ is less or equal to that of $F(u)$. Moreover $\tilde{f}$ is decreasing in s, so that by a rearrangement argument as in [2] any minimizer of $\tilde{F}$ in the class of radially decreasing functions must be concave. Finally, $\tilde{f}$ is convex in s for $s \ge 0$, and so $\tilde{F}(u)$ has a minimizer and not only an infimum. Therefore it suffices to look for minimizers of F in the class of functions $u(r)$, which obey the constraint $0 \le u \le M$ and are concave for $0 \le r \le R$. Just as in [2] one can show that a minimizing sequence u_n in this class of admissible functions converges to a minimizer of F, so that the existence of a minimizer as well as its concavity is well established. Let us remark in passing, that for $\mu_0 > 1$ the function $\tilde{f}$ is no longer concave is s, in which case an existence result would be a lot harder to prove. But according to the classical theory of friction μ_0 cannot be higher than 1, anyway.

Let us now study further properies of the minimizing profile u. As in [2] we can show that there is a well defined positive number $s_0 \geq 1$ such that $|u'(r)| \notin (0, s_0)$ for almost every $r \in [0, R)$. In other words, the slope of the curve B, G, D with respect to the vertical r-axis is either zero, like along BG of Fig. 1, or greater or equal to s_0, like in the arc $\overset{\frown}{GD}$ of the same figure. To see this, one must note that a minimizer of F satisfies $F(u) = \tilde{F}(u)$. Again this can be done as in [2].

A minimizer of $\tilde{F}$ can also be obtained by considering the associated Euler equation. One should note, however, that the derivation of the Euler equation is only justified in those points r, where $u(r) \in (0, M)$ and where u is strictly concave. But there the Euler equation reads

$$\left(r\, \frac{\mu_0 u'(r)^2 + 2u'(r) - \mu_0}{(1 + u'(r)^2)^2} \right)' = 0. \tag{2.5}$$

Hence the minimizer must have necessarily the form

$$u(r) = \begin{cases} M & \text{for } 0 \leq r \leq r_0 \ , \qquad (2.6)_\mathrm{a} \\ u(r) \ , \text{ solution of the equation} & \\ r\,\frac{\mu_0 u'^2 + 2u' - \mu_0}{(1+u'^2)^2} = c \ , & \text{for } r_0 \leq r \leq R \ . \qquad (2.6)_\mathrm{b} \end{cases}$$

where c is a positive constant and r_0 can be explicitly evaluated.

We thus conclude that, even in presence of friction, the optimum profile, *i.e.* the solution of Newton's modified problem, has a flat front of diameter r_0. Thus it resembles the curve B, G, D of Fig. 1. Let us summarize these results in a Theorem.

Theorem: *The functional F attains its minimum on the set of radially decreasing functions with values between 0 and M. Moreover a minimizer u must be concave, and the modulus of its gradient jumps from 0 to s_0 along the boundary of its blunt part and u satisfies* $(2.6)_b$.

3. Parametric representation for the meridian

In this paragraph we try to calculate the solution u of (2.5) explicitly. To this end we choose $t := -u'(r)$ with $t \in [s_0, T]$ as independent variable. Under slight abuse of notation we now think of $(r, u(r))$ as $(r(t), u(t))$, and then the boundary conditon $u(r_0) = M$ becomes

$$r(s_0) = r_0, \qquad u(s_0) = M, \tag{3.1}$$

while $u(R)$ turns into

$$r(T) = R, \qquad u(T) = 0. \tag{3.2}$$

Finally, denoting differentiation with respect to t by a dot, the initial condition $u'(r_0) = s_0$ becomes

$$\frac{\dot{u}(s_0)}{\dot{r}(s_0)} = -s_0. \tag{3.3}$$

Let us recall that M, R and $s_0 = \mu_0 + \sqrt{1+\mu_0^2}$ are given and that r_0, $T(> s_0)$ and $(r(t), u(t)) : [s_0, T] \mapsto [r_0, R] \times [0, M]$ are sought. A first integration of the Euler equation (2.5) gives

$$r(t) = \frac{C(1+t^2)^2}{\mu_0 t^2 + 2t - \mu_0}, \tag{3.4}$$

with a suitable, but at this stage unknown constant C. So in principle we already have a representation of $r(t)$. Also the function $u(t)$ can in principle be obtained, namely from the identity $\dot{u}(t) = -t\dot{r}(t)$, with $r(t)$ given by (3.4). Integration by parts gives

$$u(t) = -\int_{s_0}^{t} s\,\dot{r}(s)\,ds = \int_{s_0}^{t} r(s)\,ds - t\,r(t) + s_0 r_0 + M. \tag{3.5}$$

Calculating this integral and adjusting all the constants to the boundary conditions turns out to be tricky. Equation (3.1) has already fixed the integration constants C and $s_0 r_0 + M$ in (3.4) and (3.5). The remaining boundary conditions should (and will) suffice to determine T and r_0, resp. T and C. The details, however, are far from trivial.

Let us focus our attention on the integral $u(t) := \int_{s_0}^{t} r(s)\,ds$. We can split (3.4) into partial fractions

$$\frac{\mu_0 r(t)}{C} = \frac{t^3 + 2t + \frac{1+\sqrt{1+\mu_0^2}}{\mu_0}}{t + \frac{1+\sqrt{1+\mu_0^2}}{\mu_0}} + \frac{\frac{\sqrt{1+\mu_0^2}-1}{\mu_0}t^2 - t + \frac{2\sqrt{1+\mu_0^2}-2}{\mu_0}}{t + \frac{1-\sqrt{1+\mu_0^2}}{\mu_0}}, \tag{3.6}$$

and after a division of these polynomials in t, we arrive at

$$\begin{aligned} \frac{\mu_0 r(t)}{C} = t^2 - \frac{2}{\mu_0}t + &\left[\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^2 + \left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^2 + 1\right] \\ + \frac{\frac{\sqrt{1+\mu_0^2}-1}{\mu_0} - \left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^3}{t + \frac{1-\sqrt{1+\mu_0^2}}{\mu_0}} &- \frac{\frac{\sqrt{1+\mu_0^2}+1}{\mu_0} + \left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^3}{t + \frac{1+\sqrt{1+\mu_0^2}}{\mu_0}}. \end{aligned} \tag{3.7}$$

Equation (3.7) can now be integrated term by term (*cf.* [5]), and we obtain

$$u(t) = -t\frac{C(1+t^2)^2}{(t^2-1)\mu_0 + 2t} + \frac{C}{3\mu_0}t^3 - \frac{C}{\mu_0^2}t^2$$

$$
\begin{aligned}
&+\frac{C}{\mu_0}t\left[\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^2+\left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^2+1\right]\\
&+\frac{C}{\mu_0}\left[\frac{\sqrt{1+\mu_0^2}-1}{\mu_0}-\left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^3\right]\ln\left(t+\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)\\
&-\frac{C}{\mu_0}\left[\frac{\sqrt{1+\mu_0^2}+1}{\mu_0}+\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^3\right]\ln\left(t+\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)\\
&+K(C)\ , \qquad (3.8)
\end{aligned}
$$

with

$$
\begin{aligned}
K(C)=M&+s_0\frac{C(1+s_0^2)^2}{(s_0^2-1)\mu_0+2s_0}-\frac{C}{3\mu_0}\,s_0^3+\frac{C}{\mu_0^2}\,s_0^2\\
&-\frac{C}{\mu_0}s_0\left[\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^2+\left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^2+1\right]\\
&-\frac{C}{\mu_0}\left[\frac{\sqrt{1+\mu_0^2}-1}{\mu_0}-\left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)^3\right]\ln\left(s_0+\frac{1-\sqrt{1+\mu_0^2}}{\mu_0}\right)\\
&+\frac{C}{\mu_0}\left[\frac{\sqrt{1+\mu_0^2}+1}{\mu_0}+\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right)^3\right]\ln\left(s_0+\frac{1+\sqrt{1+\mu_0^2}}{\mu_0}\right) \qquad (3.9)
\end{aligned}
$$

and $s_0=\mu_0+\sqrt{1+\mu_0^2}$. Recalling (3.4), we see that

$$
r_0=\frac{C(1+s_0^2)^2}{\mu_0 s_0^2+2s_0-\mu_0} \qquad (3.10)
$$

is given in terms of C and μ_0. Let us now try to exploit (3.2). The condition $r(T)=R$ becomes

$$
R=\frac{C(1+T^2)^2}{\mu_0 T^2+2T-\mu_0},
$$

or

$$
C=C(T,\mu_0)=R\frac{\mu_0 T^2+2T-\mu_0}{(1+T^2)^2}, \qquad (3.11)
$$

while $u(T)=0$ can be written down as a complicated function of T, using (3.8), (3.9) and (3.11). At this point, the explicit calculation becomes hard to track. Since $u(t)$ is monotone decreasing, for given M, R and μ_0 there is a unique T such that $u(T)=0$. Once T is known, then so is C by (3.11), r_0 by (3.10), $r(t)$ by (3.4), $K(C)$ by (3.9) and $u(t)$ by (3.8).

As a result of these substitutions we obtain

$$
\begin{aligned}
\frac{M}{R} &= \frac{u(s_0) - u(T)}{R} \\
&= \frac{\mu_0 T^2 + 2T - \mu_0}{(1+T^2)^2} \Bigg\{ \frac{T(1+T^2)^2}{(T^2-1)\mu_0 + 2T} - \frac{s_0(1+s_0^2)^2}{(s_0^2-1)\mu_0 + 2s_0} \\
&\quad - \frac{1}{3\mu_0}(T^3 - s_0^3) + \frac{1}{\mu_0^2}(T^2 - s_0^2) \\
&\quad - \frac{T - s_0}{\mu_0} \left[\left(\frac{1 + \sqrt{1+\mu_0^2}}{\mu_0} \right)^2 + \left(\frac{1 - \sqrt{1+\mu_0^2}}{\mu_0} \right)^2 + 1 \right] \\
&\quad - \frac{1}{\mu_0} \left[\frac{\sqrt{1+\mu_0^2} - 1}{\mu_0} - \left(\frac{1 - \sqrt{1+\mu_0^2}}{\mu_0} \right)^3 \right] \ln \left(\frac{\mu_0 T + 1 - \sqrt{1+\mu_0^2}}{\mu_0 s_0 + 1 - \sqrt{1+\mu_0^2}} \right) \\
&\quad + \frac{1}{\mu_0} \left[\frac{\sqrt{1+\mu_0^2} + 1}{\mu_0} + \left(\frac{1 + \sqrt{1+\mu_0^2}}{\mu_0} \right)^3 \right] \ln \left(\frac{\mu_0 T + 1 + \sqrt{1+\mu_0^2}}{\mu_0 s_0 + 1 + \sqrt{1+\mu_0^2}} \right) \Bigg\}.
\end{aligned}
\tag{3.12}
$$

Let us observe that this formula, despite its appearance, maintains its meaning even in the limiting case $\mu_0 \to 0$. In order to see this, we assume that in (3.12) the factor μ_0 is small, and we expand the terms of (3.12) in powers of μ_0. Then we take $\mu_0 = 0$. At the end of this tedious but elementary computation we find, that in the limit $\mu_0 \to 0$ the ratio M/R attains the value

$$\frac{M}{R} = \frac{T}{(1+T^2)^2} \left(\frac{3}{4}T^4 + T^2 - \ln T - \frac{7}{4} \right), \tag{3.13}$$

which is exactly the result obtained by Francesca Nicolai [8] in her thesis on the inviscid Newton problem, in which $\mu_0 = 0$ and $s_0 = 1$. Another consequence of (3.12) is that M vanishes for $T = s_0$. Hence, when M is very small, T is close to s_0.

Moreover $F(u)$ can be calculated using integration by parts as follows

$$
\begin{aligned}
F(u) = \tilde{F}(u) &= 2\pi \int_0^{r_0} \rho \, d\rho + 2\pi \int_{s_0}^{T} \frac{1 + \mu_0 s}{1 + s^2} \, r(s) \, \dot{r}(s) \, ds \\
&= \pi \, r_0^2 + \frac{\pi(1 + \mu_0 T)}{1 + T^2} r^2(T) - \frac{\pi(1 + \mu_0 s_0)}{1 + s_0^2} r_0^2 + C\pi \int_{s_0}^{T} r(s) ds,
\end{aligned}
\tag{3.14}
$$

with r_0 and C determined by (3.10) and (3.11). Often, instead of $F(u)$, the ratio $C_0 = F(u)/\pi R^2$, called the "relative resistance", is considered to be a more expressive measure of the drag. Since the analytic expansion of these quantities is very involved, we calculate T and $C_0 = F(u)/\pi R^2$ numerically (using MAPLE) for $R = M = 1$ and for $\mu_0 = i/10$ with $i = 1, 2, \dots, 10$, and record the results in the following table.

μ_0	T	r_0	s_0	C_0
0	1.9168012	.35094257	1	.37481598
.1	1.9937494	.38260080	1.1049875	.41312216
.2	2.0797780	.41501996	1.2198039	.44957286
.3	2.1752978	.44754815	1.3440306	.48399428
.4	2.2804711	.47954058	1.4770329	.51625216
.5	2.3951427	.51044929	1.6180339	.54627479
.6	2.5188835	.53986088	1.7661903	.57405470
.7	2.6510621	.56750591	1.9206555	.59964100
.8	2.7909253	.59324585	2.0806248	.62312766
.9	2.9376709	.61704727	2.2453624	.64463970
1	3.0905052	.63895269	2.4142136	.66432025

The numerical results of the table describe the influence of friction on the shape and resistance of a moving body once Newton's corpuscle theory is accepted. As μ_0, the friction factor, increases from zero to one, the radius r_0 of the flat part of the meridian becomes larger and larger. The optimal profiles for various values of μ_0 are sketched in Figure 3.

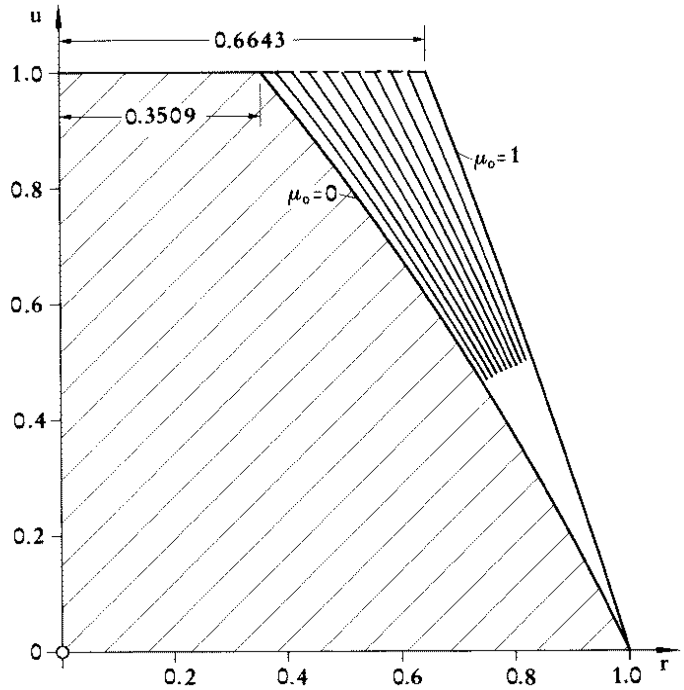

Figure 3: The optimal profiles for $\mu_0 = 0$ (left) to $\mu_0 = 1$ (right).

In mechanical terms, this means that under high friction it is more convenient to expand the flat part of the front, where the impact is direct, and to reduce the lateral, curved mantle, where friction is active. As far as this specific resistance C_0 is concerned, the table shows that it increases from a minimum of $C_0 = 0.3748$ for $\mu_0 = 0$ to a maximum of 0.6644 for $\mu_0 = 1$. This is an increase of 77%.

The graphs of Figure 3 show that the optimal u increases continuously with μ_0.

4. Asymptotic estimates.

We have so far examined the influence of different values of the factor of friction on the extent of the flat front for an optimal profile under the condition that M, the length of the projectile, and R, its caliber, are equal. There is, conversely, the complementary question of investigating the dependence of r_0, the radius of the plane region, for every value of the friction μ_0, but in the limiting cases in which M/R tends to infinity or zero. In the first case, the profile is called "slender"; in the second "bluff". The complete analysis of these two situations in absence of friction ($\mu_0 = 0$) has been carried on by Francesca Nicolai [8].

Let us first consider the "slender" case, for which $M/R \to \infty$. Then the body is very long and thin. Since u is concave $|u'(r)| > M/R$ on the curve meridian. Therefore $T = \max|u'(r)| > M/R >> 1$ for slender bodies. In other words, the slope of the profile at D (see Figure 1), denoted before by T, is very large. Thus the ratio r_0/R given by (3.10) and (3.11), namely

$$\frac{r_0}{R} = \frac{(1+s_0^2)^2}{\mu_0 s_0^2 + 2s_0 - \mu_0} \; \frac{\mu_0 T^2 + 2T - \mu_0}{(1+T^2)^2}, \tag{4.1}$$

tends to zero as $M/R \to \infty$.

Hence, instead of (4.1) we can study the ratio

$$\frac{r_0}{R} = \frac{(1+s_0^2)^2}{\mu_0 s_0^2 + 2s_0 - \mu_0} \; \frac{\mu_0 T + 2}{T^3}, \tag{4.2}$$

where s_0 still has the value $\mu_0 + \sqrt{1+\mu_0^2}$. Now we look once more at C_0. According to (3.8), (3.11) and (3.14) we see that

$$
\begin{aligned}
C_0 = & \frac{(\mu_0 T^2 + 2T - \mu_0)^2 (1+s_0^2)^4}{(1+T^2)^4 (\mu_0 s_0^2 + 2s_0 - \mu_0)^2} + \frac{(\mu_0 T^2 + 2T - \mu_0)^2 (1+T^2)^4 (1+\mu_0 T)}{(1+T^2)^5 (\mu_0 T^2 + 2T - \mu_0)^2} \\
& - \frac{(\mu_0 T^2 + 2T - \mu_0)^2 (1+s_0^2)^4 (1+\mu_0 s_0)}{(1+s_0^2)(1+T^2)^4 (\mu_0 s_0^2 + 2s_0 - \mu_0)} \\
& + \frac{(\mu_0 T^2 + 2T - \mu_0)^2}{(1+T^2)^4} \left(\left[-\frac{t(1+t^2)^2}{\mu_0 t^2 + 2t - \mu_0} + \frac{t^3}{3\mu_0} - \frac{t^2}{\mu_0^2} \right.\right. \\
& + \frac{t}{\mu_0} \left[\left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0} \right)^2 + \left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0} \right)^2 + 1 \right] \\
& + \frac{1}{\mu_0} \left[\frac{\sqrt{1+\mu_0^2} - 1}{\mu_0} - \left(\frac{1-\sqrt{1+\mu_0^2}}{\mu_0} \right)^3 \right] \ln \left(t + \frac{1-\sqrt{1+\mu_0^2}}{\mu_0} \right) \\
& \left.\left. - \frac{1}{\mu_0} \left[\frac{\sqrt{1+\mu_0^2} + 1}{\mu_0} + \left(\frac{1+\sqrt{1+\mu_0^2}}{\mu_0} \right)^3 \right] \ln \left(t + \frac{1+\sqrt{1+\mu_0^2}}{\mu_0} \right) \right] \Bigg|_{s_0}^{T} \right).
\end{aligned}
$$

If we now assume that T is very large and that μ_0 is fixed and close to zero, such that $T\mu_0 \leq 1$, we can conclude that

$$
C_0 \sim \frac{(\mu_0 T + 2)^2}{T^6} \left[\frac{T^4 (1+\mu_0 T)}{(\mu_0 T + 2)^2} + \frac{t^3}{3\mu_0} - \frac{t^2}{\mu_0^2} + \frac{4t}{\mu_0^3} - \frac{8}{\mu_0^4} \ln \left(\frac{\mu_0 t}{2} + 1 \right) \right] \Bigg|_{s_0}^{T} \tag{4.3}
$$

A power series expansion of the logarithm gives us

$$
\begin{aligned}
C_0 \sim & \frac{(\mu_0 T + 2)^2}{T^6} \left[\frac{T^4 (1+\mu_0 T)}{(\mu_0 T + 2)^2} + \frac{T^3}{3\mu_0} - \frac{T^2}{\mu_0^2} + \frac{4T}{\mu_0^3} \right. \\
& - \frac{8}{\mu_0^4} \left(\frac{\mu_0 T}{2} - \frac{\mu_0^2 T^2}{8} + \frac{\mu_0^3 T^3}{24} - \frac{\mu_0^4 T^4}{64} \right) - \frac{s_0^3}{3\mu_0} + \frac{s_0^2}{\mu_0^2} - \frac{4s_0}{\mu_0^3} \\
& \left. + \frac{8}{\mu_0^4} \left(\frac{\mu_0 s_0}{2} - \frac{\mu_0^2 s_0^2}{8} + \frac{\mu_0^3 s_0^3}{24} - \frac{\mu_0^4 s_0^4}{64} \right) \right] + O(\mu_0)
\end{aligned} \tag{4.4}
$$

where $O(\mu_0) \to 0$ as $\mu_0 \to 0$. It is worth noticing that for large T the formula maintains its meaning also in the limiting case $\mu_0 = 0$, $s_0 = 1$, in which it becomes

$$
C_0 \sim \frac{3}{2T^2}. \tag{4.5}
$$

This corresponds to the results of F. Nicolai in [8] for $\mu_0 = 0$ and $s_0 = 1$ as $T \to \infty$. In order to have an estimate of how C_0 changes with μ_0, let us take $T = 10$ in (4.3) and calculate C_0 for $\mu_0 = i/10$, where $i = 1, 2, ..., 10$. For $\mu_0 = 0$ we use the corresponding formula derived by F. Nicolai in [8, p. 81]. The numerical result which is recorded in the table below shows that, when $\mu_0 = 1$, we have an increase of the specific resistance of about 934% with respect to its value for $\mu_0 = 0$.

μ_0	C_0 (according to (4.3))	C_0 (according to (4.4))
0	.014907	.014999
.1	.028063	.031248
.2	.041210	.049995
.3	.054403	.071238
.4	.067623	.094979
.5	.080857	.121208
.6	.094098	.149922
.7	.107337	.181112
.8	.120569	.214765
.9	.133786	.250865
1	.154089	.289389

In the alternative case in which $M/R \to 0$ we have the "bluff" case, and thus we may set $T \sim s_0$ and obtain from (4.1) the asymptotic expression

$$\frac{r_0}{R} \sim 1 \quad . \tag{4.6}$$

This means that, independently of the value of the friction factor, the body tends to become a frustum cone of small height relative to its diameter. The result is not surprising at all, because, in this extreme situation, the impact is absorbed exclusively by the plane circular front, where friction does not contribute to the drag, the impact being here purely normal to the surface.

References

[1] Bryson, A., Ho, Y.C.: *Applied Optimal Control.* Lexington: Ginn (1969).

[2] Buttazzo, G., Ferone, V., Kawohl, B.: *Minimum problems over sets of concave functions and related questions.* Math. Nachrichten **173** (1995), pp. 71–89.

[3] Buttazzo, G., Kawohl, B.: *On Newton's problem of minimal resistance.* Math. Intelligencer. **15** (1993), pp. 7–12.

[4] Ferone, V., Kawohl, B.: *Bodies of Minimal Resistance under Prescribed Surface Area.* Zeitschr. Angew. Math. Mech. **79** (1999), pp. 277–280

[5] Gröbner, W., Hofreiter, N.: *Integraltafel.* Wien: Springer 1965.

[6] Miele, A.: *Theory of Aerodynamic Shapes.* New York: Academic Press 1965.

[7] Newton, I.: *Philosophiae Naturalis Principia Mathematica.* Londini 1687.

[8] Nicolai, F.: *Un Modello Variazionale per l'Ottimazione di Profili Aerodinamici.* Thesis at the University of Pisa 1993.

Author's addresses:
Bernd Kawohl, Mathematisches Institut, Universität zu Köln, D-50923 Köln, Germany; e-mail: kawohl@mi.uni-koeln.de
Dirk Horstmann, Mathematisches Institut, Universität zu Köln, D-50923 Köln, Germany; e-mail: dhorst@mi.uni-koeln.de
Piero Villaggio, Dipartimento di Ingegneria Strutturale, Via Diotisalvi, 2, I-56126 Pisa, Italy