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ON THE EVOLUTION GOVERNED BY THE INFINITY LAPLACIAN

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ABSTRACT. We investigate the basic properties of the degenerate and singular evolution equation

$$u_t = \left(D^2u\frac{Du}{|Du|}\right)\cdot\frac{Du}{|Du|},$$

which is a parabolic version of the increasingly popular infinity Laplace equation. We prove existence and uniqueness results for both Dirichlet and Cauchy problems, establish interior and boundary Lipschitz estimates and a Harnack inequality, and also provide numerous explicit solutions.

1. INTRODUCTION

In this paper, we consider the non-linear, singular and highly degenerate parabolic equation

$$u_t = \Delta_\infty u, \tag{1.1}$$

where

$$\Delta_\infty u := \left(D^2u\frac{Du}{|Du|}\right)\cdot\frac{Du}{|Du|} \tag{1.2}$$

denotes the 1-homogeneous version of the very popular infinity Laplace operator. Our goal is to establish basic results concerning existence, uniqueness and regularity of the solutions, and convince the reader that the equation is of significant mathematical interest.

The original motivation to study (1.1) stems from the usefulness of the infinity Laplace operator in certain applications. Indeed, the geometric interpretation of the viscosity solutions of the equation $-\Delta_\infty u = 0$ as absolutely minimizing Lipschitz extensions, see [1], [3], has attracted considerable interest in image processing, the main usage being in the reconstruction of damaged digital images. See e.g. [5], [29]. This so-called AMLE model has attractive properties of invariance, stability and regularity, and also has the advantage that points have positive capacity. Another related area in which (1.2) has been used is the study of shape metamorphism, see [7] and in mass transfer problems, see [15]. For numerical purposes it has been necessary to consider also the evolution equation corresponding to the infinity Laplace operator; here the main focus has been in the asymptotic behavior of the solutions of this parabolic problem with time-independent data, cf. [5], [32].

However, we claim that (1.1) also has a very interesting theory if viewed by itself and not just as an auxiliary equation connected to the infinity Laplacian. First, it is a parabolic equation with principal part in non-divergence form that, unlike for example the mean curvature evolution equation, does not belong to the class of "geometric" equations (see [6] for the definition). Thus many of the techniques used in [6], [16] are not directly applicable. Nevertheless it is used in such diverse applications as evolutionary image processing, [7] and differential games [4]. To be

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precise (1.1) arises from the fast repeated averaging of the "“forward and backwards” Hamilton-Jacobi dynamics $v_t + |Dv| = 0$ and $w_t - |Dw| = 0$. Moreover, a time dependent version of the tug-of-war game of Peres, Schramm, Sheffield and Wilson [28] leads to the backward-in-time version of (1.1), see [4]. Secondly, in the case of a one space variable, the equation (1.1) reduces to the one dimensional heat equation, see Remark 2.2 below, and, rather surprisingly, there is a connection between these two seemingly very different equations also in higher dimensions. Roughly speaking, the fact that the infinity Laplacian (1.2) is non-degenerate only in the direction of the gradient Du (and acts like the one dimensional Laplacian in that direction) causes (1.1) to behave as the one dimensional heat equation on two dimensional surfaces whose intersection with any fixed time level $t = t_0$ is an integral curve of the vector field generated by $Du(\cdot, t_0)$. This heuristic idea comes up for example in the computation of explicit solutions and in some of the proofs.

The results of this paper can be summarized as follows. We begin with a standard comparison principle in bounded domains that implies uniqueness for the Dirichlet problem. The existence of viscosity solutions with continuous boundary and initial data is established with the aid of the approximating equations

$$u_t = \varepsilon \Delta u + \frac{1}{|Du|^2 + \delta^2} \left(D^2 u \, Du \right) \cdot Du$$

and uniform continuity estimates that are derived by using suitable barriers. The Cauchy problem associated to (1.1) is also treated but only very briefly. As regards regularity, we prove interior and boundary Lipschitz estimates and obtain a Harnack inequality for the non-negative solutions of (1.1). Finally, following the work of Crandall et al. [10], [11], we show that subsolutions can be characterized by means of a comparison principle involving a two parameter family of explicit solutions of (1.1).

Although some of the results described above appear to be known to the experts of the field, see e.g. [5] and its references (without detailed proofs and with a different definition of viscosity solution), we feel that it is worthwhile to write down the proofs of our results in a self-contained and rather elaborate way. Moreover, since the formation of the theory is still in its early stages, explicit examples are important and we provide a good number of them. Note also that due to the singularity of the equation, the very definition of a solution is a non-trivial issue that needs to be discussed.

In addition to Caselles, Morel and Sbert [5], the infinity heat equation (1.1) has been studied at least by Wu [32], who obtained a variety of interesting results closely related to ours. Another parabolic version of the infinity Laplace equation

$$u_t = \left(D^2 u \, Du \right) \cdot Du$$

was investigated by Crandall and Wang in [10], but we prefer (1.1) over this one because of the closer relationship with the ordinary heat equation and the more favorable homogeneity. In fact, even Crandall and Wang find their version inconsistent with the “comparison by cones” property and take this as an indication [10, p.654] that it might not be the “right” parabolic version of the infinity-Laplace equation. Observe that the solution to our equation is amenable to comparison by cones, and that the classes of time-independent solutions of both of these equations coincide with the infinity harmonic functions, see Corollary 3.3 below.

Contents

2. Definitions and examples

Due to the singularity, degeneracy and the non-divergence form of (1.1), we are not able to use classical or distributional weak solutions as our notion of a solution. However, there is a by now standard way to define viscosity solutions for singular parabolic equations having a bounded discontinuity at the points where the gradient vanishes. We recall this definition below, and refer the reader to [16], [6] and [17] for its justification and the basic properties such as stability etc.

For a symmetric $n \times n$-matrix A, we denote its largest and smallest eigenvalue by $\Lambda(A)$ and $\lambda(A)$, respectively. That is,

$$\Lambda(A) = \max_{|\eta|=1} (A\eta) \cdot \eta$$

and

$$\lambda(A) = \min_{|\eta|=1} (A\eta) \cdot \eta.$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution* of (1.1) in Ω if, whenever $(\hat{x}, \hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(1) $u(\hat{x}, \hat{t}) = \varphi(\hat{x}, \hat{t})$,
(2) $u(x, t) < \varphi(x, t)$ for all $(x, t) \in \Omega$, $(x, t) \neq (\hat{x}, \hat{t})$

then

$$\begin{cases} \varphi_t(\hat{x}, \hat{t}) \leq \Delta_\infty \varphi(\hat{x}, \hat{t}) & \text{if } D\varphi(\hat{x}, \hat{t}) \neq 0, \\ \varphi_t(\hat{x}, \hat{t}) \leq \Lambda(D^2\varphi(\hat{x}, \hat{t})) & \text{if } D\varphi(\hat{x}, \hat{t}) = 0. \end{cases} \tag{2.1}$$

A lower semicontinuous function $v : \Omega \to \mathbb{R}$ is a *viscosity supersolution* of (1.1) in Ω if $-v$ is a viscosity subsolution, that is, whenever $(\hat{x}, \hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(1) $v(\hat{x}, \hat{t}) = \varphi(\hat{x}, \hat{t})$,
(2) $v(x, t) > \varphi(x, t)$ for all $(x, t) \in \Omega$, $(x, t) \neq (\hat{x}, \hat{t})$

then

$$\begin{cases} \varphi_t(\hat{x}, \hat{t}) \geq \Delta_\infty \varphi(\hat{x}, \hat{t}) & \text{if } D\varphi(\hat{x}, \hat{t}) \neq 0, \\ \varphi_t(\hat{x}, \hat{t}) \geq \lambda(D^2\varphi(\hat{x}, \hat{t})) & \text{if } D\varphi(\hat{x}, \hat{t}) = 0. \end{cases} \tag{2.2}$$

Finally, a continuous function $h : \Omega \to \mathbb{R}$ is a *viscosity solution* of (1.1) in Ω if it is both a viscosity subsolution and a viscosity supersolution. In points where the spatial gradient of u vanishes, one can interpret the differential equation as the differential inclusion $u_t \in [\lambda(D^2u), \Lambda(D^2u)]$, where λ and Λ are the minimal and maximal eigenvalue of the Hessian D^2u.

There are many equivalent ways to define viscosity solutions for (1.1). One of them is given in Lemma 3.2 below, and it implies, in particular, that in the case $D\varphi(\hat{x},\hat{t})=0$ we may assume that $D^2\varphi(\hat{x},\hat{t})=0$ as well. Such a relaxation is very useful in some of the proofs of this paper. Another version of the definition takes into account the heuristic principle of the parabolic equations that the future should not have any influence on the past. Mathematically this means that one should be able to determine the admissibility of a test-function φ, touching at $(\hat{x},\hat{t})$, based on what happens prior to the time $t=\hat{t}$, see Lemma 3.4.

Remark 2.2. Let $n=1$ and $\varphi\in C^2(\Omega)$. Then, if $\varphi_x(x,t)\neq 0$,

$$\Delta_\infty\varphi(x,t)=\varphi_{xx}(x,t)\frac{\varphi_x(x,t)^2}{|\varphi_x(x,t)|^2}=\varphi_{xx}(x,t),$$

and always

$$\Lambda(\varphi_{xx}(x,t))=\lambda(\varphi_{xx}(x,t))=\varphi_{xx}(x,t).$$

It follows that an upper semicontinuous function $u:\Omega\to\mathbb{R}$ is a viscosity subsolution of (1.1) in $\Omega\subset\mathbb{R}^2$ if and only if u is a viscosity subsolution of the usual heat equation $v_t=v_{xx}$. An analogous statement holds of course for the viscosity supersolutions and solutions.

Explicit examples have often a fundamental role in the formation of a mathematical theory. We present below a number of solutions that give insight to the various features of the equation (1.1). In particular, some of these examples will serve as building blocks of the general theory as we will see later in Theorem 7.1.

(a) Let $h(x,t)=f(r)g(t)$, where $r=|x|$, and assume for a moment that $x\neq 0$. Then $h_t=f(r)g'(t)$, $Dh=f'(r)g(t)\frac{x}{|x|}$ and

$$D^2h=g(t)\left(f''(r)\frac{x\otimes x}{|x|^2}+f'(r)\frac{1}{|x|}I-f'(r)\frac{x\otimes x}{|x|^3}\right).$$

Thus $h_t=\Delta_\infty h$ if and only if $f(r)g'(t)=g(t)f''(r)$, which leads us to the equations

$$f''(r)+\lambda f(r)=0 \qquad\text{and}\qquad g'(t)+\lambda g(t)=0.$$

We have $g(t)=Ce^{-\lambda t}$ and

$$f(|x|)=\begin{cases}C_1\cos(\sqrt{\lambda}|x|)+C_2\sin(\sqrt{\lambda}|x|), & \text{if } \lambda>0,\\ C_1|x|+C_2, & \text{if } \lambda=0,\\ C_1\cosh(\sqrt{-\lambda}|x|)+C_2\sinh(\sqrt{-\lambda}|x|), & \text{if } \lambda<0.\end{cases}$$

The functions

$$h(x,t)=Ce^{-\lambda t}\cos(\sqrt{\lambda}|x|),\quad \lambda>0$$

and

$$h(x,t)=Ce^{\mu t}\cosh(\sqrt{\mu}|x|),\quad \mu>0$$

are twice differentiable everywhere and satisfy the equation (in the viscosity sense) also at the points where the spatial gradient vanishes. On the contrary, the functions $Ce^{-\lambda t}\sin(\sqrt{\lambda}|x|)$ and $Ce^{\mu t}\sinh(\sqrt{\mu}|x|)$ are only viscosity sub- or supersolutions, depending on the sign of the constant in front of them. In fact, near $x=0$, these functions look like cones having vertex at the origin, and the conical shape prevents testing from one side (hence automatically a sub/supersolution), but allows test-functions with non-zero gradient and arbitrary Hessian from the other side.

One can also let

$$r=\Big(\sum_{i=1}^k x_i^2\Big)^{1/2},\qquad k\in\{1,2,\dots n\},$$

and look again for a solution in the form $h(x,t) = f(r)g(t)$. This leads to the same equation $f(r)g'(t) = g(t)f''(r)$ and hence to the same type of solutions as above. The possible singular set $r = 0$ is now a $(n-k)$ -dimensional subspace and we obtain solutions depending on k spatial variables only.

(b) Let $h(x,t) = f(r) + g(t)$, where again $r = |x|$. We must have

$$g'(t) = \lambda = f''(r),$$

and thus

$$h(x,t) = \lambda \left(\frac{1}{2}|x - x_0|^2 + (t - t_0) + C \right).$$

In particular, $h(x,t) = \frac{1}{2}|x|^2 + t$ is a solution.

(c) Next we use the scaling invariance of the equation and seek a solution in the form

$$h(x,t) = g(t)f(\xi), \qquad \xi = \frac{|x|^2}{t}.$$

Then

$$\begin{aligned} h_t(x,t) &= g'(t)f(\xi) - \frac{g(t)f'(\xi)\xi}{t}, \\ Dh(x,t) &= \frac{2g(t)f'(\xi)x}{t}, \\ D^2h(x,t) &= \frac{2g(t)f'(\xi)}{t}I + \frac{4g(t)f''(\xi)}{t^2}(x \otimes x). \end{aligned}$$

Hence h is a solution to (1.1) if

$$g'(t)f(\xi) - \frac{g(t)f'(\xi)\xi}{t} = \frac{2g(t)f'(\xi)}{t} + \frac{4g(t)f''(\xi)\xi}{t},$$

which for $t > 0$ can also be written as

$$tg'(t)f(\xi) - 2g(t)f'(\xi) = g(t)\xi\Big(f'(\xi) + 4f''(\xi)\Big).$$

The right hand side is zero if $f(\xi) = e^{-\xi/4}$. Inserting this to the left hand side leaves us with the equation

$$e^{-\xi/4}\Big(tg'(t) + \frac{1}{2}g(t)\Big) = 0,$$

whose solution is $g(t) = t^{-1/2}$. We conclude that

$$h(x,t) = \frac{1}{\sqrt{t}}e^{-\frac{|x|^2}{4t}} \tag{2.3}$$

is a solution to (1.1) in $\mathbb{R}^n \times (0,\infty)$. This solution should be compared with the fundamental solution of the linear heat equation

$$H(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}.$$

As in the first example, we may repeat the above derivation with

$$\xi = \frac{1}{t}\sum_{i=1}^{k} x_i^2 = \frac{r^2}{t}, \qquad k \in \{1, 2, \ldots n\},$$

and obtain a solution to (1.1) in the form

$$h(x,t) = \frac{1}{\sqrt{t}}e^{-\frac{r^2}{4t}}.$$

Moreover, for $t < 0$ the procedure gives

$$h(x,t) = \frac{1}{\sqrt{-t}}e^{-\frac{|x|^2}{4t}},$$

which is a solution to (1.1) in $\mathbb{R}^n \times (-\infty, 0)$.

(d) Next we seek a solution in the form

$$h(x,t) = F(\xi), \qquad \xi = \frac{|x|^2}{t}, \; t > 0.$$

Then

$$\begin{aligned} h_t(x,t) &= -\frac{F'(\xi)\xi}{t}, \\ Dh(x,t) &= \frac{2F'(\xi)x}{t}, \\ D^2h(x,t) &= \frac{2F'(\xi)}{t}I + \frac{4F''(\xi)}{t^2}(x \otimes x) \end{aligned}$$

and hence

$$h_t - \Delta_\infty h = -\frac{F'(\xi)\xi}{t} - \frac{2F'(\xi)}{t} - \frac{4F''(\xi)\xi}{t} = 0$$

if

$$\frac{d}{d\xi}\log F'(\xi) = \frac{F''(\xi)}{F'(\xi)} = -\frac{1}{2\xi} - \frac{1}{4}.$$

Integrating this gives

$$F'(\xi) = \frac{C}{\sqrt{\xi}}e^{-\xi/4},$$

i.e.,

$$h(x,t) = C\int^{|x|^2/t} \frac{1}{\sqrt{s}}e^{-s/4}\, ds = C\int^{|x|/2\sqrt{t}} e^{-s^2}\, ds.$$

Notice that this function is not differentiable at the points $(0,t)$, $t > 0$. It is a solution outside the hyperplane $\{(x,t) \in \mathbb{R}^n \times (0,\infty) : x = 0\}$ and a sub/supersolution (depending on the sign of C) in $\mathbb{R}^n \times (0,\infty)$.

3. Comparison principle and the definition of a solution revisited

For a cylinder $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain, we denote the lateral boundary by

$$S_T = \partial U \times [0,T]$$

and the parabolic boundary by

$$\partial_p Q_T = S_T \cup (U \times \{0\}).$$

Notice that both S_T and $\partial_p Q_T$ are compact sets.

The proof of the following comparison principle can be found in [6], but for reader's convenience and for later use we sketch the argument below.

Theorem 3.1. *Suppose $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain. Let u and v be a supersolution and a subsolution of (1.1) in Q_T, respectively, such that*

$$\limsup_{(x,t)\to(z,s)} u(x,t) \le \liminf_{(x,t)\to(z,s)} v(x,t) \tag{3.1}$$

for all $(z,s) \in \partial_p Q_T$ and both sides are not simultaneously ∞ or $-\infty$. Then

$$u(x,t) \le v(x,t) \qquad \textit{for all } (x,t) \in Q_T.$$

Proof. By moving to a suitable subdomain, we may assume that ∂U is smooth, $u \le v + \varepsilon$ on $\partial_p Q_T$ (u and v defined up to the boundary), u is bounded from above and v from below. All this follows from (3.1) and the compactness of the parabolic boundary $\partial_p Q_T$, cf. [22].

Also, by replacing v with $v(x,t) + \frac{\varepsilon}{T-t}$ for $\varepsilon > 0$, we may assume that v is a strict supersolution and $v(x,t) \to \infty$ uniformly in x as $t \to T$.

The proof is by contradiction. Suppose that

$$\sup_{Q_T}(u(x,t)-v(x,t))>0 \tag{3.2}$$

and let

$$w_j(x,t,y,s)=u(x,t)-v(y,s)-\frac{j}{4}|x-y|^4-\frac{j}{2}(t-s)^2.$$

Denote by (x_j,t_j,y_j,s_j) the maximum point of w_j relative to $\overline{U}\times[0,T]\times\overline{U}\times[0,T]$. It follows from (3.2) and the fact that $u<v$ on $\partial_p Q_T$ that for j large enough $x_j,y_j\in U$ and $t_j,s_j\in(0,T)$, cf. [9], Prop. 3.7. From now on, we will consider only such indexes j.

Case 1: If $x_j=y_j$, then $v-\phi$, where

$$\phi(y,s)=-\frac{j}{4}|x_j-y|^4-\frac{j}{2}(t_j-s)^2,$$

has a local minimum at (y_j,s_j). Since v is a strict supersolution and $D\phi(y_j,s_j)=0$, we have

$$0<\phi_t(y_j,s_j)-\lambda(D^2\phi(y_j,s_j))=j(t_j-s_j).$$

Similarly, $u-\psi$, where

$$\psi(x,t)=\frac{j}{4}|x-y_j|^4+\frac{j}{2}(t-s_j)^2,$$

has a local maximum at (x_j,t_j), and thus

$$0\geq\psi_t(x_j,t_j)-\Lambda(D^2\psi(x_j,t_j))=j(t_j-s_j).$$

Subtracting the two inequalities gives

$$0<j(t_j-s_j)-j(t_j-s_j)=0,$$

a contradiction.

Case 2: If $x_j\neq y_j$, we use jets and the parabolic maximum principle for semicontinuous functions. There exist symmetric $n\times n$ matrices X_j,Y_j such that Y_j-X_j is positive semidefinite and

$$\begin{aligned}
(j(t_j-s_j),j|x_j-y_j|^2(x_j-y_j),X_j)&\in\overline{\mathcal{P}}^{2,+}u(x_j,t_j),\\
(j(t_j-s_j),j|x_j-y_j|^2(x_j-y_j),Y_j)&\in\overline{\mathcal{P}}^{2,-}v(y_j,s_j).
\end{aligned}$$

See [9], [27] for the notation and relevant definitions. Using the facts that u is a subsolution and v a strict supersolution, this implies

$$\begin{aligned}
0&<j(t_j-s_j)-\Big(Y_j\frac{(x_j-y_j)}{|x_j-y_j|}\Big)\cdot\frac{(x_j-y_j)}{|x_j-y_j|}\\
&\quad-j(t_j-s_j)+\Big(X_j\frac{(x_j-y_j)}{|x_j-y_j|}\Big)\cdot\frac{(x_j-y_j)}{|x_j-y_j|}\\
&=-\Big((Y_j-X_j)\frac{(x_j-y_j)}{|x_j-y_j|}\Big)\cdot\frac{(x_j-y_j)}{|x_j-y_j|}\\
&\leq 0,
\end{aligned}$$

again a contradiction. □

The proof of the comparison principle shows that we may reduce the number of test-functions in the definition of viscosity subsolutions. This fact will become useful for example in the proof of Theorem 7.1 below.

Lemma 3.2. *Suppose $u:\Omega\to\mathbb{R}$ is an upper semicontinuous function with the property that for every $(\hat{x},\hat{t})\in\Omega$ and $\varphi\in C^2(\Omega)$ satisfying*

(1) *$u(\hat{x},\hat{t})=\varphi(\hat{x},\hat{t})$,*
(2) *$u(x,t)<\varphi(x,t)$ for all $(x,t)\in\Omega$, $(x,t)\neq(\hat{x},\hat{t})$,*

the following holds:

$$\begin{cases} \varphi_t(\hat{x},\hat{t}) \le \Delta_\infty \varphi(\hat{x},\hat{t}) & \text{if } D\varphi(\hat{x},\hat{t}) \neq 0, \\ \varphi_t(\hat{x},\hat{t}) \le 0 & \text{if } D\varphi(\hat{x},\hat{t}) = 0 \text{ and } D^2\varphi(\hat{x},\hat{t}) = 0. \end{cases} \tag{3.3}$$

Then u is a viscosity subsolution of (1.1).

The novelty in Lemma 3.2 is that nothing is required in the case $D\varphi(\hat{x},\hat{t}) = 0$ and $D^2\varphi(\hat{x},\hat{t}) \neq 0$. This implies, in particular, that if u fails to be a viscosity subsolution of (1.1), then there exist $(\hat{x},\hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ such that (1) and (2) above hold, and either

$$D\varphi(\hat{x},\hat{t}) \neq 0 \ \text{ and } \ \varphi_t(\hat{x},\hat{t}) > \Delta_\infty \varphi(\hat{x},\hat{t}),$$

or

$$D\varphi(\hat{x},\hat{t}) = 0,\ D^2\varphi(\hat{x},\hat{t}) = 0 \ \text{ and } \ \varphi_t(\hat{x},\hat{t}) > 0.$$

On the other hand, it is clear that one cannot further reduce the set of test-functions to only those with non-zero spatial gradient at the point of touching. Indeed, with such a definition, any smooth function $u(x,t) = v(t)$ would be a solution of (1.1).

Proof. Suppose u is not a viscosity subsolution but satisfies the assumptions of the lemma. Then there exist $(\hat{x},\hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ such that (1) and (2) above hold, $D\varphi(\hat{x},\hat{t}) = 0$, $D^2\varphi(\hat{x},\hat{t}) \neq 0$, and

$$\varphi_t(\hat{x},\hat{t}) > \Lambda(D^2\varphi(\hat{x},\hat{t})). \tag{3.4}$$

As in the proof of Theorem 3.1 above, we let

$$w_j(x,t,y,s) = u(x,t) - \varphi(y,s) - \frac{j}{4}|x-y|^4 - \frac{j}{2}(t-s)^2,$$

and denote by (x_j,t_j,y_j,s_j) the maximum point of w_j relative to $\overline{\Omega} \times \overline{\Omega}$. By [9], Prop. 3.7 and (1), (2), $(x_j,t_j,y_j,s_j) \to (\hat{x},\hat{t},\hat{x},\hat{t})$ as $j \to \infty$. In particular, $(x_j,t_j) \in \Omega$ and $(y_j,s_j) \in \Omega$ for all j large enough.

Again we have to consider two cases. If $x_j = y_j$, then $\varphi - \phi$, where

$$\phi(y,s) = -\frac{j}{4}|x_j-y|^4 - \frac{j}{2}(t_j-s)^2,$$

has a local minimum at (y_j,s_j). By (3.4) and the continuity of the mapping

$$(x,t) \mapsto \Lambda(D^2\varphi(x,t)),$$

we have

$$\varphi_t(x,t) > \lambda(D^2\varphi(x,t))$$

in some neighborhood of $(\hat{x},\hat{t})$. In particular, since $\varphi_t(y_j,s_j) = \phi_t(y_j,s_j)$ and $D^2\varphi(y_j,s_j) \ge D^2\phi(y_j,s_j)$ by calculus, we have

$$0 < \phi_t(y_j,s_j) - \lambda(D^2\phi(y_j,s_j)) = j(t_j - s_j)$$

for j large enough. Similarly, $u - \psi$, where

$$\psi(x,t) = \frac{j}{4}|x-y_j|^4 + \frac{j}{2}(t-s_j)^2,$$

has a local maximum at (x_j,t_j), and thus

$$0 \ge \psi_t(x_j,t_j) = j(t_j - s_j)$$

by the assumption on u; notice here that $D^2\psi(x_j,t_j) = 0$ because $x_j = y_j$. Subtracting the two inequalities gives

$$0 < j(t_j - s_j) - j(t_j - s_j) = 0,$$

a contradiction. The case $x_j \neq y_j$ is easy and goes as in the proof of Theorem 3.1. □

As a consequence of Lemma 3.2, we show that the time-independent solutions of (1.1) are precisely the infinity harmonic functions.

Corollary 3.3. *Let $Q_T = U \times (0,T)$ and suppose that $u : Q_T \to \mathbb{R}$ can be written as $u(x,t) = v(x)$ for some upper semicontinuous function $v : U \to \mathbb{R}$. Then u is a viscosity subsolution of (1.1) if and only if $-(D^2v(x)Dv(x)) \cdot Dv(x) \leq 0$ in the viscosity sense.*

Proof. Suppose first that u is a viscosity subsolution of (1.1), and let $\hat{x} \in U$ and $\psi \in C^2(U)$ be such that $v - \psi$ has a local maximum at $\hat{x}$. Then $\varphi(x,t) = \psi(x) + (t-\hat{t})^4$ is a good test-function for u at $(\hat{x},\hat{t})$. Thus if $D\psi(\hat{x}) \neq 0$, we have

$$0 = \varphi_t(\hat{x},\hat{t}) \leq \Delta_\infty \varphi(\hat{x},\hat{t}) = D^2\psi(\hat{x}) \frac{D\psi(\hat{x})}{|D\psi(\hat{x})|} \cdot \frac{D\psi(\hat{x})}{|D\psi(\hat{x})|}.$$

Hence $D^2\psi(\hat{x})D\psi(\hat{x}) \cdot D\psi(\hat{x}) \geq 0$, and since this is trivially true if $D\psi(\hat{x}) = 0$, we have shown that $-(D^2v(x)Dv(x)) \cdot Dv(x) \leq 0$ in the viscosity sense.

In order to prove the reverse implication let $(\hat{x},\hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ be such that $u(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t})$, and $u(x,t) < \varphi(x,t)$ for all $(x,t) \in \Omega$, $(x,t) \neq (\hat{x},\hat{t})$. Then $\psi(x) = \varphi(x,\hat{t})$ touches v from above at $\hat{x}$, and thus

$$0 \leq D^2\psi(\hat{x})D\psi(\hat{x}) \cdot D\psi(\hat{x}) = D^2\varphi(\hat{x},\hat{t})D\varphi(\hat{x},\hat{t}) \cdot D\varphi(\hat{x},\hat{t}).$$

Moreover, since u is independent of t, $\varphi_t(\hat{x},\hat{t}) = 0$. Hence

$$\varphi_t(\hat{x},\hat{t}) = 0 \leq D^2\varphi(\hat{x},\hat{t}) \frac{D\varphi(\hat{x},\hat{t})}{|D\varphi(\hat{x},\hat{t})|} \cdot \frac{D\varphi(\hat{x},\hat{t})}{|D\varphi(\hat{x},\hat{t})|}$$

if $D\varphi(\hat{x},\hat{t}) \neq 0$, and $\varphi_t(\hat{x},\hat{t}) \leq 0$ if $D\varphi(\hat{x},\hat{t}) = 0$ and $D^2\varphi(\hat{x},\hat{t}) = 0$. By Lemma 3.2 this implies that u is a viscosity subsolution of (1.1). □

We showed in Lemma 3.2 that a set of test-functions that is strictly smaller than the one in Definition 2.1 suffices for characterizing the viscosity subsolutions of (1.1). The next lemma establishes that for a viscosity subsolution, the inequalities (2.1) in fact hold for a set of test-functions that is strictly larger than the one in Definition 2.1.

Lemma 3.4. *Let $u : \Omega \to \mathbb{R}$ be a viscosity subsolution of (1.1) in Ω. Then if $(\hat{x},\hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that*

1. *$u(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t})$,*
2. *$u(x,t) < \varphi(x,t)$ for all $(x,t) \in \Omega \cap \{t \leq \hat{t}\}$, $(x,t) \neq (\hat{x},\hat{t})$,*

we have

$$\begin{cases} \varphi_t(\hat{x},\hat{t}) \leq \Delta_\infty \varphi(\hat{x},\hat{t}) & \text{if } D\varphi(\hat{x},\hat{t}) \neq 0, \\ \varphi_t(\hat{x},\hat{t}) \leq \Lambda(D^2\varphi(\hat{x},\hat{t})) & \text{if } D\varphi(\hat{x},\hat{t}) = 0. \end{cases} \tag{3.5}$$

Proof. Once again we argue by contradiction, and assume that there exists $(\hat{x},\hat{t}) \in \Omega$ and $\varphi \in C^2(\Omega)$ such that (1) and (2) above hold, and either

$$\varphi_t(\hat{x},\hat{t}) > \Delta_\infty \varphi(\hat{x},\hat{t}) \qquad \text{and} \qquad D\varphi(\hat{x},\hat{t}) \neq 0,$$

or

$$\varphi_t(\hat{x},\hat{t}) > \Lambda(D^2\varphi(\hat{x},\hat{t})) \qquad \text{and} \qquad D\varphi(\hat{x},\hat{t}) = 0.$$

Both alternatives imply that φ is a strict viscosity supersolution of (1.1) in $Q_\varepsilon := B_\varepsilon(\hat{x}) \times (\hat{t} - \varepsilon, \hat{t})$ for some small $\varepsilon > 0$ (see the proof of Lemma 3.2), and since $\sup_{\partial_p Q_\varepsilon}(\varphi - u) > 0$, we have $\sup_{Q_\varepsilon}(\varphi - u) > 0$ by the comparison principle. This contradicts the fact that $u(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t})$, and we are done with the proof. □

4. Existence

The main existence result we will prove is

Theorem 4.1. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain, and let $\psi \in C(\mathbb{R}^{n+1})$. Then there exists a unique $h \in C(Q_T \cap \partial_p Q_T)$ such that $h = \psi$ on $\partial_p Q_T$ and*

$$h_t = \Delta_\infty h \qquad \text{in } Q_T$$

in the viscosity sense.

The uniqueness follows from the comparison principle, Theorem 3.1. Regarding the existence, we consider the approximating equations

$$u_t = \Delta_\infty^{\varepsilon,\delta} u, \tag{4.1}$$

where

$$\Delta_\infty^{\varepsilon,\delta} u = \varepsilon \Delta u + \frac{1}{|Du|^2 + \delta^2}\Big(D^2 u\, Du\Big) \cdot Du = \sum_{i,j=1}^{n} a_{ij}^{\varepsilon,\delta}(Du) u_{ij}$$

with

$$a_{ij}^{\varepsilon,\delta}(\xi) = \varepsilon \delta_{ij} + \frac{\xi_i \xi_j}{|\xi|^2 + \delta^2}, \qquad 0 < \varepsilon \le 1, \quad 0 < \delta \le 1.$$

For this equation with smooth initial and boundary data $\psi(x,t)$, the existence of a smooth solution $h_{\varepsilon,\delta}$ is guaranteed by classical results in [25]. Our goal is to obtain a solution of (1.1) as a limit of these functions as $\varepsilon \to 0$ and $\delta \to 0$. This amounts to proving estimates for $h_{\varepsilon,\delta}$ that are independent of $0 < \varepsilon < 1$ and $0 < \delta < 1$.

4.1. Boundary regularity at $t = 0$.

Proposition 4.2. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain, and suppose that $h = h_{\varepsilon,\delta}$ is a smooth function satisfying*

$$\begin{cases} h_t = \Delta_\infty^{\varepsilon,\delta} h & \text{in } Q_T, \\ h(x,t) = \psi(x,t) & \text{on } \partial_p Q_T. \end{cases}$$

If $\psi \in C^2(\mathbb{R}^{n+1})$, then there exists $C \ge 0$ depending on $\|D^2\psi\|_\infty$ and $\|\psi_t\|_\infty$ but independent of $0 < \varepsilon \le 1$ and $0 < \delta \le 1$ such that

$$|h(x,t) - \psi(x,0)| \le Ct$$

for all $x \in U$ and $0 < t < T$. Moreover, if ψ is only continuous in x (and possibly discontinuous in t), then the modulus of continuity of h on $U \times \{0\}$ can be estimated in terms of $\|\psi\|_\infty$ and the modulus of continuity of ψ in x.

Proof. Suppose first that $\psi \in C^2(\mathbb{R}^{n+1})$, and let $w(x,t) = \psi(x,0) + \lambda t$, where $\lambda > 0$ is to be determined. We have

$$\begin{aligned} w_t - \Delta_\infty^{\varepsilon,\delta} w &= \lambda - \varepsilon \Delta \psi(x,0) - \left(D^2\psi \frac{D\psi}{|D\psi|^2 + \delta^2}\right) \cdot \frac{D\psi}{|D\psi|^2 + \delta^2} \\ &\ge \lambda - (1 + \varepsilon n)\|D^2\psi(x,0)\|_\infty \ge 0 \end{aligned}$$

if λ is large enough. Clearly $w(x,0) \ge h(x,0)$ for all $x \in U$. Moreover,

$$w(x,t) = \psi(x,0) + \lambda t \ge \psi(x,0) + \|\psi_t\|_\infty t \ge \psi(x,t)$$

for all $x \in \partial U$ and $0 < t < T$ if $\lambda \ge \|\psi_t\|_\infty$. Thus, by the comparison principle,

$$h(x,t) \le w(x,t) = \psi(x,0) + \lambda t$$

for all $x \in U$ and $0 < t < T$. By considering also the lower barrier $(x,t) \mapsto \psi(x,0) - \lambda t$, we obtain the Lipschitz estimate

$$|h(x,t) - \psi(x,0)| \le Ct, \tag{4.2}$$

where $C = \max\{(1+\varepsilon n)\|D^2\psi(x,0)\|_\infty, \|\psi_t\|_\infty\}$.

Suppose now that ψ is only continuous, and fix $x_0 \in U$. For a given $\mu > 0$, choose $0 < \tau < \operatorname{dist}(x_0, \partial U)$ such that $|\psi(x,0) - \psi(x_0,0)| < \mu$ whenever $|x - x_0| < \tau$, and consider the smooth functions

$$\psi_\pm(x,t) = \psi(x_0,0) \pm \mu \pm \frac{2\|\psi\|_\infty}{\tau^2}|x-x_0|^2.$$

It is easy to check that $\psi_- \le \psi \le \psi_+$ on the parabolic boundary of Q_T. Thus if $h_\pm$ are the unique solutions to (4.1) with boundary and initial data $\psi_\pm$ of class $C^2(\mathbb{R}^{n+1})$, respectively, we have $h_- \le h \le h_+$ in Q_T by the comparison principle. Applying the estimate (4.2) for $h_\pm$ yields

$$\begin{aligned} |h_\pm(x_0,t) - \psi_\pm(x_0,0)| &\le t\max\{\|(\psi_\pm)_t\|_\infty, (1+\varepsilon n)\|D^2\psi_\pm\|_\infty\} \\ &= t(1+\varepsilon n)\frac{4\|\psi\|_\infty}{\tau^2}, \end{aligned}$$

which implies

$$|h(x_0,t) - \psi(x_0,0)| \le \mu + (1+\varepsilon n)\frac{4\|\psi\|_\infty}{\tau^2}t.$$

The proposition is proved. □

Corollary 4.3. *Let $Q_T = U \times (0,T)$ and $h = h_{\varepsilon,\delta}$ be as in Proposition 4.2. If $\psi \in C^2(\mathbb{R}^{n+1})$, then there exists $C \ge 0$ depending on $\|D^2\psi\|_\infty$ and $\|\psi_t\|_\infty$ but independent of $0 < \varepsilon \le 1$ and $0 < \delta \le 1$ such that*

$$|h(x,t) - h(x,s)| \le C|t-s| \qquad \text{for all } x \in U \text{ and } t,s \in (0,T).$$

Moreover, if ψ is only continuous, then the modulus of continuity of h in t on $U \times (0,T)$ can be estimated in terms of $\|\psi\|_\infty$ and the modulus of continuity of ψ in x and t.

Proof. Let $v(x,t) = h(x,t+\tau)$, $\tau > 0$. Then both h and v are solutions to (4.1) in $Q_\tau := U \times (0, T-\tau)$, and hence if $\psi \in C^2(\mathbb{R}^{n+1})$, we have

$$\begin{aligned} \sup_{Q_\tau}|h-v| &= \sup_{\partial_p Q_\tau}|h-v| \\ &\le \max\{\|h(\cdot,\tau) - \psi(\cdot,0)\|_{\infty,U}, \sup_{x\in\partial U}\big(\|h(x,\cdot) - h(x,\cdot+\tau)\|_{\infty,(0,T)}\big)\} \\ &\le \max\{C\tau, \|\psi_t\|_\infty\tau\} = C\tau \end{aligned}$$

by the comparison principle and Proposition 4.2. This implies the Lipschitz estimate asserted above, and the proof for case where ψ is only continuous is analogous. □

4.2. **Regularity at the lateral boundary $S_T = \partial U \times [0,T]$.**

Proposition 4.4. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain, and suppose that $h = h_{\varepsilon,\delta}$ is a smooth function satisfying*

$$\begin{cases} h_t = \Delta_\infty^{\varepsilon,\delta} h & \text{in } Q_T, \\ h(x,t) = \psi(x,t) & \text{on } \partial_p Q_T, \end{cases}$$

where $\psi \in C^2(\mathbb{R}^{n+1})$. Then for each $0 < \alpha < 1$, there exists a constant $C \ge 1$ depending on α, $\|\psi\|_\infty$, $\|D\psi\|_\infty$ and $\|\psi_t\|_\infty$ but independent of ε and δ such that

$$|h(x,t_0) - \psi(x_0,t_0)| \le C|x-x_0|^\alpha$$

for all $(x_0,t_0) \in \partial U \times (0,T)$, $x \in U \cap B_1(x_0)$ and $\varepsilon > 0$ sufficiently small (depending on α).

Proof. Let

$$w(x,t) = h(x_0,t_0) + C|x-x_0|^\alpha - M(t-t_0),$$

where $(x_0,t_0) \in \partial U \times (0,T)$, $t_0 > 0$ and $0 < \alpha < 1$. Then a straightforward computation gives

$$\begin{aligned} w_t - \Delta_\infty^{\varepsilon,\delta} w &= -M - C\varepsilon\alpha(n+\alpha-2)|x-x_0|^{\alpha-2} - \frac{C^3\alpha^3(\alpha-1)|x-x_0|^{3\alpha-4}}{C^2\alpha^2|x-x_0|^{2\alpha-2}+\delta^2} \\ &= -M - C\alpha|x-x_0|^{\alpha-2}\left(\varepsilon(n+\alpha-2) + \frac{\alpha-1}{1+\left(\frac{\delta}{C\alpha|x-x_0|^{\alpha-1}}\right)^2}\right). \end{aligned}$$

If $|x-x_0| \le 1$ and $C \ge 1$, we have

$$\frac{1-\alpha}{1+\left(\frac{\delta}{C\alpha|x-x_0|^{\alpha-1}}\right)^2} - \varepsilon(n+\alpha-2) \ge \frac{1}{10}(1-\alpha)$$

for $\delta < 2\alpha$ and for $0 < \varepsilon \le \frac{1-\alpha}{10(n+\alpha-2)}$ if $n > 1$ and for any $\varepsilon > 0$ if $n = 1$. Thus

$$w_t - \Delta_\infty^{\varepsilon,\delta} w \ge -M + C\alpha|x-x_0|^{\alpha-2}\frac{1-\alpha}{10} \ge -M + C\alpha\frac{1-\alpha}{10} \ge 0$$

provided that ε is in the range specified above and

$$C \ge \max\{1, \tfrac{10M}{\alpha(1-\alpha)}\}.$$

Next we will show that M and C can be chosen so that $w \ge h$ on the parabolic boundary of $Q_T \cap (B_1(x_0) \times (t_0-1,t_0))$. Let us suppose first that $t_0 > 1$, and consider a point (x,t) such that $x \in (\partial U) \cap B_1(x_0)$ and $t_0 - 1 < t \le t_0$. Then, since $|x-x_0| < 1$ (and $h = \psi$ on the boundary ∂U),

$$\begin{aligned} h(x,t) &\le h(x_0,t_0) + \|D\psi\|_\infty|x-x_0| + \|\psi_t\|_\infty(t_0-t) \\ &\le h(x_0,t_0) + C|x-x_0|^\alpha + M(t_0-t) = w(x,t) \end{aligned}$$

if $C \ge \|D\psi\|_\infty$ and $M \ge \|\psi_t\|_\infty$. On the other hand, if $x \in U \cap (\partial B_1(x_0))$ and $t_0 - 1 < t \le t_0$, we have

$$w(x,t) = h(x_0,t_0) + C + M(t_0-t) \ge \|\psi\|_\infty \ge h(x,t)$$

if $C \ge 2\|\psi\|_\infty$. Finally, we consider the bottom of the cylinder. Suppose $t = t_0 - 1$ and $x \in U \cap B_1(x_0)$. Then

$$w(x,t) = h(x_0,t_0) + C|x-x_0|^\alpha + M \ge \|\psi\|_\infty \ge h(x,t)$$

if $M \ge 2\|\psi\|_\infty$.

In conclusion, we have now shown that if we choose $M \ge \max\{\|\psi_t\|_\infty, 2\|\psi\|_\infty\}$ and $C \ge \max\{\|D\psi\|_\infty, 2\|\psi\|_\infty, \frac{10M}{\alpha(1-\alpha)}\}$, then $w \ge h$ in $Q_T \cap (B_1(x_0) \times (t_0-1,t_0))$ by the comparison principle. In particular,

$$h(x,t_0) \le w(x,t_0) = \psi(x_0,t_0) + C|x-x_0|^\alpha$$

for $x \in U \cap B_1(x_0)$. The other half of the estimate claimed follows by considering the lower barrier $(x,t) \mapsto h(x_0,t_0) - C|x-x_0|^\alpha + M(t-t_0)$.

In the case when $t_0 < 1$, we consider the cylinder $Q_T \cap (B_1(x_0) \times (0,t_0))$, and notice that since $h = \psi$ on the bottom of this cylinder,

$$\begin{aligned} h(x,0) = \psi(x,0) &\le \|D\psi\|_\infty|x-x_0| + \|\psi_t\|_\infty t_0 + h(x_0,t_0) \\ &\le C|x-x_0|^\alpha + Mt_0 + h(x_0,t_0) = w(x,0) \end{aligned}$$

for $x \in U \cap B_1(x_0)$ if $C \ge \|D\psi\|_\infty$ and $M \ge \|\psi_t\|_\infty$. The rest of the argument is analogous to the previous case. $\square$

Notice that the function $w(x,t) = C|x-x_0|^\alpha - M(t-t_0)$ is *not* a viscosity supersolution of (1.1) if $\alpha = 1$. Therefore, in order to obtain Lipschitz estimates, we have to consider barriers of different type and, rather surprisingly, remove the Laplacian term from the equation.

Proposition 4.5. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain, and suppose that $h = h_\delta$ satisfies*

$$\begin{cases} h_t = \Delta_\infty^{0,\delta} h & \text{in viscosity sense in } Q_T, \\ h(x,t) = \psi(x,t) & \text{on } \partial_p Q_T. \end{cases}$$

If $\psi \in C^2(\mathbb{R}^{n+1})$, then there exists a constant $C \geq 1$ depending on $\|\psi\|_\infty$, $\|D\psi\|_\infty$ and $\|\psi_t\|_\infty$ but independent of $0 < \delta \leq 1$ such that

$$|h(x,t_0) - \psi(x_0,t_0)| \leq C|x-x_0|$$

for all $(x_0,t_0) \in \partial U \times (0,T)$, $x \in U \cap B_1(x_0)$. Moreover, if ψ is only continuous, then the modulus of continuity of h on $\partial U \times (0,T)$ can be estimated in terms of $\|\psi\|_\infty$ and the modulus of continuity of ψ.

Proof. Suppose first that $\psi \in C^2(\mathbb{R}^{n+1})$. We will use a barrier of the form

$$w(x,t) = \psi(x_0,t_0) + M(t_0-t) + C|x-x_0| - K|x-x_0|^2,$$

where $M, C, K > 0$. Then

$$\begin{aligned} w_t - \Delta_\infty^{0,\delta} w &= -M - \frac{\left(\frac{C}{|x-x_0|} - 2K\right)^3 |x-x_0|^2 - C\left(\frac{C}{|x-x_0|} - 2K\right)^2 |x-x_0|}{\left(\frac{C}{|x-x_0|} - 2K\right)^2 |x-x_0|^2 + \delta^2} \\ &= -M + \frac{2K}{1 + \left(\frac{\delta}{C-2K|x-x_0|}\right)^2} \\ &\geq -M + \frac{2K}{1 + \left(\frac{\delta}{C-2K}\right)^2} \geq 0 \end{aligned}$$

if $x \in U \cap B_1(x_0)$, $\delta \leq 1$, $2K > M$ and $C \geq 2K + \sqrt{\frac{M}{2K-M}}$.

Next we will check that $w \geq h$ on the parabolic boundary of $Q_T \cap (B_1(x_0) \times (t_0-1,t_0))$; we suppose for a moment that $t_0 > 1$. Let us first consider a point (x,t) such that $x \in (\partial U) \cap B_1(x_0)$ and $t_0 - 1 < t \leq t_0$. Then

$$\begin{aligned} h(x,t) = \psi(x,t) &\leq \psi(x_0,t_0) + \|D\psi\|_\infty |x-x_0| + \|\psi_t\|_\infty (t_0-t) \\ &\leq \psi(x_0,t_0) + (C-K)|x-x_0| + M(t_0-t) \leq w(x,t) \end{aligned}$$

if $M \geq \|\psi_t\|_\infty$ and $C \geq K + \|D\psi\|_\infty$. If $x \in U \cap (\partial B_1(x_0))$ and $t_0 - 1 < t \leq t_0$, we have

$$w(x,t) = M(t_0-t) + C - K + \psi(x_0,t_0) \geq \|\psi\|_\infty \geq h(x,t)$$

if $C \geq K + 2\|\psi\|_\infty$. Finally, if $x \in U \cap B_1(x_0)$ and $t = t_0 - 1$,

$$w(x,t) \geq M + \psi(x_0,t_0) \geq \|\psi\|_\infty \geq h(x,t)$$

if $M \geq 2\|\psi\|_\infty$. We conclude that if $M \geq \max\{2\|\psi\|_\infty, \|\psi_t\|_\infty\}$, $K > M/2$, and

$$C \geq \max\{2K + \sqrt{\tfrac{M}{2K-M}}, K + \|D\psi\|_\infty, K + 2\|\psi\|_\infty\},$$

the function w defined above is a viscosity supersolution of (4.1) with $\varepsilon = 0$ and $w \geq h$ on the parabolic boundary of $Q_T \cap (B_1(x_0) \times (t_0-1,t_0))$. Thus the comparison principle implies

$$h(x,t_0) \leq \psi(x_0,t_0) + C|x-x_0|$$

for $x \in U \cap B_1(x_0)$. As before, we obtain the full estimate by considering also the lower barrier $(x,t) \mapsto \psi(x_0,t_0) - M(t_0-t) - C|x-x_0| + K|x-x_0|^2$ with the same choice for the constants M, C and K.

The case $t_0 \le 1$ can be treated as in the proof of Proposition 4.4, by considering $Q_T \cap (B_1(x_0) \times (0,t_0))$ as the comparison domain. Note that on the bottom of this cylinder we have

$$\begin{aligned} h(x,0) = \psi(x,0) &\le \psi(x_0,t_0) + \|D\psi\|_\infty |x-x_0| + \|\psi_t\|_\infty t_0 \\ &\le \psi(x_0,t_0) + (C-K)|x-x_0| + Mt_0 \le w(x,0) \end{aligned}$$

provided that $M \ge \|\psi_t\|_\infty$ and $C \ge K + \|D\psi\|_\infty$.

Suppose now that ψ is only continuous, and fix $(x_0,t_0) \in \partial U \times (0,T)$. For a given $\mu > 0$, choose $0 < \tau < t_0$ such that $|\psi(x,t) - \psi(x_0,t_0)| < \mu$ whenever $|x-x_0| + |t-t_0| < \tau$, and consider the smooth functions

$$\psi_\pm(x,t) = \psi(x_0,0) \pm \mu \pm \frac{4\|\psi\|_\infty}{\tau^2}|x-x_0|^2 \pm \frac{4\|\psi\|_\infty}{\tau}|t-t_0|.$$

Since

$$\psi_-(x,t) \le \psi(x_0,t_0) - \mu < \psi(x,t) < \psi(x_0,t_0) - \mu \le \psi_+(x,t)$$

if $|x-x_0| + |t-t_0| < \tau$ and

$$\psi_-(x,t) \le -\|\psi\|_\infty \le \psi(x,t) \le \|\psi\|_\infty \le \psi_+(x,t)$$

otherwise, we have $\psi_- \le \psi \le \psi_+$ on the parabolic boundary of Q_T. Thus if $h_\pm$ are the unique solutions to the equation $v_t = \Delta_\infty^{0,\delta} v$ with boundary and initial data $\psi_\pm$ of class $C^2(\mathbb{R}^{n+1})$, respectively, we have $h_- \le h \le h_+$ in Q_T by the comparison principle. Applying the estimate obtained above, with the choice $K = M = 4\|\psi\|_\infty/\sigma$ for $h_\pm$ yields

$$|h_\pm(x,t_0) - \psi_\pm(x_0,t_0)| \le \max\{\frac{16\|\psi\|_\infty}{\sigma^2}, 1\}|x-x_0|.$$

Thus we obtain

$$|h(x,t_0) - \psi(x_0,t_0)| \le \mu + \max\{\frac{16\|\psi\|_\infty}{\sigma^2}, 1\}|x-x_0|.$$

The proposition is proved. □

The boundary regularity obtained above is inherited to the interior of the domain, cf. [23]:

Corollary 4.6. *Let $Q_T = U \times (0,T)$ and $h = h_\delta$ be as in Proposition 4.5. If $\psi \in C^2(\mathbb{R}^{n+1})$, then there exists $C \ge 1$ depending on $\|\psi\|_\infty$, $\|D\psi\|_\infty$ and $\|\psi_t\|_\infty$ but independent of $0 < \varepsilon \le 1$ and $0 < \delta \le 1$ such that*

$$|h(x,t) - h(y,t)| \le C|x-y| \qquad \text{for all } x,y \in U \text{ and } t \in (0,T).$$

Moreover, if ψ is only continuous, then the modulus of continuity of h in x on $U \times (0,T)$ can be estimated in terms of $\|\psi\|_\infty$ and the modulus of continuity of ψ in x and t.

Remark 4.7. In the event that the boundary data ψ is independent of the time variable t, the Lipschitz estimate is much easier to prove. Indeed, one can simply compare h with the functions $(x,t) \mapsto \psi(x_0) \pm C|x-x_0|$ where $C = \|D\psi\|_{\infty,\partial U}$ to obtain

$$|h(x,t) - \psi(x_0)| \le C|x-x_0| \qquad \text{for all } x_0 \in \partial U \text{ and } x \in U,$$

which in turn yields the interior estimate

$$|h(x,t) - h(y,t)| \le C|x-y| \qquad \text{for all } x,y \in U \text{ and } t \in (0,T).$$

4.3. **Existence of a solution to the Dirichlet problem.** Theorem 4.1 follows now easily from Corollaries 4.3 and 4.6 and the stability properties of viscosity solutions. Indeed, if $\psi \in C^2(\mathbb{R}^{n+1})$ and $h_{\varepsilon,\delta}$ is the unique smooth solution to

$$\begin{cases} v_t = \Delta_\infty^{\varepsilon,\delta} v & \text{in } Q_T, \\ v(x,t) = \psi(x,t) & \text{on } \partial_p Q_T, \end{cases}$$

then Corollary 4.3, Proposition 4.4 and the comparison principle imply that the family $(h_{\varepsilon,\delta})$ is equicontinuous and uniformly bounded. Hence, up to a subsequence, $h_{\varepsilon,\delta} \to h_\delta$ as $\varepsilon \to 0$ and h_δ is the unique solution to

$$\begin{cases} v_t = \Delta_\infty^{0,\delta} v & \text{in the viscosity sense in } Q_T, \\ v(x,t) = \psi(x,t) & \text{on } \partial_p Q_T, \end{cases}$$

by the stability properties of viscosity solutions. Next we apply Corollaries 4.3 and 4.6 and conclude as above that $h_\delta \to h$ uniformly as $\delta \to 0$ and h is a viscosity solution to (1.1) with boundary and initial data ψ. The existence for a general continuous data ψ follows by approximating the data by smooth functions and using Corollaries 4.3 and 4.6.

4.4. **On the Cauchy problem.** Let us next very briefly discuss the Cauchy problem associated to (1.1).

Theorem 4.8. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function. Then there exists a unique bounded solution $h : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ to the Cauchy problem*

$$\begin{cases} h_t = \Delta_\infty h & \textit{in the viscosity sense in } \mathbb{R}^n \times (0,T), \\ h(x,0) = \psi(x) & \textit{for all } x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

Moreover, the modulus of continuity of h in $\mathbb{R}^n \times (0,T)$ can be estimated in terms of the modulus of continuity of ψ in $\mathbb{R}^n$ and $\sup_{\mathbb{R}^n} |\psi|$.

The solution to (4.3) can be constructed as a limit of functions h_r that satisfy

$$\begin{cases} (h_r)_t = \Delta_\infty h_r & \text{in the viscosity sense in } B_r(0) \times (0,T), \\ h_r(x,t) = \psi(x) & \text{for all } (x,t) \in \partial_p(B_r(0) \times (0,T)). \end{cases}$$

Due to the boundedness and uniform continuity of ψ, we have uniform continuity estimates for h_r in x and t and thus is follows from Ascoli-Arzela and the stability of viscosity solutions that the sequence (h_r) converges to a bounded solution of (4.3) as $r \to \infty$. Regarding uniqueness, we state a comparison principle that follows from the result proved in [18]:

Theorem 4.9. *Let u and v be a viscosity subsolution and a viscosity supersolution, respectively, of (1.1) in $\mathbb{R}^n \times (0,T)$ such that there exists $K > 0$ and a modulus of continuity ω so that*

- (A1) *$u(x,t) \le K(|x|+1)$ and $v(x,t) \ge -K(|x|+1)$ for all $(x,t) \in \mathbb{R}^n \times (0,T)$;*
- (A2) *$u(x,0) - v(y,0) \le \omega(|x-y|)$ for all $x, y \in \mathbb{R}^n$;*
- (A3) *$u(x,0) - v(y,0) \le K(|x-y|+1)$ for all $x, y \in \mathbb{R}^n$.*

Then $u \le v$ in $\mathbb{R}^n \times (0,T)$.

Indeed, in order to apply Theorem 2.1 of [18], it is enough to notice that by Lemma 3.4 the functions u and v are a viscosity sub- and supersolution of (1.1) in $\mathbb{R}^n \times (0,T']$ (which is not an open set) for every $0 < T' < T$.

Remark 4.10. The uniqueness part of Theorem 4.8 can only hold if we impose some conditions on the growth of the solution $h(x,t)$ as $|x| \to \infty$. Indeed, since for $n=1$ the equation (1.1) is nothing but the classical heat equation, the well-known counterexample of Tihonov [31], [13] shows that there exists a non-vanishing solution to (4.3) with $\psi \equiv 0$. By adding dummy variables, we obtain a counterexample to the uniqueness also in higher dimensions. It would be interesting to know if the optimal growth rate that guarantees uniqueness for (4.3) is $\mathcal{O}(e^{a|x|^2})$ as in the case of the heat equation.

5. An interior Lipschitz estimate

In this section, we establish an interior Lipschitz estimate for the solutions of (1.1) using Bernstein's method. Such an estimate was first obtained by Wu [32] for smooth solutions (see also [14]). We follow his ideas and show a similar estimate for the solutions of the approximating equation (4.1) with constants independent of ε and δ, and thereby extend Wu's result to all solutions of (1.1).

Proposition 5.1. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain. There exists a constant $C > 0$, independent of $0 < \varepsilon \leq 1$ and $0 < \delta \leq 1/2$, such that if $h = h_{\varepsilon,\delta} \in C^1(\overline{Q}_T)$ is a bounded, smooth solution of the approximating equation (4.1) in Q_T, then*

$$|Dh(x,t)| \leq C\left(1 + \frac{\|h\|_\infty}{\operatorname{dist}((x,t),\partial_p Q_T)^2}\right)$$

for all $(x,t) \in Q_T$.

Proof. Let us denote

$$v = \left(|Dh|^2 + \delta^2\right)^{1/2}$$

and consider the function

$$w(x,t) = \zeta(x,t)v(x,t) + \lambda h(x,t)^2,$$

where $\lambda \geq 0$ and ζ is a smooth, positive function that vanishes on the parabolic boundary of Q_T. Let (x_0,t_0) be a point where w takes its maximum in $\overline{Q}_T$, and let us first suppose that this point is not on the parabolic boundary $\partial_p Q_T$. Then at that point, since the matrix $(a_{ij}^{\varepsilon,\delta}(Dh))_{ij}$ is positive definite, we have

(5.1)
$$\begin{aligned} 0 \leq w_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)w_{ij} =& \zeta\left(v_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)v_{ij}\right) + v\left(\zeta_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_{ij}\right) \\ &+ 2\lambda h\left(h_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)h_{ij}\right) - 2\sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_j v_i \\ &- 2\lambda \sum a_{ij}^{\varepsilon,\delta}(Dh)h_i h_j. \end{aligned}$$

Notice that that the third term on the right hand side is zero because h is a solution to (4.1). In order to estimate the first term, we need to derive a differential inequality for v. To this end, note first that differentiating (4.1) with respect to x_k leads to the equation

$$h_{tk} = \varepsilon \Delta h_k + \frac{1}{v^2}\sum_{i,j} h_i h_j h_{ijk} + \frac{2}{v^2}\sum_{i,j} h_i h_{jk} h_{ij} - \frac{2}{v^4}\sum_{i,j}(h_i h_j h_{ij})\sum_l (h_l h_{lk}).$$

Multiplying this with $\frac{h_k}{v}$ and adding from 1 to n yields

$$v_t = \frac{\varepsilon}{v}\sum h_k h_{iik} + \frac{1}{v^3}\sum h_i h_j h_k h_{ijk} + \frac{2}{v^3}\sum h_i h_{ij} h_k h_{jk} - \frac{2}{v^5}\left(\sum h_i h_j h_{ij}\right)^2.$$

Since

$$v_{ij} = \frac{1}{v}\sum_k h_{ik}h_{jk} + \frac{1}{v}\sum_k h_k h_{ijk} - \frac{1}{v^3}\sum_k (h_k h_{ik}) \sum_l (h_l h_{jl}),$$

we thus have that

$$\begin{aligned} v_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon,\delta}(Dh) v_{ij} &= \frac{1}{v^3}\sum_j \Big(\sum_i h_i h_{ij}\Big)^2 - \frac{1}{v^5}\Big(\sum_{i,k} h_i h_k h_{ik}\Big)^2 \\ &\quad - \frac{\varepsilon}{v}\sum_{i,j} h_{ij}^2 + \frac{\varepsilon}{v^3}\sum_k \Big(\sum_i h_i h_{ik}\Big)^2 \\ &\le (1+\varepsilon)\frac{|Dv|^2}{v}. \end{aligned} \tag{5.2}$$

Using (5.2) and the fact the h is a solution to the approximating equation in (5.1) then gives

$$\begin{aligned} 0 \le{}& \zeta(1+\varepsilon)\frac{|Dv|^2}{v} + v\left(\zeta_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_{ij}\right) - 2\sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_j v_i \\ & - 2\lambda |Dh|^2 \left(\varepsilon + \frac{|Dh|^2}{|Dh|^2+\delta^2}\right). \end{aligned} \tag{5.3}$$

In order to estimate the various terms above, we notice that since $0 = w_i = \zeta_i v + \zeta v_i + 2\lambda h h_i$ at (x_0,t_0), we have

$$\zeta v_i = -\zeta_i v - 2\lambda h h_i.$$

Hence

$$\begin{aligned} \zeta\frac{|Dv|^2}{v} = \frac{\sum(\zeta v_i)^2}{\zeta v} &= \frac{v|D\zeta|^2}{\zeta} + 4\lambda\frac{h}{\zeta} D\zeta\cdot Dh + 4\lambda^2\frac{h^2}{\zeta v}|Dh|^2 \\ &\le \frac{v}{\zeta}\left(|D\zeta|^2 + 4\lambda|h||D\zeta| + 4(\lambda h)^2\right) \\ &\le \frac{6v}{\zeta}\left(|D\zeta|^2 + (\lambda h)^2\right) \end{aligned}$$

and

$$\begin{aligned} -2\sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_j v_i &= \frac{2v}{\zeta}\left(\varepsilon|D\zeta|^2 + \frac{(Dh\cdot D\zeta)^2}{v^2}\right) + 4\lambda\frac{h(Dh\cdot D\zeta)}{\zeta}\left(\varepsilon + \frac{|Dh|^2}{v^2}\right) \\ &\le \frac{2v}{\zeta}(1+\varepsilon)|D\zeta|^2 + 4(1+\varepsilon)\frac{(\lambda h)v|D\zeta|}{\zeta} \\ &\le \frac{4(1+\varepsilon)v}{\zeta}\left(|D\zeta|^2 + (\lambda h)^2\right). \end{aligned}$$

Moreover, using Young's inequality,

$$\begin{aligned} v\left(\zeta_t - \sum a_{ij}^{\varepsilon,\delta}(Dh)\zeta_{ij}\right) &\le v\left(|\zeta_t| + (1+n\varepsilon)|D^2\zeta|\right) \\ &\le \frac{1}{5}\lambda v^2 + \frac{5}{4\lambda}\left(|\zeta_t| + (1+n\varepsilon)|D^2\zeta|\right)^2. \end{aligned}$$

Thus (5.3) implies

$$\begin{aligned} 2\lambda|Dh|^2\left(\varepsilon + \frac{|Dh|^2}{|Dh|^2+\delta^2}\right) &\le \frac{10(1+\varepsilon)v}{\zeta}\left(|D\zeta|^2 + (\lambda h)^2\right) + \frac{1}{5}\lambda v^2 \\ &\quad + \frac{5}{4\lambda}\left(|\zeta_t| + (1+n\varepsilon)|D^2\zeta|\right)^2 \\ &\le \frac{500}{\lambda\zeta^2}\left(|D\zeta|^2 + (\lambda h)^2\right)^2 + \frac{2}{5}\lambda v^2 \\ &\quad + \frac{5}{4\lambda}\left(|\zeta_t| + (1+n)|D^2\zeta|\right)^2. \end{aligned} \tag{5.4}$$

If $|Dh(x_0,t_0)| \geq 1$ and $0 < \delta \leq 1/2$, then

$$\begin{aligned} 2\lambda |Dh|^2 \left(\varepsilon + \frac{|Dh|^2}{|Dh|^2+\delta^2}\right) &= 2\lambda v^2 \frac{|Dh|^2}{|Dh|^2+\delta^2}\left(\varepsilon + \frac{|Dh|^2}{|Dh|^2+\delta^2}\right) \\ &\geq 2\lambda v^2 \frac{1}{1+\delta^2}\left(\varepsilon + \frac{1}{1+\delta^2}\right) \geq 2\lambda v^2 \left(\frac{4}{5}\right)^2. \end{aligned}$$

Thus in (5.4) we can move the term $\frac{2}{5}\lambda v^2$ to the left-hand side, then divide by λ and multiply by ζ^2 to obtain

$$\frac{22}{25}\zeta^2 v^2 \leq \frac{500}{\lambda^2}\left(|D\zeta|^2 + (\lambda h)^2\right)^2 + \frac{5\zeta^2}{4\lambda^2}\left(|\zeta_t| + (1+n)|D^2\zeta|\right)^2,$$

that is,

$$(\zeta v)^2 \leq \frac{C}{\lambda^2}\left(\left(|D\zeta|^2 + (\lambda h)^2\right)^2 + \zeta^2\left(|\zeta_t| + (1+n)|D^2\zeta|\right)^2\right)$$

at the point (x_0,t_0). Now let $\lambda = \|h\|_\infty^{-1}$, fix $(x,t) \in Q_T$ and choose ζ so that $\zeta(x,t) = 1$ and

$$\max\{\|D\zeta\|_\infty, \|\zeta_t\|_\infty\} \leq \frac{1}{\operatorname{dist}((x,t),\partial_p Q_T)}.$$

Then

$$\begin{aligned} |Dh(x,t)| \leq w(x,t) &\leq w(x_0,t_0) = \zeta(x_0,t_0)v(x_0,t_0) + \lambda h(x_0,t_0)^2 \\ &\leq \frac{C}{\lambda}\left(\|D\zeta\|_\infty^2 + \lambda^2\|h\|_\infty^2 + \|D^2\zeta\|_\infty + \|\zeta_t\|_\infty\right) + \lambda\|h\|_\infty^2 \\ &\leq C\|h\|_\infty\left(1 + \frac{1}{\operatorname{dist}((x,t),\partial_p Q_T)^2}\right) \end{aligned}$$

with a constant $C \geq 1$ depending only on n. On the other hand, if $|Dh(x_0,t_0)| < 1$, then

$$\begin{aligned} |Dh(x,t)| &\leq v(x,t) \leq w(x,t) \leq w(x_0,t_0) = \zeta(x_0,t_0)v(x_0,t_0) + \lambda h(x_0,t_0)^2 \\ &\leq \|\zeta\|_\infty\sqrt{1+\delta^2} + \|h\|_\infty. \end{aligned}$$

Finally, if it happens that the maximum point (x_0,t_0) of w is on the parabolic boundary of Q_T, then

$$|Dh(x,t)| \leq v(x,t) \leq w(x,t) \leq w(x_0,t_0) = \lambda h(x_0,t_0)^2 \leq \|h\|_\infty,$$

because ζ vanishes on $\partial_p Q_T$. □

Corollary 5.2. *Let $Q_T = U \times (0,T)$, where $U \subset \mathbb{R}^n$ is a bounded domain. There exists a constant $C > 0$ such that if $h \in C(Q_T)$ is a viscosity solution of (1.1) in Q_T, then*

$$|Dh(x,t)| \leq C\left(1 + \frac{\|h\|_\infty}{\operatorname{dist}((x,t),\partial_p Q_T)^2}\right)$$

for almost every $(x,t) \in Q_T$.

Proof. Let $V' \subset\subset V \subset\subset U$ be open, $\sigma' > \sigma > 0$ and $Q_1 = V \times (\sigma, T-\sigma)$, $Q_2 = V' \times (\sigma', T-\sigma')$. Let also $h_{\varepsilon,\delta}$ satisfy

$$\begin{cases} (h_{\varepsilon,\delta})_t = \Delta_\infty^{\varepsilon,\delta} h_{\varepsilon,\delta}, & \text{in } Q_1, \\ h_{\varepsilon,\delta}(x,t) = h(x,t), & \text{on } \partial_p Q_1. \end{cases}$$

By Proposition 5.1 and the maximum principle,

$$|Dh_{\varepsilon,\delta}(x,t)| \leq C\left(1 + \frac{\|h\|_\infty}{\operatorname{dist}((x,t),\partial_p Q_2)^2}\right)$$

for any $(x,t) \in Q_2$ with a constant $C \geq 1$ independent of ε and δ. Using Ascoli-Arzela, we conclude that the functions $h_{\varepsilon,\delta}$ converge locally uniformly as $\varepsilon \to 0$ and

$\delta \to 0$ to a locally Lipschitz continuous function $\tilde{h}$ that by the stability properties of the viscosity solutions satisfies

$$\begin{cases} \tilde{h}_t = \Delta_\infty \tilde{h}, & \text{in the viscosity sense in } Q_1, \\ \tilde{h}(x,t) = h(x,t), & \text{on } \partial_p Q_1. \end{cases}$$

The comparison principle implies that $\tilde{h} = h$ in Q_1, and hence we have

$$|Dh(x,t)| \le C\left(1 + \frac{\|h\|_\infty}{\operatorname{dist}((x,t), \partial_p Q_2)^2}\right)$$

for a.e. $(x,t) \in Q_2$. Since the constant C can be taken to be independent of the subdomains used in the argument, the asserted estimate follows. □

Remark 5.3. We do not know whether solutions of (1.1) are differentiable in x or not. This question is still largely open also in elliptic case, although Savin [30] has recently shown the C^1 regularity for infinity harmonic functions in two dimensions. The "worst" example known to us is the time-independent solution

$$h(x,t) = \sum_{i=1}^{n} a_i |x_i|^{4/3}, \qquad a_1^3 + \cdots + a_n^3 = 0,$$

which is in $C^{1,1/3}$. This solution belongs to a family of quasi-radial solutions of the infinity Laplacian constructed by Aronsson [2]. It would be interesting to know if such a family of solutions exists for (1.1) as well.

6. The Harnack inequality

In this section, we prove the Harnack inequality for nonnegative viscosity solutions of (1.1). The proof is based on the ideas of Krylov and Safonov [24] and DiBenedetto [12], [13]. In fact, the argument below follows closely the proof of the Harnack inequality for the solutions of the heat equation given in [13].

Theorem 6.1. *Let h be a nonnegative viscosity solution of the infinity heat equation (1.1) in $\Omega \subset \mathbb{R}^{n+1}$. Then there exists a constant $c > 0$ such that whenever $(x_0, t_0) \in \Omega$ is such that $B_{4r}(x_0) \times (t_0 - (4r)^2, t_0 + (4r)^2) \subset \Omega$, we have*

$$\inf_{x \in B_r(x_0)} h(x, t_0 + r^2) \ge c h(x_0, t_0).$$

Proof. Using the change of variables

$$x \to \frac{x - x_0}{r}, \qquad t \to \frac{t - t_0}{r^2},$$

and replacing h by $h/h(0,0)$, we may assume that $(x_0, t_0) = (0,0)$, $r = 1$ and $h(0,0) = 1$. For $s \in (0,1)$, let $Q_s = B_s(0) \times (-s^2, 0)$ and

$$M_s = \sup_{x \in Q_s} h(x), \qquad N_s = \frac{1}{(1-s)^\beta},$$

where $\beta > 1$ is chosen later. Since h is continuous in Q_1, the equation $M_s = N_s$ has a well-defined largest root $s_0 \in [0,1)$, and there exists $(\hat{x}, \hat{t}) \in \overline{Q}_{s_0}$ such that $h(\hat{x}, \hat{t}) = (1 - s_0)^{-\beta}$.

Next let $\rho = (1 - s_0)/2 > 0$, and notice that since

$$Q_\rho(\hat{x}, \hat{t}) := B_\rho(\hat{x}) \times (\hat{t} - \rho^2, \hat{t}) \subset Q_{\frac{1+s_0}{2}},$$

we have

$$\sup_{Q_\rho(\hat{x},\hat{t})} h \le \sup_{Q_{\frac{1+s_0}{2}}} h \le N_{\frac{1+s_0}{2}} = \frac{2^\beta}{(1-s_0)^\beta}.$$

We now apply the interior Lipschitz estimate of Corollary 5.2 and conclude that there exists $C \geq 1$ such that for a.e. $(x,t) \in Q_{\rho/4}(\hat{x},\hat{t})$

$$
\begin{aligned}
|Dh(x,t)| &\leq C\left(1 + \frac{\sup_{Q_\rho(\hat{x},\hat{t})} h}{\operatorname{dist}((x,t), \partial_p Q_\rho(\hat{x},\hat{t}))}\right) \leq C\left(1 + \frac{2^\beta (1-s_0)^{-\beta}}{(\frac{3}{4}\rho)^2}\right) \\
&\leq \frac{9 \cdot 2^\beta C}{(1-s_0)^{\beta+2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
h(x,\hat{t}) &\geq h(\hat{x},\hat{t}) - \sup_{Q_{\frac{\rho}{4}}(\hat{x},\hat{t})} |Dh(x,t)||x-\hat{x}| \geq \frac{1}{(1-s_0)^\beta} - \frac{9 \cdot 2^\beta C}{(1-s_0)^{\beta+2}}|x-\hat{x}| \\
&\geq \frac{1}{2(1-s_0)^\beta} = \frac{1}{2} h(\hat{x},\hat{t})
\end{aligned}
$$

for all $x \in B_{\rho/4}(\hat{x})$ such that $|x-\hat{x}| < \frac{(1-s_0)^2}{18 \cdot 2^\beta C}$.

In the last step of the proof, we expand the set of positivity by using a comparison function

$$
\Psi(x,t) = \frac{MR^4}{((t-\hat{t}) + R^2)^2}\left(4 - \frac{|x-\hat{x}|^2}{(t-\hat{t}) + R^2}\right)_+^2,
$$

where $M = \frac{1}{2(1-s_0)^\beta}$ and $R = \frac{(1-s_0)^2}{36 \cdot 2^\beta C}$. A straightforward computation as in [13], Lemma 13.1 shows that Ψ is a viscosity subsolution of (1.1) in $\mathbb{R}^n \times (\hat{t}, \infty)$; here Lemma 3.2 can be used to take care of the critical points. Moreover,

$$
h(x,\hat{t}) \geq M \geq \frac{1}{16}\Psi(x,\hat{t}) \qquad \text{in } B_{2R}(\hat{x}),
$$

and

$$
h(x,t) \geq 0 = \Psi(x,t) \qquad \text{if } |x-\hat{x}| \geq 2\sqrt{R^2 + (t-\hat{t})}.
$$

Therefore the comparison principle implies that $h \geq \frac{1}{16}\Psi$ in $B_4(0) \times (\hat{t}, 4)$. In particular, in order to complete the proof, it suffices to show that $\Psi(x,1) \geq c > 0$ for all $x \in B_1(0)$. To this end, we first note that since for such x

$$
|x-\hat{x}|^2 \leq (1+s_0)^2 = (2-(1-s_0))^2,
$$

and $1 \leq (1-\hat{t}) \leq 2$, $R \leq 1$, we have

$$
4 - \frac{|x-\hat{x}|^2}{(1-\hat{t}) + R^2} \geq \frac{4 + 4\gamma^2(1-s_0)^4 - (2-(1-s_0))^2}{(1-\hat{t}) + R^2} \geq 1 - s_0,
$$

where we have denoted $\gamma = (36 \cdot 2^\beta C)^{-1}$. Consequently,

$$
\Psi(x,1) \geq \frac{1}{2(1-s_0)^\beta} \frac{\gamma^4 (1-s_0)^8}{((1-\hat{t}) + R^2)^2}(1-s_0)^2 \geq \frac{\gamma^4}{18}(1-s_0)^{10-\beta},
$$

and hence, by choosing $\beta = 10$, we obtain

$$
\Psi(x,1) \geq \frac{1}{18 \cdot (36 \cdot 2^{10} C)^4} > 0,
$$

where $C \geq 1$ is the constant from Corollary 5.2. $\square$

Remark 6.2. We do not know whether the estimate obtained in Theorem 6.1 remains valid for continuous nonnegative viscosity supersolutions of (1.1). The only place where we used the fact that h is a solution was when we applied the interior Lipschitz estimate of Corollary 5.2. In the elliptic case, i.e. for the equation $-\Delta_\infty u = 0$, it is known that a Harnack inequality holds also for nonnegative supersolutions, see e.g. [3].

7. Characterization of subsolutions á la Crandall

In the case of the stationary version of (1.1), a large number of estimates for the sub- and supersolutions can be derived from the fact that these sets of functions are characterized via a comparison property that involves a special class of solutions, cone functions, see [8], [3]. This kind of a characterization of subsolutions is known also for the Laplace equation [11] and the ordinary heat equation [10], [26], and in these cases the set of comparison functions is formed by using the fundamental solutions of these equations.

In this section, we prove an analogous result for the subsolutions of (1.1). To this end, let us denote

$$\Gamma(x,t)=\frac{1}{\sqrt{t}}e^{-\frac{|x|^2}{4t}},\ t>0,$$

and recall that Γ is a viscosity solution to (1.1) in $\mathbb{R}^n\times(0,\infty)$. We say that a function u satisfies the parabolic comparison principle with respect to the functions

$$W(x,t)=W_{x_0,t_0}(x,t)=-\Gamma(x-x_0,t-t_0),\qquad (x_0,t_0)\in\mathbb{R}^{n+1},$$

in $\Omega\subset\mathbb{R}^{n+1}$ if it holds that whenever $Q=B_r(\hat{x})\times(\hat{t}-r^2,\hat{t})\subset\subset\Omega$ and $t_0<\hat{t}-r^2$, we have

$$\sup_Q(u-W_{x_0,t_0})=\sup_{\partial_p Q}(u-W_{x_0,t_0}).$$

Note that this is equivalent to the condition

$$u\le W_{x_0,t_0}+c\ \text{ on } \partial_p Q \quad\text{implies}\quad u\le W_{x_0,t_0}+c\ \text{ in } Q,$$

where $c\in\mathbb{R}$ is a constant.

Theorem 7.1. *An upper semicontinuous function $u:\Omega\to\mathbb{R}$ is a viscosity subsolution of (1.1) in Ω if and only if u satisfies the parabolic comparison principle with respect to the functions*

$$W(x,t)=W_{x_0,t_0}(x,t)=-\Gamma(x-x_0,t-t_0),$$

where $t>t_0$ and $x_0\in\mathbb{R}^n$.

Proof. Since W_{x_0,t_0} is a solution of (1.1) in $\mathbb{R}^n\times(t_0,\infty)$, the necessity of the comparison condition follows from Theorem 3.1.

For the converse, suppose that u satisfies the parabolic comparison principle with respect to all the functions W_{x_0,t_0}, but u is not a viscosity subsolution of (1.1). Then we may assume, using Lemma 3.2 and the translation invariance of the equation, that there exists $\varphi\in C^2(\mathbb{R}^{n+1})$ such that $u-\varphi$ has a local maximum at $(0,0)$,

$$a=\varphi_t(0,0),\ q=D\varphi(0,0),\ X=D^2\varphi(0,0),$$

and

$$\begin{cases} a>(X\hat{q})\cdot\hat{q}, & \text{if } q\neq 0,\\ a>0\quad\text{and } X=0, & \text{if } q=0,\end{cases}\tag{7.1}$$

where $\hat{q}=q/|q|$. We want show that there exist $t_0<0$ and $x_0\in\mathbb{R}^n$ such that

$$\begin{aligned}&\frac{\partial}{\partial t}W_{x_0,t_0}(0,0)<a,\quad DW_{x_0,t_0}(0,0)=q\quad\text{and}\\ &D^2W_{x_0,t_0}(0,0)>X.\end{aligned}\tag{7.2}$$

Indeed, if we can find x_0,t_0 such that (7.2) holds, then by Taylor's theorem it follows that the origin is the unique maximum point of $u-W_{x_0,t_0}$ over $B_\delta(0)\times(-\delta^2,0]$ for $\delta>0$ small enough. Thus u fails to satisfy the parabolic comparison principle with respect to the family W_{x_0,t_0}, and we obtain a contradiction.

By computing the derivatives of W_{x_0,t_0} we see that (7.2) amounts to finding x_0, t_0 such that

$$\begin{aligned} &\text{(I)} \quad a > \left(\frac{1}{2} + \frac{|x_0|^2}{4t_0}\right)(-t_0)^{-3/2} e^{\frac{|x_0|^2}{4t_0}}, \\ &\text{(II)} \quad q = -\frac{x_0}{2}(-t_0)^{-3/2} e^{\frac{|x_0|^2}{4t_0}}, \\ &\text{(III)} \quad X < \left(\frac{1}{2}I + \frac{1}{4t_0} x_0 \otimes x_0\right)(-t_0)^{-3/2} e^{\frac{|x_0|^2}{4t_0}}. \end{aligned} \tag{7.3}$$

We consider separately the cases $q = 0$ and $q \neq 0$.

Case 1: $q = 0$. In this case, condition (II) is clearly satisfied if we choose $x_0 = 0$, and then the two remaining conditions can be written as

$$0 < \frac{1}{2}(-t_0)^{3/2} < a; \tag{7.4}$$

recall that by Lemma 3.2, we were able to assume that $X = 0$. Because $a > 0$ by (7.1), there exists $t_0 < 0$ so that (7.4) holds.

Case 2: $q \neq 0$. Note that (II) implies $x_0 = rq$ for some $r < 0$. Let us denote

$$\tau = \frac{1}{2}(-t_0)^{-3/2} e^{\frac{|x_0|^2}{4t_0}}, \quad \sigma = -\frac{|x_0|^2}{2t_0}.$$

Then $\tau > 0$, $\sigma > 0$, and (I)-(III) can be rewritten as

$$\begin{aligned} &\text{(I)} \quad a > \tau(1 - \sigma), \\ &\text{(II)} \quad q = -\tau x_0, \\ &\text{(III)} \quad X < \tau\left(I + \frac{1}{2t_0} x_0 \otimes x_0\right) = \tau\left(I - \sigma \hat{x}_0 \otimes \hat{x}_0\right), \end{aligned}$$

where $\hat{x}_0 = x_0/|x_0|$. We simplify things further by noting that $r = -\frac{1}{\tau}$. Then the conditions above reduce to

$$\begin{aligned} &\text{(I)} \quad \sigma > ra + 1, \\ &\text{(II)} \quad x_0 = rq, \\ &\text{(III)} \quad I + rX > \sigma \hat{q} \otimes \hat{q}. \end{aligned}$$

In order to investigate (III), we write a vector $p \in \mathbb{R}^n$ in the form $p = \alpha\hat{q} + q^\perp$, where $\alpha \in \mathbb{R}$ and $\hat{q} \cdot q^\perp = 0$. Then, for any $0 < \varepsilon < 1$,

$$\begin{aligned} (I + rX)p \cdot p - \sigma(\hat{q} \otimes \hat{q})p \cdot p &= \alpha^2\left(1 + rX\hat{q} \cdot \hat{q} - \sigma\right) + |q^\perp|^2 \\ &\quad + r\left(2\alpha X\hat{q} \cdot q^\perp + Xq^\perp \cdot q^\perp\right) \\ &\geq \alpha^2\left(1 + rX\hat{q} \cdot \hat{q} - \sigma + \varepsilon r\|X\|^2\right) \\ &\quad + \left(1 + r\|X\| + \frac{1}{\varepsilon}r\right)|q^\perp|^2. \end{aligned} \tag{7.5}$$

We choose first $\varepsilon > 0$ so small that

$$X\hat{q} \cdot \hat{q} + \varepsilon\|X\|^2 < a;$$

here we used (7.1). Next we choose $r < 0$ so that

$$1 + r\|X\| + \frac{1}{\varepsilon}r > 0 \qquad \text{and} \qquad X\hat{q} \cdot \hat{q} + \varepsilon\|X\|^2 < -\frac{1}{r}$$

and then $\sigma > 0$ so that

$$X\hat{q} \cdot \hat{q} + \varepsilon\|X\|^2 < \frac{\sigma - 1}{r} < a;$$

note that since $X\hat{q}\cdot\hat{q}+\varepsilon\|X\|^2 < -\frac{1}{r}$, we can take σ to be positive. By these choices we have

$$\begin{cases} 1+rX\hat{q}\cdot\hat{q}-\sigma+\varepsilon r\|X\|^2 > 0, \\ 1+r\|X\|+\frac{1}{\varepsilon}r > 0, \end{cases}$$

and hence $I+rX > \sigma\hat{q}\otimes\hat{q}$ by (7.5), i.e., (III) holds. Also, by the choice of σ, we have $\sigma > 1+ra$, i.e., (I) holds.

Finally, we notice that by choosing r and σ we actually chose x_0 and t_0 as well. First recall that $x_0 = rq$, and thus x_0 is determined by r and the function φ. Also, since σ and x_0 are now known and $\sigma = -\frac{|x_0|^2}{2t_0}$, the point $t_0 < 0$ has been determined as well. □

Remark 7.2. The main difference between Theorem 7.1 and the corresponding results for the heat equation is that above the comparison functions are single translates of the "fundamental solution" Γ, whereas in the case of the heat equation one has to take linear combinations of at least n copies of the heat kernel with different poles (see [10], [26] for details). The same is true also for the elliptic counterparts of these equations, see [11]. Note that if $n = 1$, then our result slightly improves the one obtained in [10].

The proof of Theorem 7.1 is to a great extent an adaptation of the arguments in [11] and [10] to our situation. In [10], the authors obtained a similar type of characterization for the subsolutions of the equation

$$v_t(x,t) = \big(D^2v(x,t)Dv(x,t)\big)\cdot Dv(x,t),$$

which is another parabolic version of the infinity Laplace equation.

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