ON THE OPTIMAL INSULATION OF CONDUCTORS

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Abstract: We coat a conductor with an insulator and equate the effectiveness of this procedure with the rate at which the body dissipates heat when immersed in an ice–bath. In the limit, as the thickness and conductivity of the insulator approach zero, the dissipation rate approaches the first eigenvalue of a Robin problem with a coefficient determined by the shape of the insulator. Fixing the mean of the shape function we search for the shape with the least associated Robin eigenvalue. We offer exact solutions for balls and, for general domains, establish existence and necessary conditions and report on the results of a numerical method.

1. Introduction

We suppose that a homogeneous conductor occupies the smooth open bounded connected set $\Omega \in \mathbf{R}^d$. With $\varepsilon > 0$ and a nonnegative function h on $\partial\Omega$ (whose outer normal we denote by n) we define the insulated conductor

$$\Omega_\varepsilon(h) \equiv \overline{\Omega} \cup \{x + \delta h(x) n(x) : x \in \partial\Omega,\ 0 < \delta < \varepsilon\}.$$

If the conductivity of Ω is 1 while that of $\Omega_\varepsilon(h) \setminus \Omega$ is ε then in solving the heat equation on $\Omega_\varepsilon(h)$, subject to zero Dirichlet data, one is lead to consider the eigenvalue problem

$$\begin{aligned} -\nabla \cdot (1_\Omega + (1 - 1_\Omega)\varepsilon)\nabla u = \lambda u, &\quad x \in \Omega_\varepsilon(h), \\ u = 0, &\quad x \in \partial\Omega_\varepsilon(h). \end{aligned} \tag{1.1}$$

where 1_Ω denotes the characteristic function of Ω. In particular, the least eigenvalue, let us denote it by $\lambda_1(\Omega, h, \varepsilon)$, governs the rate at which $\Omega_\varepsilon(h)$ dissipates heat and hence is a natural measure of the quality of the insulation procedure. Hence, we seek out those h that render $\lambda_1(\Omega, h, \varepsilon)$ small. Our search is considerably simplified if we first permit the thickness and conductivity of the insulator to simultaneously approach zero. More precisely, we recall that Friedman [F] has established that

$$\lambda_1(\Omega, h, \varepsilon) \to \xi_1(\Omega, h) \quad \text{as} \quad \varepsilon \to 0$$

where $\xi_1(\Omega, h)$ is the least eigenvalue of the Robin problem,

$$\begin{aligned} -\Delta u = \xi u, &\quad x \in \Omega, \\ u + h\partial u/\partial n = 0, &\quad x \in \partial\Omega. \end{aligned} \tag{1.2}$$

Friedman went on to pose the problem of minimizing $\xi_1(\Omega,h)$ over those h of fixed mean. The present note provides the solution to this problem under the additional assumption that h satisfies a pointwise lower bound. More precisely, we address

$$\hat{\xi}_1(\Omega) = \inf_{h\in ad_0(\partial\Omega)} \xi_1(\Omega,h) \tag{1.3}$$

where

$$ad_0(\partial\Omega) \equiv \{h : \alpha|\Omega|/|\partial\Omega| \le h(x),\ \int_{\partial\Omega} h\,dx = h_0|\Omega|\} \tag{1.4}$$

and $0<\alpha<h_0<\infty$ are both fixed throughout. As h has the dimension of length the convention specified in (1.4) renders α and h_0 dimensionless.

Let us dispense with a few preliminaries before summarizing the main results. To begin, the weak formulation of (1.2) reads

$$\int_\Omega \nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} h^{-1}uv\,dx = \xi\int_\Omega uv\,dx, \qquad \forall\, v\in H^1(\Omega), \tag{1.5}$$

and gives rise to the associated variational characterization

$$\xi_1(\Omega,h) = \inf_{u\in H_1^1(\Omega)} \mathcal{R}(u,h), \qquad \mathcal{R}(u,h)\equiv \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} h^{-1}u^2\,dx. \tag{1.6}$$

where $H_1^1(\Omega)$ is the class of $H^1(\Omega)$ functions with $L^2(\Omega)$ norm one. It follows from (1.6), see, e.g., Gilbarg and Trudinger [GT, Theorem 8.38], that $\xi_1(\Omega,h)$ is simple and may be associated with a nonnegative eigenfunction. A useful lower bound on $\xi_1(\Omega,h)$ follows from Hölders inequality

$$\int_{\partial\Omega} h^{-1}u^2\,dx \ge \frac{\left(\int_{\partial\Omega} u\,dx\right)^2}{\int_{\partial\Omega} h\,dx}. \tag{1.7}$$

For, using (1.7) in (1.6) one finds

$$\xi_1(\Omega,h) \ge \inf_{u\in H_1^1(\Omega)} \int_\Omega |\nabla u|^2\,dx + \frac{1}{h_0|\Omega|}\left(\int_{\partial\Omega} u\,dx\right)^2 \qquad \forall\, h\in ad_0(\partial\Omega). \tag{1.8}$$

If we denote by $\zeta_1(\Omega,h_0)$ the value of this *infimum* and by v_1 a function at which this infimum is attained we find $\zeta_1(\Omega,h_0)$ and v_1 to be the first eigenpair of the nonlocal problem

$$\begin{aligned} -\Delta v &= \zeta v \quad \text{in } \Omega,\\ h_0|\Omega|\frac{\partial v}{\partial n} + \int_{\partial\Omega} v\,dx &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.9}$$

In §2 we find that $\zeta_1(B,h_0|B|/|\partial B|) = \xi_1(B,h_0|B|/|\partial B|)$ when B is a ball and so establish that an insulator of constant thickness is optimal for such domains.

In §3 we show, via the direct method, that the *infimum* in (1.3) is attained for arbitrary smooth Ω. We produce the associated necessary conditions in §4, and, in §5, describe and test a numerical method.

Finally, let us recall that Buttazzo [B] has investigated the related problem of choosing h so to maximize the heat

$$\int_\Omega wf\,dx$$

where w is the temperature of the insulated body in the presence of the heat source f, i.e., w satisfies

$$\begin{aligned} -\Delta w &= f \quad x \in \Omega, \\ w + h\partial w/\partial n &= 0 \quad x \in \partial\Omega. \end{aligned}$$

Our analyses differ in that Buttazzo was able to establish existence of an optimal h without imposing a pointwise lower bound, i.e., he permits $\alpha = 0$. He produced explicit solutions for rods subject to nonnegative f and for planar annular domains under constant f. He did not address the numerical treatment of more general domains. It is exactly out of numerical necessity that we have imposed a pointwise lower bound on h. More precisely, the lack of a lower bound on h introduces infinite components (see, e.g., (5.3)) in the stiffness matrix stemming from the standard finite element approximation to (1.5). Moreover, our numerical results suggest that the bound, α, may always be chosen small enough, with respect to h_0, to guarantee that the best h, in the sense of (1.3), remains positively above α.

2. **Best Insulation of Balls**

If $\Omega = B_a$ is a ball of radius a in $\mathbf{R}^d$ then each radial solution of (1.9) is necessarily of the form

$$v(r;\zeta) = (\sqrt{\zeta}r)^{1-d/2} J_{d/2-1}(\sqrt{\zeta}r)$$

where ζ is a zero of

$$z \mapsto v(a;z) + h_0 \frac{|B_a|}{|\partial B_a|}\frac{dv}{dr}(a;z).$$

Let us denote these zeros, in increasing order, by $\{\zeta_{0,j}\}_{j=1}^\infty$. Note that these are exactly the radial eigenfunctions, and associated eigenvalues, of (1.2) with $h = h_0|B_a|/|\partial B_a|$.

Next we note that every nonradial Neumann eigenfunction of the Laplacian on B_a is also an eigenfunction of (1.9). Now, if v is an eigenfunction of (1.9) that is not a nonradial Neumann eigenfunction then, perforce, it is orthogonal to each of these and hence lies in the span of the radial Neumann eigenfunctions. The upshot being that v is radial. Having already characterized the radial eigenfunctions it follows that the spectrum of (1.9) is

$$\{\zeta_{0,j} : j = 1, \ldots, \infty\} \cup \{\mu_{i,j} : i = 1, \ldots, \infty,\ j = 1, \ldots, \infty\}$$

where the $\mu_{i,j}$ are Neumann eigenvalues (nondecreasing in j for fixed i) associated with nonradial eigenfunctions. It is not difficult to show that $\zeta_{0,1}$ is the smallest member of the spectrum and therefore $\zeta_1 = \zeta_{0,1}$. Observing now that (1.9) and (1.2) with $h = h_0|B_a|/|\partial B_a|$ have precisely the same radial eigenfunctions and associated eigenvalues we find $\xi_1(B_a, h_0|B_a|/|\partial B_a|) = \zeta_1(B_a, h_0|B_a|/|\partial B_a|)$. This fact, together with (1.8), yields

Proposition 2.1. *Insulators of constant thickness are best for balls. More precisely,* $\xi_1(h_0|B_a|/|\partial B_a|, B_a) \le \xi_1(h, B_a)$ *for all* $h \in ad_0(\partial B_a)$.

3. Existence of Optimal Insulators

The arguments of the previous section were of course custom fit to balls. Though $\zeta_1(\Omega, h_0) \le \xi_1(\Omega, h)$ for all $h \in ad_0(\partial\Omega)$ indeed holds for all smooth domains, Ω, our ignorance of $\zeta_1(\Omega, h_0)$ together with our doubt that equality may be attained for domains other than the disk, leads us to pursue less constructive means in the proof of existence of minimal h. In the following we shall drop the argument Ω when the domain is understood. For example, $\hat{\xi}_1$ will denote the value of the infimum in (1.3).

Proposition 3.1. *If Ω is smooth, open, bounded and connected then there exists an* $\hat{h} \in ad_0(\partial\Omega)$ *such that* $\xi_1(\hat{h}) = \hat{\xi}_1$.

Proof: Suppose that $\{h_n\} \subset ad_0(\partial\Omega)$ is a minimizing sequence, i.e., $\xi_1(h_n) \to \hat{\xi}_1$. As $\{h_n^{-1}\}$ is uniformly bounded in $L^\infty(\partial\Omega)$ it has a subsequence, that we neglect to relabel, that converges in the weak* topology to some function g, i.e., $h_n^{-1} \overset{*}{\rightharpoonup} g$. We write $h \equiv g^{-1}$ and suppose that u_n is the positive first eigenfunction, associated with h_n, normalized such that

$$\int_\Omega u_n^2\,dx = 1 \quad \text{and} \quad \int_\Omega |\nabla u_n|^2\,dx + \int_{\partial\Omega} h_n^{-1} u_n^2\,dx = \xi_1(h_n). \tag{3.1}$$

From (3.1) it follows that $\{u_n\}_n$ is bounded in $H^1(\Omega)$ and hence that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ and the traces $u_n|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$. These observations permit us to pass to the limit in the weak form

$$\int_\Omega \nabla u_n \cdot \nabla v\,dx + \int_{\partial\Omega} h_n^{-1} u_n v\,dx = \xi_1(h_n) \int_\Omega u_n v\,dx,$$

and so conclude that $\hat{\xi}_1$ and u constitute an eigenpair for h. As u is positive it follows that $\hat{\xi}_1 = \xi_1(h)$. It remains only to check whether h lies in ad_0. The pointwise bound offers no obstacle. Regarding the integral constraint we note that the convexity of $t \mapsto t^{-1}$ allows us to invoke Dacorogna [D, Theorem 1.1] in claiming

$$\int_{\partial\Omega} h\,dx = \int_{\partial\Omega} g^{-1}\,dx \le \liminf_n \int_{\partial\Omega} (h_n^{-1})^{-1}\,dx = h_0|\Omega|.$$

Hence, at worst, h uses less than all the available insulator. Now let $\hat{h}$ be any function that is pointwise greater than h but that lies in ad_0. From (1.6) it follows that $\xi_1(\hat{h}) \le \xi_1(h)$. Now, as $\hat{\xi}_1 = \xi_1(h)$, our claim follows. ∎

4. Necessary Conditions

We denote by $\hat{u}_1 \in H_1^1(\Omega)$ the positive eigenfunction associated with $\hat{h}$, note that one may always interchange the order of two minimizations, and so record

$$\xi_1(\hat{h}) = \mathcal{R}(\hat{u}_1, \hat{h}) = \min_{h \in ad_0} \min_{u \in H_1^1(\Omega)} \mathcal{R}(u, h) = \min_{u \in H_1^1(\Omega)} \min_{h \in ad_0} \mathcal{R}(u, h).$$

In other words,

$$\mathcal{R}(\hat{u}_1,\hat{h}) = \min_{u\in H_1^1(\Omega)} \mathcal{R}(u,\hat{h}) \quad \text{and} \quad \mathcal{R}(\hat{u}_1,\hat{h}) = \min_{h\in ad_0} \mathcal{R}(\hat{u}_1,h).$$

The former simply states that $\hat{u}_1$ is an eigenfunction corresponding to $\hat{h}$. The latter however informs us that $\hat{h}$ solves the much simpler optimization problem

$$\int_{\partial\Omega} \hat{h}^{-1}\hat{u}_1^2\,dx = \min_{h\in ad_0} \int_{\partial\Omega} h^{-1}\hat{u}_1^2\,dx. \tag{4.1}$$

We make it still simpler by exchanging the volume constraint for a Lagrange multiplier. More precisely, from the Lagrange Multiplier Rule of Clarke [Cl, Theorem 6.1.1] we deduce that (4.1) implies the existence of $\nu_1 \geq 0$ and $|\nu_1| + |\nu_2| > 0$ such that

$$\int_{\partial\Omega} \hat{h}^{-1}(\nu_1\hat{u}_1^2 + \nu_2\hat{h}^2)\,dx = \min_{h\in ad} \int_{\partial\Omega} h^{-1}(\nu_1\hat{u}_1^2 + \nu_2 h^2)\,dx, \tag{4.2}$$

where

$$ad \equiv \{h : \alpha|\Omega|/|\partial\Omega| \leq h(x)\}.$$

From $\nu_1\hat{u}_1^2 \geq 0$ we deduce from (4.2) that $\nu_2 < 0$. More precisely, if $\nu_2 > 0$ then the term in parentheses in (4.2) is positive and so (4.2) requires that $\hat{h} \equiv \infty$, in contradiction of the integral constraint.

Similarly, as $\nu_2 < 0$ if $\nu_1 = 0$ then (4.2) implies that $\hat{h} \equiv \alpha|\Omega|/|\partial\Omega|$, again in violation of the integral constraint.

As a result, $\nu^2 \equiv -\nu_2/\nu_1$ is well defined and we may deduce from (4.2), along the lines say of Cea and Malanowski [CM], the following pointwise necessary conditions,

$$\alpha|\Omega|/|\partial\Omega| = \hat{h}(x) \Rightarrow \hat{u}_1(x) \leq \nu\hat{h}(x), \tag{4.3}$$

$$\alpha|\Omega|/|\partial\Omega| < \hat{h}(x) \Rightarrow \hat{u}_1(x) = \nu\hat{h}(x). \tag{4.4}$$

With respect to the best insulated ball, where $\hat{h}$ and the trace of $\hat{u}_1$ are constant, we note that (4.4) is trivially satisfied along the entire boundary. In the general case, these conditions state that where $\hat{h}$ is off of its bound it coincides with (a multiple of) its associated eigenfunction, $\hat{u}_1$. Recalling (1.2), this implies that

$$\frac{\partial\hat{u}_1}{\partial n} = \text{constant} \tag{4.5}$$

there. Conditions on Ω sufficient to guarantee the solvability (together with questions pertaining to the inheritance of symmetry) of $-\Delta u = \xi u$ subject to (4.5) on $\partial\Omega$ have yet to be specified.

5. A Numerical Approach

As the analytical necessary conditions fail to characterize the optimal h we turn to its numerical approximation. Our approach is similar to that promoted by Cox and Uhlig [CU].

We write $\partial\Omega$ as the closure of the disjoint union of m open edges, $\{\Gamma_j\}_{j=1}^m$, and restrict h to

$$h(s) = \sum_{j=1}^m H_j 1_{\Gamma_j}(s),$$

where $H \in \mathbf{R}^m$ satisfies

$$\alpha|\Omega|/|\partial\Omega| \le H_j \quad \text{and} \quad \sum_{j=1}^m H_j|\Gamma_j| = h_0|\Omega|. \tag{5.1}$$

In order to compute ξ_1 at such an h we restrict our search to eigenvectors of the form

$$u(x) = \sum_{i=1}^p U_i T_i(x)$$

where $p < \infty$ and each T_i is a piecewise linear hat function. On substituting this expansion into the weak form (1.3) with v running through the T_i we arrive at the $p \times p$ eigensystem

$$(K + Q(H))U = \Xi M U, \tag{5.2}$$

where K and M are independent of H while

$$Q_{ij}(H) = \int_{\partial\Omega} T_i T_j / h\, dx = \sum_{k=1}^m \frac{1}{H_k}\int_{\Gamma_k} T_i T_j\, ds. \tag{5.3}$$

Let us denote the least eigenvalue of (5.2) by $\Xi_1(H)$. As this approximation procedure respects the symmetry of the original problem we retain a variational characterization,

$$\Xi_1(H) = \min_{U\cdot MU=1} \mathcal{R}(U,H), \qquad \mathcal{R}(U,H) \equiv U\cdot KU + U\cdot Q(H)U. \tag{5.4}$$

As the approximation is refined, i.e., as m and p become large, $\Xi_1(H)$ tends to $\xi_1(h)$ and so $\Xi_1(H)$ may be assumed simple. This, recalling Cox [Co], permits the computation of the gradient of Ξ_1 in terms of the representation in (5.4). More precisely,

$$\frac{\partial \Xi_1(H)}{\partial H_k} = U_H \cdot \frac{\partial Q(H)}{\partial H_k} U_H,$$

where, U_H is the minimizer (eigenvector) in (5.4) associated with $\Xi_1(H)$, and, recalling (5.3),

$$\frac{\partial Q_{ij}(H)}{\partial H_k} = \frac{-1}{H_k^2}\int_{\Gamma_k} T_i T_j\, ds.$$

With AD_0 denoting those H satisfying (5.1), we have now assembled all the ingredients necessary for a numerical attack on the finite dimensional optimization problem

$$\min_{H \in AD_0} \Xi_1(H). \tag{5.5}$$

We have implemented such an attack in Matlab, using `pdeeig`, within its Partial Differential Equations Toolbox, in the computation of $\Xi_1(H)$ and U_H, and `constr`, within its Optimization Toolbox, in the resolution of (5.5). We present detailed results for two model domains.

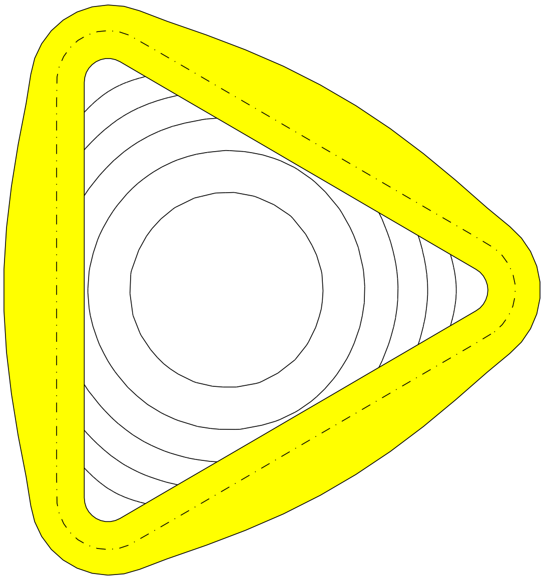

Figure 1. Best insulation, $\alpha = 0.1$.

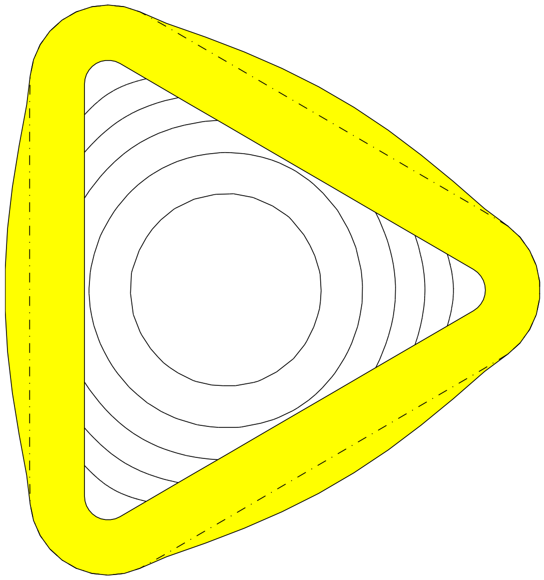

Figure 2. Best insulation, $\alpha = 0.2$.

In Figure 1 we have plotted a rounded equilateral triangle together with its associated best insulation for the parameter choices

$$\alpha = 0.1 \quad \text{and} \quad h_0 = 0.25. \tag{5.6}$$

The best H is here depicted by the shaded region. We have magnified it in order to better depict its variation in thickness. The contours within the domain are those of the associated eigenvector, U_H. The dot–dash line within the insulated region corresponds to the line $h(x) = \alpha|\Omega|/|\partial\Omega|$. Hence, given (5.6), the best H achievable by `constr` from a random start lies completely off of its lower bound. We have found that by increasing α one can arrive at an H that attains α. In particular, with

$$\alpha = 0.2 \quad \text{and} \quad h_0 = 0.25, \tag{5.7}$$

the best insulation of the rounded equilateral triangle assumes the form depicted in Figure 2. For each of these runs, we partitioned the boundary into $m = 60$ segments and partitioned the domain into $p = 1177$ elements.

As our final example we depart from convexity and consider the rounded cross. We plot it in Figure 3 along with its (magnified) best H and associated U_H contours under the parameter choice

$$\alpha = 0.05 \quad \text{and} \quad h_0 = 0.1. \tag{5.8}$$

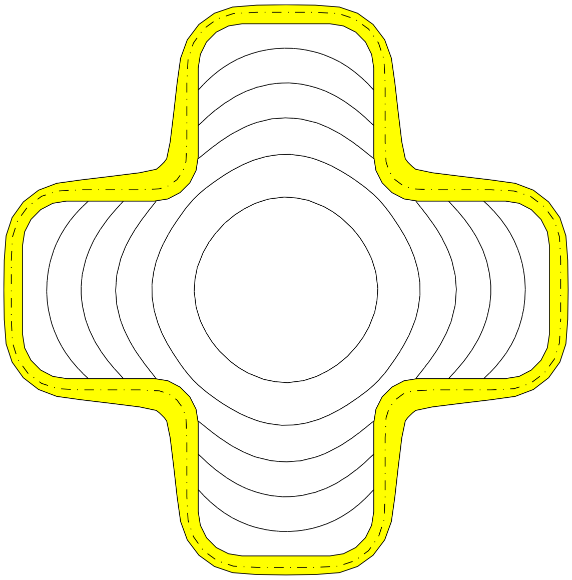

Figure 3. Best insulation of a cross.

As in the first case, the best insulator is everywhere off of its lower bound. In this case we partitioned the boundary into $m = 96$ segments and the domain into $p = 1913$ elements.

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7. References

[B] Buttazzo, G., *Thin insulating layers: The optimization point of view*, in *Material Instabilities in Continuum Mechanics and Related Problems*, J.M. Ball, ed., Oxford Univ. Press, Oxford, 1988, pp. 11-19.

[CM] Cea, J. and Malanowski, K., *An example of a max–min problem in partial differential equations*, SIAM J. Control 8(3), 1970, pp. 305–316.

[Cl] Clarke, F., *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.

[Co] Cox, S., *The generalized gradient at a multiple eigenvalue*, J. of Functional Analysis 33, 1995, pp.30–40.

[CU] Cox, S.J. and Uhlig, P.X., *Where best to hold the drum fast*, SIAM J. Optimization, to appear.

[D] Dacorogna, B., Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals, Lecture Notes in Math 922, Springer, Berlin, 1982.

[F] Friedman, A., *Reinforcement of the principal eigenvalue of an elliptic operator*, Arch. Rat. Mech. Anal. 73, 1980, pp. 1–17.

[GT] Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983.