CIRCULAR SYMMETRIZATION AND EXTREMAL ROBIN CONDITIONS

Steven J. Cox† and Bernhard Kawohl‡

† Department of Computational & Applied Mathematics, Rice University, 6100 Main St., Houston, TX 77005. `cox@rice.edu`

‡ Mathematisches Institut, Universität Köln, 50923 Köln. `kawohl@thales.mi.uni-koeln.de`

Abstract: We study the average temperature in a homogeneous disk subject to uniform heating in its interior and Newton's law of cooling, $hv + \partial v/\partial n = 0$, on its boundary. More precisely, among those h taking values in a prescribed interval, and of prescribed mean, we identify the minimizer and a maximizer of the average temperature. The latter characterization makes use of circular symmetrization. We include an elementary proof of the fact that such symmetrization may not increase the Dirichlet energy of any H^1 function on the disk.

Key words: Average temperature, third type, dual formulation.

AMS subject classifications: 35K20, 49J20, 49K20.

1. **Introduction**

We consider the following model problem on the disk, D, of radius 1. We specify a unit source of heat throughout D, immerse D in an ice bath, and suppose that heat flows across the boundary according to Newton's law of cooling,

$$hv + \partial v/\partial n = 0 \tag{1.1}$$

where v denotes temperature and h is conductivity. Our objective is to determine the numerical range of the average temperature in D as h varies over mixtures of two pure phases, h_1 and h_2, subject to a prescribed length fraction, γ. More precisely, we extremize, with respect to h, the average temperature

$$\mathcal{T}(h) \equiv \int_D v_h \, dx$$

where v_h satisfies (1.1) and

$$-\Delta v = 1 \quad \text{in } D, \tag{1.2}$$

and h is constrained to lie in

$$ad^*_\gamma \equiv \{h_1 \le h(x) \le h_2, \ \int_{\partial D} h \, ds = 2\pi(h_1\gamma + h_2(1-\gamma))\},$$

with γ lying strictly between 0 and 1. From the variational characterization

$$\mathcal{T}(h)=2\max_{v\in H^1(D)}E(v,h),\quad E(v,h)\equiv\int_D v\,dx-\tfrac{1}{2}\int_D|\nabla v|^2\,dx-\tfrac{1}{2}\int_{\partial D}hv^2\,ds,\tag{1.3}$$

we obtain the trivial bounds

$$\mathcal{T}(h_2)\leq\mathcal{T}(h)\leq\mathcal{T}(h_1)\qquad\forall h\in ad_\gamma^*.\tag{1.4}$$

We offer an immediate sharpening of the lower bound.

Proposition 1.1. *$\mathcal{T}$ attains its unique minimum at $\underline{h}_\gamma\equiv h_1\gamma+h_2(1-\gamma)$.*

Proof: The temperature associated with $\underline{h}_\gamma$ is

$$\underline{v}_\gamma=\frac{1}{2\underline{h}_\gamma}+\frac{1-r^2}{4}.$$

Now $\mathcal{T}(\underline{h}_\gamma)=2E(\underline{v}_\gamma,\underline{h}_\gamma)=2E(\underline{v}_\gamma,h)\leq\mathcal{T}(h)$ for each $h\in ad_\gamma^*$. If in fact $2E(\underline{v}_\gamma,h)=\mathcal{T}(h)$ for some admissible h then $\underline{v}_\gamma$ is the temperature for this h. As a result, $\underline{v}_\gamma$ and h satisfy (1.1). As $\underline{v}_\gamma$ is radial it follows that h must be constant. ∎

Now, to maximize $\mathcal{T}$ over ad_γ^* is simply to maximize E over $H^1(D)\times ad_\gamma^*$. We accomplish this through circular symmetrization of v. More precisely, we demonstrate that such rearrangement decreases the Dirichlet integral while preserving the mean and that if h is then subjected to the rearrangement antipodal to that of v we may decrease the boundary integral in E as well. We are using decrease here in its equivocal form – 'does not increase.' The decrease in these two terms of E can be traced back to results of Polya and Szegö [9]. Their bare–handed proof of the decrease of the Dirichlet integral, [9, p. 154], however, was conducted only for "sufficiently smooth" v. Sperner [10], replacing the arguments of Polya and Szegö with a systematic use of geometric measure theory, offered a proof for Lipschitz v. Though Kawohl [8, Thm 2.31] revisited the original argument he did not completely free it of nagging smoothness hypotheses. For this reason we offer, in §2, a selfcontained, elementary proof that circular symmetrization decreases the Dirichlet integral of every $H^1(D)$ function.

In §3 we return to the maximization of $\mathcal{T}$. We show that $\mathcal{T}$ is convex and that it attains its maximum at an extreme point of ad_γ^*. Given the symmetry established in §2 this maximal h, call it $\overline{h}_\gamma$, is shown to take the value h_1 on a circular arc of length $2\pi\gamma$ and the value h_2 on the complementary arc. The dual characteriztion of $\mathcal{T}$ produces a simple upper bound for $\mathcal{T}(\overline{h}_\gamma)$. Combining this bound with that of Prop. 1.1, yields

$$\frac{\pi}{2}\left(\frac{1}{h_1\gamma+h_2(1-\gamma)}+\frac{1}{4}\right)\leq\mathcal{T}(h)\leq\frac{\pi}{2}\left(\frac{\gamma}{h_1}+\frac{1-\gamma}{h_2}+\frac{1}{4}\right)\qquad\forall h\in ad_\gamma^*.\tag{1.5}$$

Regarding existing studies of extremal Robin problems we can cite only Buttazzo [3] and Cox and Uhlig [4]. Though Buttazzo also sought to maximize the average temperature he was working in a context where it was natural to specify not the mean of h but the mean of its reciprocal. Roughly speaking, his maximization problem then corresponds to our minimization problem. In particular, his maximizing h is constant when the conductor is a disk. Unlike the present study, Buttazzo treats conductors of arbitrary shape.

2. Circular Symmetrization

Given $v \in H^1(D)$ its polar representative $u(r,t) \equiv v(x)$, where $x = r(\cos t, \sin t)$, is defined on $B = \{(r,t) : 0 < r < 1,\ -\pi < t \leq \pi\}$. Given a set $A \subset (-\pi, \pi]$ we denote by $A^* = (-|A|/2, |A|/2)$ the symmetrization of A with respect to 0. Applying this to u at each fixed r produces the symmetrically increasing and decreasing rearrangements

$$u^\vee(r,t) \equiv \inf\{c : t \in \{\theta : u(r,\theta) \leq c\}^*\} \quad \text{and} \quad u^\wedge(r,t) \equiv u^\vee(r, \pi - |t|).$$

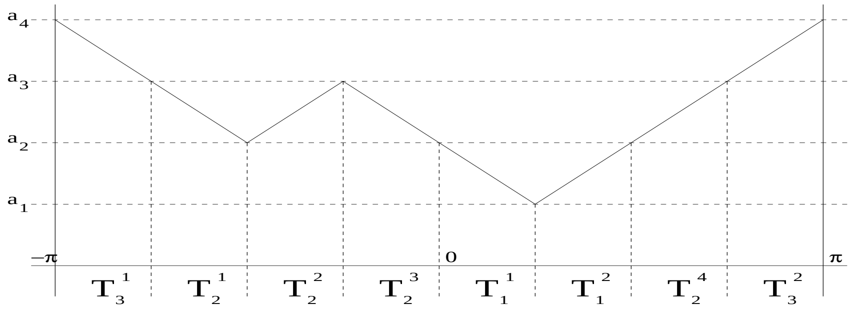

Figure 1. The graph of $t \mapsto u(r,t)$.

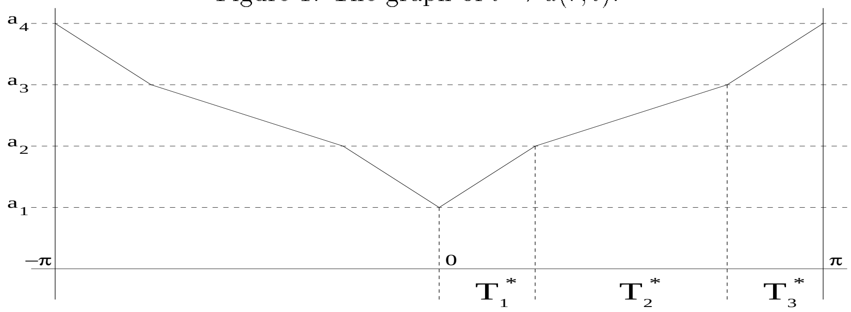

Figure 2. The graph of $t \mapsto u^\vee(r,t)$.

In figures 1 and 2 we have plotted a piecewise affine function and its symmetrically increasing rearrangement. The levels, a_j, and subintervals, T_j^i, will appear in our proof of the decrease of the Dirichlet integral.

The corresponding circular rearrangements of v are simply $v^\vee(x) = u^\vee(r,t)$ and $v^\wedge(x) = u^\wedge(r,t)$.

Proposition 2.1. *If $v \in H^1(D)$ then $v^\vee \in H^1(D)$ and*

$$\int_D |\nabla v|^2\,dx \geq \int_D |\nabla v^\vee|^2\,dx \tag{2.1}$$

Proof: As $v \in H^1(D)$ it follows from Adams [1, Thm. 3.18] that there exists a sequence $\{v_n\} \subset C^\infty(\overline{D})$ for which $v_n \to v$ in $H^1(D)$. Now, if (2.1) is assumed to hold for each v_n then it follows that $\{v_n^\vee\}$ is uniformly bounded in $H^1(D)$. As a result there exists a $v_\infty \in H^1(D)$ and a subsequence $\{v_{n_k}^\vee\} \subset \{v_n^\vee\}$ such that $v_{n_k}^\vee \to v_\infty$ weakly in $H^1(D)$ and strongly in $L^2(D)$. As $\|v_{n_k}^\vee - v^\vee\|_2 \leq \|v_{n_k} - v\|_2$, see, e.g., [8, p. 25], it follows that $v_\infty = v^\vee$. Now, recalling the weak lower semicontinuity of the Dirichlet integral, we arrive at

$$\int_D |\nabla v_\infty|^2\,dx \leq \liminf_{n_k} \int_D |\nabla v_{n_k}^\vee|^2\,dx \leq \liminf_{n_k} \int_D |\nabla v_{n_k}|^2\,dx = \int_D |\nabla v|^2\,dx.$$

Hence, it suffices to prove (2.1) for $v \in C^\infty(\overline{D})$. For such v the corresponding polar representative, $u(r,t) \equiv v(x)$, lies in $C(\overline{B}) \cap C^\infty(B)$. Recalling Ekeland and Temam [6, Thm. X.2.1] there exists a sequence of piecewise affine functions, $\{u_n\}$, such that $u_n \to u$ uniformly in B, $\nabla u_n \to \nabla u$ locally uniformly in B and $\|\nabla u_n\|_\infty \leq \|\nabla u\|_\infty$. Arguing as above, we see that it suffices to prove

$$\int_B \{u_r^2 + (u_t/r)^2\}\,r\,dr\,dt \geq \int_B \{(u_r^\vee)^2 + (u_t^\vee/r)^2\}\,r\,dr\,dt \tag{2.2}$$

for piecewise affine u. We shall find it convenient to suppose that

$$u_t(r,t) \neq 0 \quad \text{a.e. in } B. \tag{2.3}$$

If u should fail this test we simply turn our attention to $u^\varepsilon \equiv u + \varepsilon|t|$, where ε is positive but less than the minimum of the (finite number) of nonzero values of $|u_t|$.

Now, for fixed r, the function $t \mapsto u(r,t)$ is differentiable except at, at most, a finite number of points, $\{t_j(r)\}_{j=1}^M$. We label the associated values $a_j(r) = u(r, t_j(r))$ and suppose that the t_j are ordered in such a way that a_j increases with j. We then define

$$T_j(r) \equiv \{t \in (-\pi, \pi] : a_j < u(r,t) < a_{j+1}\} \quad j = 1, \ldots, M-1.$$

Associated to each $T_j(r)$ we define

$$N_j(r,a) \equiv \text{cardinality of } \{t : u(r,t) = a\}$$

for those a satisfying

$$a_j(r) < a < a_{j+1}(r). \tag{2.4}$$

By construction, this number is even and *independent of* a. Accordingly, we partition $T_j(r)$ into $N_j(r)$ subintervals, $\{T_j^i(r)\}$, on each of which $t \mapsto u_t(r,t)$ does not change sign while on adjacent (in i) intervals $u_t(r,t)$ assumes different signs. Recall that the a_j and the T_j^i are labeled for a concrete example in fig. 1. The upshot is that $t \mapsto u(r,t)$ is now invertible on each T_j^i. Accordingly, the implicit function theorem guarantees the existence of a smooth function

$$\tau_j^i : (r-\varepsilon, r+\varepsilon) \times (a_j(r), a_{j+1}(r)) \mapsto T_j^i$$

for which

$$u(r, \tau_j^i(r,a)) = a. \tag{2.5}$$

Upon differentiating this equation we find

$$\frac{\partial u}{\partial t}(r,t) = \frac{1}{\partial \tau_j^i/\partial a} \quad \text{and} \quad \frac{\partial u}{\partial r}(r,t) = \frac{-\partial \tau_j^i/\partial r}{\partial \tau_j^i/\partial a}. \tag{2.6}$$

Regarding the rearrangement of u, we note that

$$u^\vee(r, \pm\tau_j^*(r,a)) = a, \quad a_j(r) < a < a_{j+1}(r), \tag{2.7}$$

where

$$\tau_j^*(r,a) \equiv \frac{1}{2}|\{t : u(r,t) \leq a\}|, \quad a_j(r) < a < a_{j+1}(r).$$

We note that $t \mapsto u^\vee(r,t)$ is affine on each

$$T_j^* = \{t > 0 : a_j < u^\vee(r,t) < a_{j+1}\}, \quad j = 1, \ldots, M-1,$$

as seen, e.g., in figure 2. If $u_t(r,t) < 0$ for $t \in T_j^1$ then it is not difficult to show that

$$\tau_j^*(r,a) = \frac{1}{2}\sum_{i=1}^{N_j}(-1)^i \tau_j^i(r,a) \tag{2.8}$$

while if $u_t(r,t) > 0$ for $t \in T_j^1$ then (2.8) holds with the opposite sign. Without loss we shall assume (2.8). For $t \in T_j^*$ we may differentiate (2.7) and arrive at

$$\frac{\partial u^\vee}{\partial t} = \frac{1}{\partial \tau_j^*/\partial a} \quad \text{and} \quad \frac{\partial u^\vee}{\partial r} = \frac{-\partial \tau_j^*/\partial r}{\partial \tau_j^*/\partial a}. \tag{2.9}$$

In order to express these derivatives in terms of τ_j^i we simply differentiate (2.8). In fact, we require only,

$$\left|\frac{\partial \tau_j^*}{\partial a}\right| = \frac{1}{2}\sum_{i=1}^{N_j}\left|\frac{\partial \tau_j^i}{\partial a}\right| \quad \text{and} \quad \left|\frac{\partial \tau_j^*}{\partial r}\right| = \frac{1}{2}\left|\sum_{i=1}^{N_j}(-1)^i\frac{\partial \tau_j^i}{\partial r}\right|, \tag{2.10}$$

for, to establish (2.2) we need only show, for each j, that

$$\int_{T_j}(u_r^2 + (u_t/r)^2)\,dt \geq \int_{T_j^*}((u_r^\vee)^2 + (u_t^\vee/r)^2)\,dt$$

or, equivalently, that

$$\int_{a_j}^{a_{j+1}}\sum_{i=1}^{N_j}\left\{\left|\frac{\partial \tau_j^i}{\partial r}\right|^2 + \frac{1}{r^2}\right\}\left|\frac{\partial \tau_j^i}{\partial a}\right|^{-1} da \geq 2\int_{a_j}^{a_{j+1}}\left\{\left|\frac{\partial \tau_j^*}{\partial r}\right|^2 + \frac{1}{r^2}\right\}\left|\frac{\partial \tau_j^*}{\partial a}\right|^{-1} da. \tag{2.11}$$

This will be done by pointwise comparison of the integrands. In particular,

$$\sum_{i=1}^{N_j}\left\{\left|\frac{\partial \tau_j^i}{\partial r}\right|^2 + \frac{1}{r^2}\right\}\left|\frac{\partial \tau_j^i}{\partial a}\right|^{-1} \geq \left(\sum_{i=1}^{N_j}\left|\frac{\partial \tau_j^i}{\partial a}\right|\right)^{-1}\left[\sum_{i=1}^{N_j}\left\{\left|\frac{\partial \tau_j^i}{\partial r}\right|^2 + \frac{1}{r^2}\right\}^{1/2}\right]^2 \tag{2.12}$$

$$\geq \left(\sum_{i=1}^{N_j}\left|\frac{\partial \tau_j^i}{\partial a}\right|\right)^{-1}\left(\left|\sum_{i=1}^{N_j}(-1)^i\frac{\partial \tau_j^i}{\partial r}\right|^2 + \frac{N_j^2}{r^2}\right) \tag{2.13}$$

$$= \frac{1}{2}\left|\frac{\partial \tau_j^*}{\partial a}\right|^{-1}\left(4\left|\frac{\partial \tau_j^*}{\partial r}\right|^2 + \frac{N_j^2}{r^2}\right) \tag{2.14}$$

$$\geq 2\left\{\left|\frac{\partial \tau_j^*}{\partial r}\right|^2 + \frac{1}{r^2}\right\}\left|\frac{\partial \tau_j^*}{\partial a}\right|^{-1}. \tag{2.15}$$

The first inequality, (2.12), is Cauchy's. More precisely, if x and y both lie in the strict positive orthant of $\mathbf{R}^{N_j}$ then one may define their componentwise square roots, $\sqrt{x}$ and $\sqrt{y}$, as well as the componentwise quotient, $\sqrt{x}/\sqrt{y}$. In terms of the standard inner product and norm on $\mathbf{R}^{N_j}$, (2.12) is simply

$$\langle\sqrt{x}/\sqrt{y}, \sqrt{y}\rangle \leq \|\sqrt{x}/\sqrt{y}\|\|\sqrt{y}\|.$$

The second inequality, (2.13), is the triangle inequality in $\mathbf{R}^2$. Namely,

$$\sum_{i=1}^{N_j}\|z_i\| \geq \left\|\sum_{i=1}^{N_j} z_i\right\| \quad \text{where} \quad z_i = \left[(-1)^i\frac{\partial \tau_j^i}{\partial r} \quad 1/r\right].$$

The equality, (2.14), follows immediately from (2.10). The final equality, (2.15), follows on recalling that $N_j \geq 2$. This completes the proof of the Theorem. ∎

As $v^\wedge$ is merely a rotation of $v^\vee$ they possess the same Dirichlet integral, and so

$$\|\nabla v^\wedge\|_2 \leq \|\nabla v\|_2, \tag{2.16}$$

as well. We note that for equality to hold in (2.2) for piecewise affine u equality must hold in (2.15), i.e., N_j must equal two for each j. If such is the case then unless

$$\partial \tau_j^1/\partial r = -\partial \tau_j^2/\partial r \qquad \forall\, j, \tag{2.17}$$

we find strict inequality in (2.12) and (2.13). If however $N_j = 2$ and (2.17) holds for each j then $t \mapsto u^\vee(r,t)$ is merely a rotation of $t \mapsto u(r,t)$. More generally, Denzler [5] has recently shown that equality holds in (2.16) for $v \in C^\infty(D)$ iff $v^\wedge$ is a rotation of v.

3. Maximizing $\mathcal{T}$

From (2.16) comes the fact that symmetrically increasing h produce the largest average temperatures.

Proposition 3.1. $\mathcal{T}(h) \leq \mathcal{T}(h^\vee)$.

Proof: Given $h \in ad_\gamma^*$ we denote its associated temperature by v_h. As $v_h^\wedge$ and $h^\vee$ are oppositely ordered it follows from [9, p. 153] that

$$\int_{\partial D} h^\vee |v_h^\wedge|^2\, ds \leq \int_{\partial D} h v_h^2\, ds.$$

This, together with (2.16), explains $\mathcal{T}(h) = 2E(v_h, h) \leq 2E(v_h^\wedge, h^\vee) \leq \mathcal{T}(h^\vee)$. ∎

For arbitrary $h \in ad_\gamma^*$ nothing, beyond the fact that it is symmetrically increasing, can be said of $h^\vee$. If, however, h is drawn from the set of classical mixtures

$$ad_\gamma \equiv \{h = h_1\chi + h_2(1-\chi),\ \chi(x) \in \{0,1\},\ \int_{\partial D} \chi\, ds = 2\pi\gamma\},$$

then

$$h^\vee(t) = \overline{h}_\gamma(t) = \begin{cases} h_1 & \text{if } |t| < \gamma\pi, \\ h_2 & \text{otherwise.} \end{cases}$$

We now establish the fact that, when maximizing $\mathcal{T}$ over ad_γ^*, it suffices to search only over ad_γ.

Proposition 3.2. $\mathcal{T}(h) \leq \mathcal{T}(\overline{h}_\gamma)$ *for each* $h \in ad_\gamma^*$.

Proof: We recall, see, e.g., Bauer [2], that a bounded convex function on a compact convex set attains its maximum at an extreme point. The convexity of ad_γ^* is obvious. An upper bound on $\mathcal{T}$ was given in (1.4). As $\mathcal{T}$, recall (1.3), is a maximum of functions which are affine in h it follows that $\mathcal{T}$ is convex. Finally, Friedland [7] has shown that ad_γ^* is compact with respect to the weak* topology on $L^\infty(\partial D)$ and that ad_γ comprises its set of extreme points. Hence, the maximum of $\mathcal{T}$ over ad_γ is no less than the maximum of $\mathcal{T}$ over ad_γ^*. It follows from Prop. 3.1 that the former is achieved at $\overline{h}_\gamma$. ∎

Having characterized the maximizer let us now consider the maximum value, $\mathcal{T}(\overline{h}_\gamma)$. Unable to produce this number by hand we resort to numerical approximation along the lines established by Uhlig [11]. More precisely, in the context of Matlab's PDE toolbox, we

(i) Define the polygon with vertices $(\cos(2\pi j/N), \sin(2\pi j/N))$ for $j = 1, \ldots, N$.

(ii) Triangulate the polygon.

(iii) Assemble and solve the corresponding finite element representation of $-\Delta v = 1$ subject to $h_1 v + \partial v/\partial n = 0$ on γN contiguous boundary edges and $h_2 v + \partial v/\partial n = 0$ on the remainder. This returns v_N, an approximation to the temperature at the triangle vertices.

(iv) Compute $\tilde{v}_N$, the interpolation onto triangle centers of v_N. Compute A_N, the vector of triangle areas. Finally, the approximate average temperature is the scalar product $\mathcal{T}_N(\overline{h}_\gamma) = \tilde{v}_N \cdot A_N$.

In figure 3 we indicate the performance of steps (i)–(iii) by plotting the contours of the corresponding v_N for the following parameter values:

$$h_1 = 1 \quad h_2 = 10 \quad \gamma = .5 \quad N = 150.$$

The corresponding triangulation consisted of 2751 vertices.

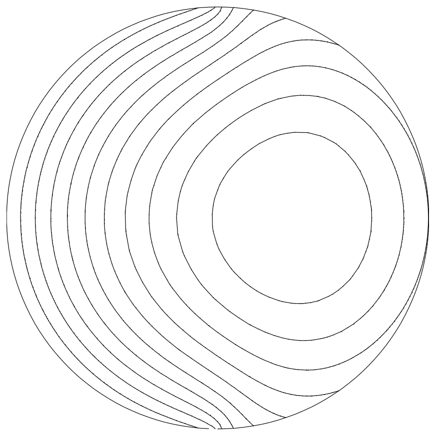

Figure 3. Isotherms associated with $\overline{h}_{1/2}$.

Repeating this calculation for $\gamma_j = j/150$, $j = 0, \ldots, 150$, and implementing step (iv) in each case, we arrive at the set of pairs $\{\gamma_j, \mathcal{T}_N(\overline{h}_{\gamma_j})\}$ constituting the upper curve in figure 4.

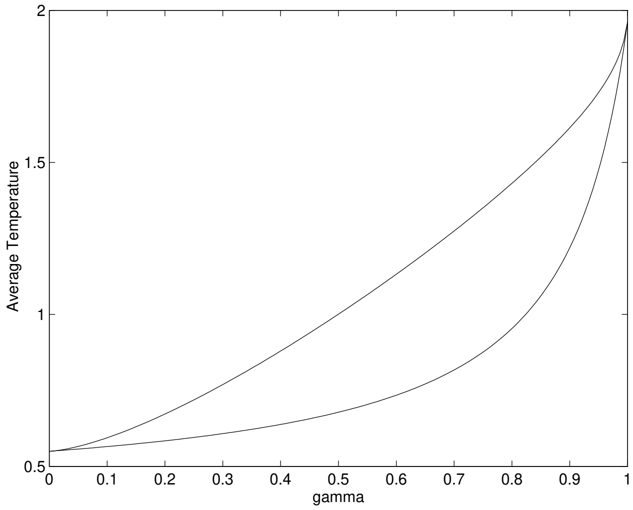

Figure 4. Minimal and Maximal average temperatures *vs.* γ.

For comparison purposes we have also plotted the minimal average temperature. Recall that Prop. 1.1 yields the exact value

$$\mathcal{T}(\underline{h}_\gamma) = \int_D \underline{v}_\gamma \, dx = \frac{\pi}{2}\left(\frac{1}{h_1\gamma + h_2(1-\gamma)} + \frac{1}{4}\right).$$

This is obviously convex in γ. The figure indicates that this is also true of $\gamma \mapsto \mathcal{T}(\overline{h}_\gamma)$. An immediate consequence of such convexity is

$$\mathcal{T}(h) \le \mathcal{T}(\overline{h}_\gamma) \le \gamma\mathcal{T}(\overline{h}_1) + (1-\gamma)\mathcal{T}(\overline{h}_0) = \frac{\pi}{2}\left(\frac{\gamma}{h_1} + \frac{1-\gamma}{h_2} + \frac{1}{4}\right) \qquad (3.1)$$

for each $h \in ad^*_\gamma$. Although we have not been able to analytically confirm the convexity of $\gamma \mapsto \mathcal{T}(\overline{h}_\gamma)$ we may nonetheless derive (3.1) as a consequence of the dual characterization of $\mathcal{T}$.

Proposition 3.3. $\mathcal{T}(h) = \min\limits_{\substack{q\in L^2(D)^2 \\ \nabla\cdot q = 1}} \int_D |q|^2 \, dx + \int_{\partial D} \frac{(q\cdot n)^2}{h} \, ds.$

Proof: This is a direct application of Ekeland and Temam [6, Thm. III.4.2]. We set $V = H^1(D)$ and $Y = L^2(D)^2 \times L^2(\partial D)$ and define

$$\Lambda : V \to Y, \quad \Lambda u = (u_{x_1}, u_{x_2}, \sqrt{h}u|_{\partial D})$$

Now, following the sign convention of [6] we write our original characterization, (1.3), as

$$-\mathcal{T}(h) = 2\min_{u\in V}\{F(u) + G(\Lambda u)\} \quad \text{where} \quad F(u) = -\int_D u \, dx \quad \text{and} \quad G(p) = \tfrac{1}{2}\|p\|_Y^2.$$

Invoking [6, Remark III.4.2] and shadowing their Neumann example, [6, §IV.2], we find

$$-\mathcal{T}(h) = 2 \max_{p^* \in Y^*} \{-F^*(\Lambda^* p^*) - G^*(-p^*)\}, \tag{3.2}$$

where Y^*, the dual of Y, is simply Y itself while

$$F^*(\Lambda^* p^*) = \begin{cases} 0 & \text{if } \Lambda^* p^* + 1 = 0 \\ +\infty & \text{otherwise.} \end{cases} \quad \text{and} \quad G^*(p^*) = \tfrac{1}{2}\|p^*\|_Y^2. \tag{3.3}$$

The constraint, $\Lambda^* p^* + 1 = 0$, reads $\langle 1, v\rangle + \langle p^*, \Lambda v\rangle = 0$ for each $v \in V$, i.e.,

$$\langle 1, v\rangle + \langle p_1^*, v_{x_1}\rangle + \langle p_2^*, v_{x_2}\rangle + \langle p_3^*, \sqrt{h} v|_{\partial D}\rangle = 0 \qquad \forall\, v \in V.$$

Setting $q^* \equiv (p_1^*, p_2^*)$ and integrating by parts brings

$$\int_D \{1 - \nabla \cdot q^*\} v\, dx + \int_{\partial D} \{q^* \cdot n + \sqrt{h} p_3^*\} v\, ds = 0 \qquad \forall\, v \in V.$$

Hence, $\nabla \cdot q^* = 1$ and $p_3^* = -(q^*|_{\partial D} \cdot n)/\sqrt{h}$. Substituting this into (3.3) and (3.2) we arrive at

$$-\mathcal{T}(h) = \max_{\substack{q^* \in L^2(D)^2 \\ \nabla \cdot q^* = 1}} \left\{ -\int_D |q^*|^2\, dx - \int_{\partial D} (q^* \cdot n)^2/h\, ds \right\}.$$

Dropping the * and multiplying through by -1 completes the derivation. ∎

Taking $q(x) = x/2$ and $h = \overline{h}_\gamma$ in the dual formulation brings

$$\mathcal{T}(\overline{h}_\gamma) \le \frac{\pi}{8} + \frac{1}{4} \int_{\partial D} \frac{1}{\overline{h}_\gamma}\, ds = \frac{\pi}{2} \left(\frac{\gamma}{h_1} + \frac{1-\gamma}{h_2} + \frac{1}{4} \right), \tag{3.4}$$

and so establishes (3.1).

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5. References

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Bauer, H., *Sur le prolongement des formes linéaire positives dans un espace vectorial ordonné*, C.R. Acad. Sci. Paris **244**, 1957, pp. 289–292.

[3] Buttazzo, G., *Thin insulating layers: The optimization point of view*, in *Material Instabilities in Continuum Mechanics and Related Mathematical Problems*, J.M. Ball, ed., Oxford University Press, Oxford, 1988, pp. 11–19.

[4] Cox, S.J. and Uhlig, P.X., *Where best to hold a drum fast*, SIAM J. on Optimization, to appear.

[5] Denzler, J., *Windows of given area with minimal heat diffusion*, Trans. A.M.S., to appear.

[6] Ekeland, I. and Temam, R., *Convex Analysis and Variational Problems*, North–Holland, Amsterdam, 1976.

[7] Friedland, S., *Extremal eigenvalue problems defined for certain classes of functions*, Arch. Rat. Mech. Anal., **67**(1), 1977, pp. 73–81.

[8] Kawohl, B., *Rearrangements and Convexity of Level Sets in PDE*, LNM # 1150, Springer, Berlin, 1985.

[9] Pólya, G. and Szegö, G., *Isoperimetric Inequalities in Mathematical Physics*, Annals of Math. Studies, no. 27, Princeton, 1951.

[10] Sperner, E., *Spherical symmetrization and eigenvalue estimates*, Math. Z. 176, 1981, pp. 75–86.

[11] Uhlig, P., *Numerical methods for extremal Robin problems*, Tech. Report, Dept. Computational & Appl. Math., Rice University, 1997.