Convexity and Starshapedness of Level Sets for Some Nonlinear Parabolic Problems

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by

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Abstract. *We consider some degenerate parabolic problems on a convex (or starshaped) ring and investigate the convexity (and starshapedness) of level sets of solutions. Our results imply in particular the convexity (or starshapedness) of certain free boundaries in space and time. Other nonlinear parabolic problems are also discussed.*

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§1. Introduction

In this paper we study some qualitative properties of the level sets of certain nonlinear parabolic problems. We show that some qualitative properties of the initial data $u_0(x)$, namely convexity and starshapedness of the level sets $\{x \in \Omega | u_0(x) \geq c\}$ are preserved for all positive times. Both properties have been extensively studied in the last years for elliptic equations, but for parabolic problems only few results of this type seem to be mentioned in the literature. We shall comment on these at the end of this introduction.

Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and let $G \subset \Omega$ be a compact, simply connected set with smooth boundary ∂G. Consider the following nonlinear degenerate diffusion problem (P) in a ring domain $\Omega \setminus G$.

$$(P)\begin{cases} u_t - \Delta_p u + f(u) = 0 & \text{in } (0,\infty) \times (\Omega \setminus G), \quad (1.1) \\ u \equiv 1 & \text{on } (0,\infty) \times G, \quad (1.2) \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \quad (1.3) \\ u(0,x) = u_0(x) & \text{in } \Omega, \quad (1.4) \end{cases}$$

where

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) \qquad \text{and } p > 1.$$

Notice that problem (P) is really a boundary value problem on $(0,\infty)\times$ $(\Omega \setminus G)$ with constant boundary values on the interior boundary, and that for $p = 2$ the pseudo-Laplace operator Δ_p becomes the Laplace operator. Problems of this type are related to the study of non-Newtonian fluids (see Diaz and Herrero [21]). Throughout the paper we shall assume that f is the sum of a monotone nondecreasing (or maximal monotone) map, and a Lipschitz continuous map, and that $f(s) \geq 0$ for $s \geq 0$. Moreover the initial condition (1.4) and boundary condition (1.2) should be compatible in the sense that $u_0(x) \equiv 1$ on G, and $u_0 = 0$ on $\partial\Omega$.

The term $f(u)$ describes absorption phenomena and is of particular interest in the study of chemical catalysts. Aris [2] gives $f(u) = u^q$ as a typical example for a reaction of order q. If $q < p - 1$ the absorption dominates the diffusion and the solution u can be zero in a subset of $\Omega \setminus G$; this set is then called the **dead core.** Existence, uniqueness and regularity results for problem (P) as well as for more general problems have been provided by many authors. We refer for instance to Alt-Luckhaus [1] and its references.

The main objective of this paper is to prove that for every fixed $t > 0$ the level sets $\Omega_c(t) := \{x \in \Omega | u(t,x) \geq c\}$ are convex, provided they are convex for $t = 0$. As a preliminary step we first establish the starshapedness of these level sets in Section 2. Notice that the p-Laplacian operator is degenerate where $|\nabla u| = 0$ and $p \neq 2$. We shall in fact prove under natural assumptions on u_0 and f that $x \cdot \nabla u < 0$ in $(0,T) \times (\Omega \setminus G)$. For $p = 2$ the proof of this result is easy. For $p \neq 2$, however, the arguments are quite involved and make use of suitable barrier functions which were inspired by the work of Lewis [37]. Incidentally, starshapedness of level sets implies Lipschitz continuity of their boundaries. In particular, if there are free boundaries, they have to be Lipschitz continuous. One can also look at level sets $\Omega_c := \{(t,x) \in (0,\infty) \times \Omega | u(t,x) \geq c\}$ in space and time and investigate their starshapedness. This is also done in Section 2, see Remark 1 and Proposition 1.

In Section 3 we study the convexity of level sets. A nonnegative continuous function w on a convex set C has convex level sets $\{z \in C | w(x) \geq c, c \in I\!R\}$ if and only if

$$w((z_1 + z_2)/2) \geq \min\{w(z_1), w(z_2)\} \quad \text{in } C \times C.$$

In this case w is called **quasiconcave**. Notice that concavity implies quasiconcavity, but quasiconcavity does not imply concavity. In fact there are strictly convex functions which are quasiconcave. There are two ways of defining the convex set C in our context. Either we interpret $(0,\infty) \times \Omega$ as a convex set C and then attempt to show that the quasiconcavity function

$$Q(t_1, t_2, x_1, x_2) := u((t_1 + t_2)/2, (x_1 + x_2)/2) - \min\{u(t_1, x_1), u(t_2, x_2)\}$$

is nonnegative for any pair $(t_1, x_1), (t_2, x_2)$ of points in $(0,\infty) \times \Omega$. This would imply the convexity of the level sets Ω_c in space and time. Or we fix a positive t and try to prove the convexity of level sets $\Omega_c(t)$ in space only. In this case $C = \{t\} \times \Omega$ and one would have to show that the quasiconcavity function

$$Q_t(x_1, x_2) := u(t, (x_1 + x_2)/2) - \min\{u(t, x_1), u(t, x_2)\}$$

is nonnegative for any fixed $t > 0$ and any pair x_1, x_2 of points in Ω.

This approach has its origin in work of Gabriel [27] on the convexity of level sets of harmonic functions. Some variants of it have been used in the stationary setting by several authors: Lewis [37] considered p-harmonic functions on convex rings, Caffarelli and Spruck [14] as well as Kawohl [30] treated the case $p = 2$, f and f' nonnegative, and in [31] the case $p \neq 2$, f and f' nonnegative was discussed under a nondegeneracy assumption. For parabolic problems on "convex rings" we are aware only of the results of Borell [9]. He studied the special case $p = 2$, $f \equiv 0$ and $u_0 \equiv 0$ in $\Omega \setminus G$, and we shall generalize it to f and f' nonnegative. The case of nontrivial u_0 or $p \neq 2$ can also be handled under some technical assumptions which seem to be hard to verify. We shall comment on those difficulties and on the problems connected with proving that Q_t is nonnegative.

Our methods of proof also apply to some other nonlinear parabolic problems. The results on starshapedness remain valid for interior obstacle problems or for (1.1) replaced by

$$u_t - \Delta(\phi(u)) + f(u) = 0 \tag{1.5}$$

with ϕ monotone nondecreasing. If in addition ϕ is concave, such as in fast diffusion problems, we also prove convexity of level sets. In fact, since (1.5) includes equation (1.1) for $p = 2$, our exposition in Section 3 will first exhibit the results for problem (P_ϕ), i.e. Problem (P) with (1.5) instead of (1.1), and then for problem (P).

To conclude the introduction let us briefly make some bibliographical remarks. Friedman and Kinderlehrer proved starshapedness of the free boundary for the Stefan problem [25]. This is a special case of equation (1.5) with $\phi(s) = k(s-a)^+$ and k and a positive. In [10] Brascamp and Lieb showed that the linear heat equation under Dirichlet boundary conditions (on an interior convex domain Ω) preserves log-concavity of the initial data. Therefore $v = -\log u$ is a convex function of x for any fixed $t > 0$,provided tis is true for $t = 0$. It is easy to see that log-concavity is stronger than quasiconcavity and weaker than concavity. Using a Trotter-Kato product formula P.L. Lions [38] extended their results to positive solutions of semilinear equations

$$u_t - \Delta u + f(u) = 0 \quad \text{in } (0,\infty) \times \Omega$$

with f satisfying $f''(s)s - f'(s)s + f(s) \geq 0$ for $s \geq 0$, and with convex Ω and vanishing Dirichlet data. Korevaar gave a different and more general proof of this result in [35] by introducing a concavity function. A slight extension of his result on parabolic equations is remarked in [33].

The problems which were treated by Caffarelli and Friedman [13], Kawohl [33] and Friedman and Phillips [26] are interior problems and require totally different methods than the ones presented here.

§2. Starshapedness of level sets

A set $D \subset I\!R^n$ is called **starshaped with respect to** $x^0 \in D$ iff for any $x \in D$ the line segment $\{y = \lambda x^0 + (1-\lambda)x, 0 \leq \lambda \leq 1\}$ is contained in D. For brevity of notation we call a set **starshaped** iff it is starshaped with respect to the origin.

Theorem 1. *Let $G \subset\subset \Omega$ and G and Ω be starshaped. Let $p > 1$ and let f be the sum of a continuous nondecreasing and a Lipschitz-continuous function and suppose that $f(s) \geq 0$ for $s \geq 0$. Let $u_0 \in W^{1,p}(\Omega \setminus G)$ be given and suppose*

$$0 \leq u_0 \leq 1 \quad \text{in } \Omega, \quad u_0 \equiv 1 \quad \text{on } G, \tag{2.1}$$

$$\text{the level sets of } u_0 \text{ are starshaped ,} \tag{2.2}$$

$$\Delta_p u_0 - f(u_0) \geq 0 \quad \text{in } \mathcal{D}'(\Omega \setminus G). \tag{2.3}$$

If $u \in C((0,\infty) : L^\infty(\Omega)) \cap L^\infty((0,\infty) : W^{1,p}(\Omega))$ is the solution of problem (P), *then we have*

$$x \cdot \nabla u(t,x) \leq 0 \quad \text{for a.e. } t > 0 \quad \text{and } x \in \Omega. \tag{2.4}$$

Moreover, if $u \in C((0,\infty) \times \Omega)$, then the level sets $\{x \in \Omega | u(t,x) \geq c\}$ of $u(t,\cdot)$ are starshaped for every $t > 0$.

Proof.

The existence and uniqueness of a weak solution (in $C((0,\infty): L^\infty(\Omega)) \cap L^\infty((0,\infty) : W^{1,p}(\Omega))$ is a well known result (see for instance Alt-Luckhaus [1]). The continuity of u can be shown for increasing f and continuous u_0 by accretiveness on the space $X = C(\Omega)$, see Diaz [19]. More general results on regularity are given in Di Benedetto and Friedman [23] and Wiegner [43]. The maximum principle holds for (P), since for instance, the realization on $L^1(\Omega \setminus G)$ of the operator $Au = -\Delta_p u + f(u)$ is T-accretive [19], and so we conclude that $0 \leq u \leq 1$ in $(0,\infty) \times \Omega$. Without loss of generality we may assume the initial data to be smooth. In that case $u_t \geq 0$ a.e. in $(0,\infty) \times \Omega$, see [16]. Indeed other wise we approach u_0 by a sequence u_{0n} satisfying (2.1)(2.2)(2.3). If each of the associated solutions u_n has starshaped level sets, then

$$S_n(t,x,s) = u_n(t,sx) - u_n(t,x) \geq 0 \qquad \text{for } t \geq 0, s \in [0,1] \qquad \text{and a.e. } x \in \Omega.$$

Thus in the limit

$$S(t,x,s) = u(t,sx) - u(t,x) \geq 0 \qquad \text{for } t \geq 0, s \in [0,1] \qquad \text{and a.e. } x \in \Omega.$$

To derive (2.4) we define $v(t,x) := u(t,sx)$ for a fixed $s \in [0,1]$, and $(\Omega \setminus G)_s := \{x \in \Omega \setminus G | sx \in \Omega \setminus G\}$. Notice that

$$v(0,x) \geq u(0,x) \qquad \text{in } (\Omega \setminus G)_s \tag{2.6}$$

and

$$v(t,x) \geq u(t,x) \qquad \text{on } (0,\infty) \times \partial(\Omega \setminus G)_s \tag{2.7}$$

hold. On the other hand we have

$$v_t - \Delta_p v + f(v) = \tilde{g} \qquad \text{in } \mathcal{D}'((0,\infty) \times (\Omega \setminus G)_s) \tag{2.8}$$

with

$$\tilde{g}(t,x) = u_t(t,sx) - s^p \Delta_p u(t,sx) + f(u(t,sx)) \tag{2.9}$$

and so $\tilde{g} \geq 0$ on $(0,\infty) \times (\Omega \setminus G)_s$. Another application of the parabolic comparison principle yields $v(t,x) \geq u(t,x)$ for any $t \geq 0$ and a.e. $x \in \Omega$, as desired.∎

Remark 1.

In the proof of Theorem 1 we have derived $u_t \geq 0$ as well as $x \cdot \nabla u \leq 0$ in $(0,T) \times (\Omega \setminus G)$. Thus the level sets $\{(t,x) \in (0,T) \times \Omega | u(t,x) \geq c\}$ are starshaped with respect to the point $(T,0) \in I\!R \times I\!R^n$ in space and time for any $T > 0$.

Remark 2.

Problem (P) can give rise to free boundaries such as

$$\mathcal{F}(t) := \partial\{x \in \Omega | u(t,x) > 0\} \cap (\Omega \setminus G).$$

The occurrence of such a free boundary can be caused by the degeneracy of the equation (for $p > 2$ and general f) or by sufficiently strong absorption, e.g. $f(s) = |s|^{q-1}s$ and $(p-1) > q > 0$. Results on the existence of such free boundaries have been obtained by many different authors. We refer to the expository paper [18] by Diaz and its references.

Notice that solutions of problem (P) can (for $p > 2$) have the property that they are constant and nonzero on sets of positive measure. The boundaries of such sets can be again considered as free boundaries. For an elliptic version of this phenomenon see Diaz [19, p.41].

In any case it is of interest to study the regularity of the free boundary. According to the above theorem those free boundaries are starshaped with respect to the origin. From this property we may deduce that $\mathcal{F}(t)$ is Lipschitz continuous in space and that $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}(t)$ is locally Lipschitz continuous in space and time.

Corollary 1.

Assume the hypotheses of Theorem 1 as well as condition (2.10):

$$\left.\begin{array}{l}\textit{The interior of } G \textit{ contains the origin, and } \Omega \\ \textit{and } G \textit{ are starshaped with respect to an open} \\ \textit{neighborhood of the origin.}\end{array}\right\} \qquad (2.10)$$

Then for each $t \geq 0$ the free boundary $\mathcal{F}(t)$ is Lipschitz continuous in x, and $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}(t)$ is locally Lipschitz continuous in (t, x).

Proof.

By Theorem 1 any ray originating in (t, x^0) with $x^0 \in B_\delta(0) \subset I\!R^n$ and $0 < \delta$ intersects $\mathcal{F}(t)$ at most once, provided δ is sufficiently small. If one varies x^0 in $B_\delta(0)$ one finds that $\mathcal{F}(t)$ cannot lie inside certain cones with vertex in $\mathcal{F}(t)$, and hence $\mathcal{F}(t)$ is Lipschitz continuous. To prove the analogous result for $\mathcal{F}$ one has to recall Remark 1 and apply the cone type argument in $(0,T) \times I\!R^n$ with T sufficiently large. ∎

In the next section on convexity we shall require that the spatial gradient of u is nonzero in $(0,\infty) \times (\Omega \setminus G)$. This is not difficult to show for $p = 2$, and a simple proof is included in Proposition 1 below. For general $p > 1$ we shall now derive the strict inequality $x \cdot \nabla u(t,x) < 0$ under suitable assumptions. We shall use the following barrier function $v(x)$ defined by

$$v(x) := \begin{cases} \alpha |x - z|^{(p-n)/(p-1)} + \beta, & \text{if } p \neq n, \\ \alpha \log |x - z| + \beta, & \text{if } p = n, \end{cases} \qquad (2.11)$$

for $x \in B_\delta(z) \setminus \overline{B_{\delta/2}(z)}$, $v(x) := 1$ on $\overline{B_{\delta/2}(z)}$, $v(x) := 0$ on $I\!R^n \setminus B_\delta(z)$, where z will be chosen later, and

$$\alpha := \begin{cases} (2^{(m-p)/(p-1)} - 1)^{-1}\delta^{(n-p)/(p-1)}, & \text{if } p \neq n. \\ -(log 2)^{-1}, & \text{if } p = n. \end{cases}$$

and

$$\beta := \begin{cases} [1 - 2^{(N-p)/(p-1)}]^{-1}, & \text{if } p \neq n. \\ \log\delta/log 2, & \text{if } p = n. \end{cases}$$

It is a simple exercise to check that $-\Delta_p v = 0$ in $B_\delta(z) \setminus \overline{B_{\delta/2}(z)}$, that v is continuous and $v \in W^{1,p}(I\!R^n)$. Moreover

$$\nabla v(x)|\nabla v(x)|^{-1} \to (z - x_0)|z - x_0|^{-1} \tag{2.12}$$

as $x \to x_0 \neq z$ in $B_\delta(z)$, and

$$|\nabla v(x)| \geq c > 0 \qquad \text{for } x \in B_\delta(z) \setminus \overline{B_{\delta/2}(z)} \tag{2.13}$$

Theorem 2.

Let u be the solution to problem (P) *and in addition to the assumptions of Theorem 1 suppose that the interior of G is not empty, and that G and Ω are convex and satisfy the following uniform sphere condition.*

$$\left.\begin{array}{l} \text{There exists a } \delta > 0 \text{ such that for any } x_0 \in \\ \partial G \cup \partial\Omega \text{ there is a } z \text{ with } x_0 \in B_\delta(z) \text{ and} \\ B_\delta(z) \subset \Omega \setminus G. \end{array}\right\} \tag{2.14}$$

Moreover, we assume

$$f \in C^1([0,1]), \qquad f' \geq 0, \qquad f(0) = f'(0) = 0, \tag{2.15}$$

$$u_0 \in C^1(\Omega \setminus G) \quad \text{and } x \cdot \nabla u_0(x) < 0 \quad \text{in } \Omega \setminus G, \tag{2.16}$$

and

$$u_0(x) \geq \epsilon v(x) \quad \text{for some } \epsilon > 0 \quad \text{and for } d(x, \partial\Omega) \leq \delta \tag{2.17}$$

Then the solution $u \in C((0,\infty) \times \Omega)$ of problem (P) *satisfies the nondegeneracy condition*

$$x \cdot \nabla u(t,x) < 0 \quad \text{in } (0,\infty) \times (\Omega \setminus G).$$

Proof.

We shall adopt some ideas from Lewis [37], who considered the stationary problem with vanishing f. First we obtain some estimates on the growth of $u(t, sx)$ with respect to s, when $x \in \partial\Omega \cup \partial G$. Then we shall estimate from above the function $u(t, sx) - u(t, x)$ when s decreases to one. This will give us the result.

Without loss of of generality we can restrict the proof to any finite time $t < T$, say.

Let $K = \{x \in \Omega \backslash G | d(x, \partial\Omega \cup \partial G \geq \delta/2\}$. Because of the choice of u_0 ($0 < u_0 < 1$ on $\Omega \setminus G$) and f, there is a constant $A > 0$ such that

$$\min\{1 - u(t,x), u(t,x)\} \geq A \quad \text{for } x \in K, t \in [0,T]. \tag{2.18}$$

Moreover, from the continuity of u, A decreases to zero as δ goes to zero. Given $x_0 \in \partial\Omega \cup \partial G$ we choose z according to (2.14) and consider $v(x)$.

From the convexity of Ω and G and (2.13) we see that

$$|(z - x_0) \cdot x_0| \geq \tau |z - x_0||x_0|,$$

where $\tau > 0$ is independent of x_0 and z. Then, if $x_0 \in \partial G$ we have that $sx_0 \in B(z,\delta) \setminus \overline{B(z,\delta/2)}$ for $1 < s \leq s_0$ and suitable s_0, and

$$v(sx_0) \geq 2^{-1}\tau(s-1)c|x_0| \geq \mu|s-1|. \tag{2.19}$$

Similarly, if $x_0 \in \partial\Omega$, then $s^{-1}x_0 \in B(z,\delta) \setminus \overline{B(z,\delta/2)}$ and

$$v(x_0/s) \geq \mu|s-1| \quad \text{for } 1 < s \leq s_0. \tag{2.20}$$

Notice that s_0 and μ can be chosen independent of $x_0 \in \partial\Omega \cup \partial G$.

If $x_0 \in \partial G$, it is clear from (2.18) that

$$u(t,x) \leq 1 - Av(x), \quad \text{for } x \in \partial(B(z,\delta) \setminus \overline{B(z,\delta/2)}), t \in (0,T],$$

while $-\Delta_p u = -u_t - f(u) \leq 0$ in $Q_T := (0,T) \times (\Omega \setminus \bar{G})$. Then by the comparison principle we have

$$u(t,x) \leq 1 - Av(x) \quad \text{for } x \in B(z,\delta) \setminus B(z,\delta/2) \quad \text{and } t \in (0,T] \tag{2.21}$$

If $x_0 \in \partial\Omega$ we consider the test function $\underline{v}(t,x) = Av(x) - C_1 t$ with $C_1 > 0$ to be chosen later. On $\partial(B(z,\delta) \setminus B(z,\delta/2)) \times (0,T)$ we have $u \geq \underline{v}$. On the other hand, due to (2.17) we may suppose without loss of generality that $u_0(x) \geq Av(x)$ on $B(z,\delta) \setminus B(z,\delta/2)$. Finally

$$\underline{v}_t - \Delta_p \underline{v} + f(\underline{v}) = -C_1 + f(Av - C_1 t) \leq C_1 + f(A) \leq 0,$$

if we choose now $C_1 = f(A)$. Then again by the comparison principle

$$Av(x) - f(A)t \leq u(t,x) \quad \text{for } x \in B(z,\delta) \setminus B(z,\delta/2) \quad \text{and } t \in (0,T] \tag{2.22}$$

Now we may conclude from (2.21) and (2.19) that, if $x_0 \in \partial G$

$$1 - u(t, sx_0) \geq A\mu(s-1) \quad \text{for } 1 < s \leq s_0 \tag{2.23}$$

and analogously, if $x_0 \in \partial\Omega$, from (2.22) and (2.20)

$$u(t, x_0/s) \geq Av(x_0/s) - f(A)t \geq A\mu(s-1) - f(A)t \tag{2.24}$$

Since $f'(0) = 0$ we may assume A to be small enough so that $f(A)/A < \mu(s-1)/2T$ and then (2.24) leads to

$$u(t, x_0/s) \geq \frac{A}{2}\mu(s-1) \quad \text{for } 1 < s \leq s_0. \tag{2.25}$$

Next fix $1 < s \leq s_0$ and let $\Omega_s = \{y \in \Omega | sy \in \Omega\}$. Set $w(t,x) := u(t, sx)$. Then

$$\begin{aligned} w_t - \Delta_p w + f(w) =& 0 \qquad \text{in } (0,T) \times (\Omega_s \setminus G), \\ w \leq& 1 \qquad \text{on } (0,T) \times G, \\ w =& 0 \qquad \text{on } (0,T) \times \partial\Omega_s, \\ w(0,x) =& u_0(sx) \text{ on } \Omega_s. \end{aligned}$$

We want to estimate $u(t, sx) - u(t,x)$ from above, i.e. we want to show that there is a positive function ψ depending on t (and s) such that $\underline{w}(t,x) = w(t,x) + \psi(t) \leq u(t,x)$ on $(0,T) \times (\Omega_s \setminus G)$. The following considerations are aimed at the construction of ψ. On $(0,T) \times \partial G$ and $(0,T) \times \partial\Omega_s$ we may use (2.23) and (2.25) to conclude that $\underline{w} \leq u$ on $(0,T) \times \partial(\Omega_s \setminus G)$, provided

$$\psi(t) \leq \frac{A}{2}\mu(s-1) \quad \text{for any } t \in [0,T]. \tag{2.26}$$

Since $\underline{w} \leq u$ should hold on the parabolic boundary of $(0,T) \times (\Omega_s \setminus G)$ we set $\omega = \min\{|x \nabla u_0(x)| \, |x \in \overline{\Omega \setminus G}\}$, then $u_0(sx) - u_0(x) \leq \omega(s-1)$ and $\underline{w}(0,x) \leq u_0(x)$ in $\Omega_s \setminus G$ is implied by the condition

$$\psi(0) \leq \omega(s-1). \tag{2.27}$$

This is how ψ depends on s. Finally

$$\underline{w}_t - \Delta_p \underline{w} + f(\underline{w}) = f(w) + f(w + \psi(t)) + \psi'(t) \leq 0,$$

if ψ satisfies

$$\psi'(t) \leq -H\psi(t), \quad \text{with } H = max\{f'(\xi) | \xi \in [0,1] \tag{2.28}$$

In fact $-H\psi(t) \leq \min\{f(w) - f(w + \psi(t)) | w \in [0,1]\}$, and condition (2.28) holds by taking $\psi(t) = c\exp\{-Ht\}$ for any $c > 0$. Conditions (2.26) and (2.27) are satisfied by choosing $c = (s-1)\min\{\frac{A}{2}\mu, \omega\}$. Then, by the comparison principles we conclude that $\underline{w} \leq u$ on $(0,T) \times (\Omega_s \setminus G)$, i.e. that

$$u(t, sx) - u(t,x) \leq -(s-1)e^{-Ht}\min\{\frac{A}{2}\mu, \omega\} \leq \nu(s-1)$$

for $t \in (0,T), x \in \Omega_s \setminus G$, and consequently

$$x \cdot \nabla u(t,x) \leq \lim_{s \to 1}(u(t,sx) - u(t,x))(s-1)^{-1} \leq \nu$$

as desired. This completes the proof of Theorem 2.

Remark 3.

Several generalizations are possible:
For instance we can replace the interior boundary condition (1.2) by

$$u(t,x) = w(t) \quad \text{on } (0,\infty) \times G,$$

with $w \in W^{1,1}_{\text{loc}}(0,\infty)$; $w \geq 0$, $w' \geq 0$, $0 \leq u_0 \leq w(0)$ in Ω, and $u_0 \equiv w_0(0)$ on G. Then the conclusions of Theorems 1 and 2 remain true.

Or we can let G shrink to a point $\{0\}$ and obtain starshapedness of level sets for solutions which are singular at the origin.

In the remainder of this section we shall derive starshapedness results for two other nonlinear parabolic problems. The first one describes another kind of non-linear diffusion phenomena.

Let Ω and G be as in Problem (P) and let ϕ be a continuous nondecreasing (or even maximal monotone) mapping with $\phi(0) = 0$. Let f be the sum of a monotone nondecreasing or maximal monotone map and a Lipschitz-continuous map, and suppose that $f(s) \geq 0$ for every $s \geq 0$. Consider the problem (P_ϕ)

$$(\mathrm{P}_\phi)\begin{cases} u_t - \Delta\phi(u) + f(u) = 0 & \text{in } (0,\infty) \times (\Omega \setminus G) \qquad (1.5), \\ \phi(u) \equiv 1 & \text{on } (0,\infty) \times G \qquad (2.29), \\ \phi(u) = 0 & \text{on } (0,\infty) \times \partial\Omega \qquad (2.30), \\ u(0,x) = u_0(x) & \text{on } \Omega. \qquad (1.4). \end{cases}$$

It is tacitly understood that u_0 is compatible with the boundary conditions; i.e. $\phi(u_0) \equiv 1$ on G, $\phi(u_0) = 0$ on $\partial\Omega$.

An important special case of problem (P_ϕ) is $f \equiv 0$ and $\phi(s) = |s|^{m-1}s$ with $m > 0$. The case $m > 1$ appears in the modelling of porous media flow, $m = 1$ corresponds to the linear heat equation and $0 < m < 1$ is related to some problems of fast diffusion in plasma physics. Another special case is $\phi(s) = k(s-a)^+$ for some positive numbers k and a. It models the one-phase Stefan problem. As for the problems of existence, uniqueness and regularity we refer to [1, 8, 6, 24] and to their bibliographies.

Theorem 3.

Let $G \subset\subset \Omega$, and let G and Ω be starshaped. Let $u_0 \in L^\infty(\Omega)$ be given with $\phi(u_0) \in H^1(\Omega \setminus G)$ and suppose

$$0 \leq \phi(u_0) \leq 1 \quad \text{a.e.in } \Omega, \tag{2.31}$$

$$\textit{the level sets of } u_0 \textit{ are starshaped}, \tag{2.32}$$

$$\Delta\phi(u_0) - f(u_0) \geq 0 \quad \text{in } \mathcal{D}'(\Omega \setminus G) \tag{2.33}$$

If $u \in C((0,\infty) : L^1(\Omega))$ is the solution of problem (P_ϕ), then

$$u(t, sx) \geq u(t,x) \quad \text{for any } t > 0 \quad \text{and a.e. } x \in \Omega.$$

Moreover, if $u \in C((0,\infty) \times \Omega)$, then the level sets $\{x \in \Omega | u(x,t) \geq c\}$ are starshaped for every $t > 0$.

Proof.

The existence and uniqueness of a weak solution follows e.g. from abstract semigroup theory as in Benilan [6] or by easy modifications of well-known results such as the ones of Alt and Luckhaus [1]. Under suitable assumptions on f, ϕ and u_0 one knows that $u \in C((0,\infty) \times \Omega)$, see e.g. Di Benedetto [22]. By the maximum principle we know that

$$0 \leq \phi(u(t,x)) \leq 1 \quad \text{in } (0,\infty) \times \Omega.$$

As a consequence of (2.33) we have

$$u_t \geq 0 \quad \text{in } \mathcal{D}'((0,\infty) \times \Omega).$$

This can be shown after approximating the data and applying the maximum principle to the approximate solutions, or by abstract monotonicity arguments, see e.g. Damlamian [16]. To complete the proof, we define $v(t,x) := u(t,sx)$ for a fixed $s \in [0,1]$ and $(\Omega \setminus G)_s := \{x \in \Omega \setminus G | sx \in \Omega \setminus G\}$. Now we observe that (2.6) and (2.7) hold, and that

$$v_t - \Delta\phi(v) + f(v) = \tilde{g}(t,x) \quad \text{in } \mathcal{D}'((0,\infty) \times (\Omega \setminus G)_s),$$

with

$$\tilde{g}(t,x) = u_t(t,sx) - s^2 \Delta\phi(u(t,sx)) + f(u(t,sx)) \quad \text{a.e. in } (0,\infty) \times (\Omega \setminus G)_s),$$

The remainder of the proof is identical to the one of Theorem 1. Notice, however, that one has to justify the applicability of the comparison principle via a regularization. Since this is more or less standard, we omit the details and refer to Diaz [17, Lemma 1].∎

In order to derive convexity of level sets for problem (P_ϕ) we shall need the following stronger result on radial derivatives of smooth solutions.

Proposition 1.

Let u be the solution to Problem (P_ϕ), and in addition to the assumptions of Theorem 3 suppose that u_0 and ϕ are of class C^2 and f is of class C^1. Furthermore suppose that ϕ' is bounded below by a positive constant. Then for any $T > 0$

$$x \cdot \nabla\phi(u(t,x)) < 0 \quad \text{in } (0,T) \times (\Omega \setminus G). \tag{2.34a}$$

If in addition $u_o(x) = 0$ in $\Omega \setminus G$, then

$$t \cdot (\phi(u(t,x)))_t + x \cdot \nabla\phi(u(x,t)) \leq 0 \quad \text{in } (0,T) \times (\Omega \setminus G). \tag{2.34b}$$

Proof.

As in the proof of Theorem 3 we know that $u_t \geq 0$. Let $v = \phi(u)$. Then

$$\psi'(v)v_t - \Delta v + f(\psi(v)) = 0$$

where $\psi = \phi^{-1}$. Next we consider $w(t,x) = x \cdot \nabla v(t,x)$ and calculate

$$\begin{aligned}\Delta w =& x \cdot \nabla(\Delta v) + 2\Delta v \\ =&(x \cdot \nabla v)_t \psi'(v) + (x \cdot \nabla v)[\psi''(v)v_t + f'(\psi(v))\psi'(v)] + \\ &+ 2\psi'(v)v_t + 2f(\psi(v)) \\ \geq& a(t,x)w_t + b(t,x)w,\end{aligned}$$

where a and b are in $L^\infty(0,T) \times (\Omega \setminus G)$ and a is bounded below by a positive constant. Since $x \cdot \nabla v \leq 0$ on the parabolic boundary of $(0,T) \times (\Omega \setminus G)$, the conclusion (2.34a) follows from the strong maximum principle. To prove (2.34b) we note that $(tv_t)_t = v_t + tv_{tt}$, and, using the notation $z(t,x) = tv_t(t,x) + x \cdot \nabla v(t,x)$, we obtain a similar differential inequality for z as for w.∎

Remark 4.

It is well known that the solution u of (P_ϕ) can have a free boundary which is defined by $\mathcal{F}(t) = \partial\{x \in \Omega | u(t,x) > 0\} \cap (\Omega \setminus G)$. Again the occurrence of such a free boundary can be caused by the degeneracy of the equation, i.e. if $\phi'(0) = 0$ and if ϕ satisfies

$$\int_0^1 \frac{\phi'(s)}{s}\, ds < \infty,$$

or by sufficiently strong absorption such as $f(s) = |s|^{q-1}s$ for $\phi(s) = |s|^{m-1}s$ with $m > q > 0$. We refer to [17, 20] for details.

Corollary 1 as well as Remarks 2 and 3 remain valid for Problem (P_ϕ). Thus we recover the result of Friedman and Kinderlehrer [25] on the Stefan problem. Recent results for the porous medium equation ($\phi(u) = u^m$ with $m > 1$) on $(0,T) \times I\!R^n$ are due to Caffarelli, Vazquez and Wolanski [15] and to Bardi [5]. They prove the starshapedness of the support of $u(t,\cdot)$ as a preliminary step to other qualitative results. Notice that our theorems do not apply to their Cauchy problems.

Problems (P) and (P_ϕ) can be interpreted as interior obstacle problems in Ω with the following obstacle.

$$\psi(x) = \chi_G(x) = \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{if } x \in \Omega \setminus G. \end{cases}$$

The coincidence set $\{x \in \Omega | u(x,t) = \psi(x)\}$ always contains G, but can also contain regions near $\partial\Omega$. Such is the case when a dead core occurs.

Therefore it seems natural to extend our results to more general obstacles $\psi(x)$. To simplify our exposition we shall restrict ourselves to a linear (but inhomogeneous) differential equation.

For the remainder of this section let Ω be a bounded open set of $I\!R^n$ with smooth boundary $\partial\Omega$. Let $\psi \in H^1(\Omega)$ be a "stationary obstacle" with $\psi \leq 0$ on $\partial\Omega$. Consider the so-called **obstacle problem** (P_ψ):

$$
(\mathrm{P}_\psi)\begin{cases} u_t - \Delta u \geq g(t,x), u \geq \psi, (u_t - \Delta u - g)(u-\psi) = 0 & \text{a.e. in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}
$$

Here $u_0(x)$ and $g(t,x)$ are given functions and u is unknown. Different notions of weak solutions to this problem can be introduced, all of them being in fact regular solutions under additional assumptions on the data ψ, g and u (see e.g. Brezis [11] and Friedman [24]).

Theorem 4.
Let $\psi \in H^1(\Omega)$ have the properties $\psi \leq 0$ on $\partial\Omega$, $\psi^+(x) := \max\{\psi(x), 0\} \not\equiv 0$ in Ω and (2.35).

$$\text{For every } c \geq 0 \text{ the set } \{x \in \Omega | \psi(x) \geq c\}\text{- is starshaped} \tag{2.35}$$

Let $u_0 \in H^1(\Omega)$ be given and satisfy

$$u_0(x) \geq \psi^+(x) \quad \text{a.e. in } \Omega, \tag{2.36}$$
$$x \cdot \nabla u_0(x) \leq 0 \quad \text{a.e. in } \Omega. \tag{2.37}$$

Let $g \in W^{1,1}((0,T) : L^2(\Omega))$ be given and satisfy

$$x \cdot \nabla g(t,x) + 2g(t,x) \leq 0 \quad \text{in } \mathcal{D}'(Q_T), \tag{2.38}$$
$$g_t \geq 0 \quad \text{a.e. in } Q_T, \tag{2.39}$$
$$\Delta u_0 + g(0,x) \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.40}$$

If $u \in C([0,T] : L^2(\Omega)) \cap L^1((0,T) : H^1(\Omega))$ is the solution of the obstacle problem, then for every $t \in [0,T]$ we have $x \cdot \nabla u(t,x) \leq 0$ a.e. $x \in \Omega$. Moreover, if $u \in C([0,T] \times \Omega)$, then the level sets $\{x \in \Omega | u(x,t) \geq c\}$ are starshaped for every $t \in [0,T]$.

Proof.
The existence and uniqueness of a solution $u \in C([0,T] : L^2(\Omega))$ can be established by applying the results of Brezis [11], and its continuity is guaranteed if $\psi \in C^{1,1}(\Omega)$, $g \in C^\alpha([0,T] \times \Omega)$, and $u_0 \in C^\alpha(\overline{\Omega})$, see Friedman [24].

In fact without loss of generality it suffices to prove the theorem under additional regularity assumptions on the data:

$$\psi \in C^{1,1}(\Omega) \cap H^2(\Omega), \quad g \in C^\alpha([0,T] \times \Omega) \cap W^{1,1}([0,T] : L^2(\Omega)), \\ \text{and } u_0 \in W^{2,\infty}(\Omega) \cap C^\alpha(\overline{\Omega}) \cap H^1_0(\Omega). \tag{2.41}$$

Otherwise we approximate ψ, g and u_0 by regular functions ψ_n, g_n and u_{0n} which satisfy the assumptions of Theorem 4 and argue as in Theorem 1 when it comes to passing to the limit. Under the given regularity assumptions we note that $u \in C^\beta((0,T) \times \Omega)$ and that u_{x_i}, $u_{x_i x_j}$ and u_t belong to $L^\infty_{\text{loc}}((0,T) \times \Omega)$. We claim

$$u_t \geq 0 \qquad \text{a.e. in } (0,T) \times \Omega. \tag{2.42}$$

To prove (2.42) we apply the parabolic maximum principle to the function $v = u_t$ on the noncoincidence set $N = \{(t,x) \in Q_T | u(t,x) > \psi(x)\}$. By our assumptions (essentially (2.40)) we have $v(t,x) \geq 0$ on the parabolic boundary of N and $v_t - \Delta v = g_t \geq 0$ in N. Therefore, using (2.39) and the maximum principle (2.42) follows. Let us remark that (2.42) and (2.36) imply the nonnegativity of $u(t,x)$ in $(0,T) \times \Omega$.

Now we consider the function $w(t,x) = x \cdot \nabla u(t,x)$ and want to show

$$w(t,x) = x \cdot \nabla u(t,x) \leq 0 \qquad \text{in } Q_T, \tag{2.43}$$

which implies the conclusion of Theorem 4. A simple computation shows

$$w_t - \Delta w = -2u_t + 2g + x \cdot \nabla g \leq 0 \qquad \text{in } N.$$

Here we have used (2.38) and (2.42). Moreover $w \leq 0$ on the parabolic boundary of N because of (2.35), (2.37) and the fact that $u \geq 0$ in $(0,T) \times \Omega$ while $u = 0$ on $(0,T) \times \partial\Omega$. Consequently $w < 0$ in N by the strong maximum principle. Thus $w \leq 0$ on $(0,T) \times \Omega$ and the proof is complete.∎

Notice that we have in fact shown a stronger result for regular data g, u_0 and ψ:

Corollary 2.

Let ψ, u_0 and g satisfy (2.35)-(2.41). Then for any $t \in (0,T)$ we have

$$x \cdot \nabla u(t,x) < 0 \qquad \text{a.e. in } \{x \in \Omega | u(t,x) > \psi(x)\}.$$

Remark 5.

Sakaguchi has recently studied the shape of the coincidence set in the stationary obstacle problem. For special obstacles $\psi(x)$, which are homogeneous polynomials, he has been able to prove starshapedness of the coincidence set [40, 41]. In general, however, one cannot expect to get such a result even for an arbitrary concave obstacle.

§3. Convexity of level sets

Now we shall prove the convexity of level sets for problem (P_ϕ) in convex rings.

$$(\mathrm{P}_\phi)\begin{cases} u_t - \Delta\phi(u) + f(u) = 0 & \text{in } (0,\infty) \times (\Omega \setminus G) \quad (1.5),\\ \phi(u) \equiv 1 & \text{on } (0,\infty) \times G \quad (2.29),\\ \phi(u) = 0 & \text{on } (0,\infty) \times \partial\Omega \quad (2.30),\\ u(0,x) = u_0(x) & \text{on } \Omega \quad (1.4).\end{cases}$$

Let us recall that we have to show that

$$Q(t_1,t_2,x_1,x_2) := u((t_1+t_2)/2,(x_1+x_2)/2) - \min\{u(t_1,x_1),u(t_2,x_2)\}$$

or equivalently, that

$$\tilde{Q}(t_1,t_2,x_1,x_2) := \phi(u((t_1+t_2)/2,(x_1+x_2)/2)) - \min\{\phi(u(t_1,x_1)),\phi(u(t_2,x_2))\}$$

is nonnegative for any pair $(t_1,x_1),(t_2,x_2)$ of points in $(0,\infty)\times\Omega$.

Theorem 5.
Let Ω and G be convex with smooth boundaries. Let f be continuous and nondecreasing with $f(0)\geq 0$. Let ϕ be increasing and concave and suppose

$$\phi(u_0)\equiv 0 \text{ in } \Omega\setminus G, \quad \phi(u_0)\equiv 1 \text{ on } G \tag{3.1}$$

Then, if $u\in C((0,\infty)\times\Omega)$ is the solution of Problem (P_ϕ), it is quasiconcave in x and t.

Proof.
Without loss of generality we may assume that ϕ and f are sufficiently regular and ϕ' and f' are positive. If this is not the case we can approximate ϕ and f by regular functions ϕ_n and f_n which also satisfy the assumptions of Theorem 6. If $f_n(s)\to f(s)$ and $\phi_n(s)\searrow\phi(s)$ uniformly on compact sets, we conclude as in Damlamian [16, Theorem 2.3], that $u_n \rightharpoonup u$ weakly in $W^{1,2}([0,T],H^{-1}(\Omega\setminus G))$ and $\phi_n(u_n(t,\cdot))\rightharpoonup\phi(u(t,\cdot))$ weakly in $H^1(\Omega\setminus G)$. So we can choose a subsequence, still denoted by u_n, such that $\phi_n(u_n(t,\cdot))\to\phi(u(t,\cdot))$ in $L^2(\Omega)$ and pointwise for a.e. x in Ω and every $t\in[0,T]$, where T is arbitrarily large. Once we manage to prove the theorem for regular data, we are done, because the quasiconcavity functions

$$\tilde{Q}_n(t_1,t_2,x_1,x_2) := \\ \phi_n((u_n(t_1+t_2)/2,(x_1+x_2)/2)) - \min\{\phi_n(u_n(t_1,x_1)),\phi_n(u_n(t_2,x_2))\}$$

are all nonnegative and converge pointwise for every $(t_1,t_2)\in(0,T]\times(0,T]$ and a.e.$(x_1,x_2)\in\Omega\times\Omega$ to the quasiconcavity function $\tilde{Q}$ of $\phi(u)$. Finally we can pass with T to ∞.

Thus it remains to prove Theorem 5 for regular data and under strict monotonicity assumptions. But now the solution of problem (P_ϕ) is classical again, see [36].

Suppose that the quasiconcavity function $\tilde{Q}$ of $\phi(u)$ is not nonnegative. Then $\tilde{Q}$ approaches a global infimum in a pair of points $z_1=(t_1,x_1)$ and $z_2=(t_2,x_2)$ with t_1,t_2 nonnegative and finite and with $x_1,x_2\in\bar{\Omega}$. Otherwise, if $t_1\to\infty$, then $(t_1+t_2)/2\to\infty$ and, under abuse of notation, since $u_t\geq 0$,

$$\tilde{Q}(t_1,t_2,x_1,x_2) = \\ \phi(u(\infty,(x_1+x_2)/2) - \min\{\phi(u(\infty,x_1),\phi(u(t_2,x_2))\} \geq \tilde{Q}(\infty,\infty,x_1,x_2).$$

The last expression, though, is nonnegative, because u tends asymptotically to the solution of the stationary problem [36a] and the latter one is quasiconcave, see [32]. Therefore we may assume that there is a $T > 0$ such that $\tilde{Q}$ approaches its global infimum in $(t_1, x_1), (t_2, x_2)$ and $0 \leq t_1, t_2 < T$. The following considerations will show that

$$t_1 > 0 \quad \text{and} \quad t_2 > 0. \tag{3.2}$$

In fact if t_1 and t_2 are zero, then $\tilde{Q}$ cannot be negative because of (3.1). If t_2 is positive and $t_1 = 0$ and if $x_1 \in \Omega \setminus G$, then $\tilde{Q}$ is again nonnegative. Finally if $t_1 = 0$ and $x_1 \in G$ we can use (2.34b) to conclude that $\inf \tilde{Q} \geq 0$.

Next I will show that

$$x_1, x_2 \quad \text{and} \quad (x_1 + x_2)/2 \in (\Omega \setminus G) \tag{3.3}$$

and

$$u(z_1) = u(z_2). \tag{3.4}$$

Property (3.3) follows from the starshapedness of level sets. To prove (3.4) suppose that $u(z_1) < u(z_2)$. Then locally near z_1, z_2 the quasiconcavity function $\tilde{Q}$ of $v = \phi(u)$ would have the representation

$$\tilde{Q}(\xi_1, \xi_2) = v((\xi_1 + \xi_2)/2) - v(\xi_1),$$

and the spatial gradients of $\tilde{Q}$ would have to vanish at the points z_1 and z_2, i.e. $\frac{1}{2}\nabla v((t_1 + t_2)/2, (x_1 + x_2)/2) = 0 = \nabla v(t_1, x_1)$. This would contradict (2.34a).

We can now vary the points (z_1, z_2) in time and space and deduce some useful relations between derivatives of $v = \phi(u)$ in z_1, z_2 and $(z_1 + z_2)/2$. Using the notation $Dv = (v_t, \nabla v)$ for the gradient in $I\!R^{1+n}$ and $A = |Dv((z_1 + z_2)/2)|$, $B = |Dv(z_1)|$, $C = |Dv(z_2)|$ one can derive the following lemma.

Lemma 1.

Under the above assumptions $Dv(z_1), Dv(z_2)$ and $Dv((z_1 + z_2)/2)$ are all parallel and point in the same direction. Moreover

$$\frac{1}{A} = \frac{1}{2}\left(\frac{1}{B} + \frac{1}{C}\right) \tag{3.5}$$

and

$$\frac{1}{A^2}\Delta(u((t_1 + t_2)/2), (z_1 + z_2)/2)) \geq \frac{\mu}{B^2}\Delta(u(t_1, z_1)) + \frac{1-\mu}{C^2}\Delta(u(t_2, z_2)), \tag{3.6}$$

where $\mu = C/(B + C) \in (0, 1)$.

The proof of this Lemma is fairly lengthy. It uses the implicit function theorem in $I\!R^{1+n}$ and can be found in this form in papers of Gabriel [27], Lewis [37] and Kawohl [30, 32].

Now we proceed with the proof of Theorem 5. We use $v((z_1+z_2)/2) < v(z_1) = v(z_2)$, the monotonicity of ϕ and f and (3.5) as well as the differential equation

$$u_t - \Delta v + f(u) = 0$$

to arrive at

$$\frac{1}{A^2} u_t((z_1+z_2)/2) > \frac{\mu}{B^2} u_t(z_1) + \frac{1-\mu}{C^2} u_t(z_2). \tag{3.7}$$

Because $u_t > 0$ and because of (3.5) we have

$$\frac{1}{v_t((z_1+z_2)/2)} = \frac{1}{2v_t(z_1)} + \frac{1}{2v_t(z_2)}. \tag{3.8}$$

If ϕ' is monotone decreasing, (i.e. ϕ is concave), then (3.8) implies

$$\frac{1}{u_t((z_1+z_2)/2)} \geq \frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}. \tag{3.9}$$

Now (3.7),(3.9) and the definition of μ give

$$0 > \frac{\mu}{B^2} u_t(z_1) + \frac{1-\mu}{C^2} u_t(z_2) - \frac{1}{A^2}\left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}\right)^{-1}$$

$$= A\left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}\right)^{-1} \left[\left(\frac{1}{2B^3} u_t(z_1) + \frac{1}{2C^3} u_t(z_2)\right)\left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}\right) - \frac{1}{A^3}\right].$$

To complete the proof of Theorem 5 we have to show that $[...] \geq 0$, but

$$\left(\frac{1}{2B^3} u_t(z_1) + \frac{1}{2C^3} u_t(z_2)\right)\left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}\right)$$
$$\geq \left(\frac{1}{2}B^{-3/2} + \frac{1}{2}C^{-3/2}\right)^2 \geq (A^{-3/2})^2 = A^{-3},$$

which is the desired contradiction. ∎

Remark 6.

Theorem 5 applies to fast diffusion problems with $\phi(s) = s^m$, $0 \leq m \leq 1$. Problems of this nature were introduced in Gurtin and MacCamy [29] to model spatial spread of some biological populations. Notice that Theorem 5 does not apply to the porous medium equation with $\phi(s) = s^m$, $m > 1$.

Remark 7.

In Theorem 5 the function f does not have to be continuous. If for instance or $\phi(s) \equiv s$ and if f is the maximal monotone mapping defined by

$$f(s) = \begin{cases} \emptyset & \text{if } s < 0, \\ (-\infty, 1] & \text{if } s = 0, \\ \{1\} & \text{if } s > 0, \end{cases}$$

then we can approximate f by continuous monotone functions and show that the corresponding solution is quasiconcave. Problems of this type arise in the modelling of galvanization processes (see [42] for the stationary case), and one can show (see [4, 20]), that for sufficiently large Ω the solution will have compact support in Ω. We have in fact shown that the support is convex in space and time.

Let us now turn to problem (P) with the pseudo-Laplace operator. Here we encounter various technical difficulties. Recall that for the proof of Theorem 5 we needed various inequalities. In the present case they will be assumptions.

$$u_t \geq 0 \tag{3.10}$$

$$x \cdot \nabla u < 0 \tag{3.11}$$

$$\text{Q cannot approach its infimum at } t_1 = 0 \text{ or } t_2 = 0 \tag{3.12}$$

Recall that we can verify (3.10), and under suitable assumptions on u_0 we can also show (3.11), see Theorem 2. But if $u_0 = 0$ in $\Omega \setminus G$, which was useful for proving (3.12), then we do not have (3.11) available. Nevertheless it is instructive to see that the structure of the differential equation (1.1) is still sufficiently good to carry our argument over to this case.

Theorem 6.
Let Ω and G be convex with smooth boundaries. Let u_0 satify the assumptions of Theorem 1 and suppose that (3.11) and (3.12) hold. Let $f \in C([0,1])$ be nondecreasing and $f(0) \geq 0$.

Then if $u \in C((0,\infty) \times \Omega)$ is the solution of problem (P), *u is quasiconcave in* x and t.

Proof.
We proceed as before along the lines of the proof of Theorem 5 and suppose that the quasiconcavity function Q of u is not nonnegative. Then with obvious modifications one can show that Q attains a negative minimum in a pair $(t_1, x_1) = z_1, (t_2, x_2) = z_2$ of points such that (3.2)(3.3) and (3.4) hold. For the following Lemma we recall the notation $A = |Du((z_1 + z_2)/2)|$ etc. and introduce $a = |\nabla u((z_1 + z_2)/2)|, b = |\nabla u(z_1)|, c = |\nabla u(z_2)|$.

Lemma 2.
Under the above assumptions $\nabla u(z_1), \nabla u(z_2)$ and *$\nabla u((z_1 + z_2)/2)$ are all parallel and point in the same direction. Moreover*

$$\frac{1}{a} = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) \tag{3.13}$$

and

$$\frac{1}{a^p}\Delta_p(u((t_1+t_2)/2),(z_1+z_2)/2)) \geq \frac{\mu}{b^p}\Delta_p(u(t_1,z_1)) + \frac{1-\mu}{c^p}\Delta_p(u(t_2,z_2)), \tag{3.14}$$

where $\mu = c/(b+c) \in (0,1)$.

For the proof of Lemma 2 we notice that (3.13) follows from Lemma 1. If we observe that

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C},$$

we can derive (3.14) as it was done in the work of Lewis [37] and Kawohl [31].

To complete the proof of Theorem 6 we mimic the computations which follow (3.7) and obtain

$$\frac{1}{a^p} u_t((z_1 + z_2)/2) > \frac{\mu}{b^p} u_t(z_1) + \frac{1-\mu}{c^p} u_t(z_2). \tag{3.15}$$

This is left as an exercise to the reader. One merely has to replace A, B, C by a, b, c and the exponents 2 and 3 by p and $p+1$ to reach a contradiction. ∎

Remark 8.

Does the approach that leads to Theorems 5 and 6 apply as well to the obstacle problem (P_ψ)? This seems to be an open problem even in the stationary setting, cf. [32, p.111]. The main difficulty lies in proving that all three extremal points z_1, z_2 and $(z_1+z_2)/2$ are contained in the noncoincidence set $\{u > \psi\}$.

Remark 9.

As mentioned in the introduction one could also study the quasiconcavity of u in space (for fixed t) and investigate the sign of

$$Q_t(x_1, x_2) := u(t, (x_1 + x_2)/2) - \min\{u(t, x_1), u(t, x_2)\},$$

hoping that $Q_0 \geq 0$ implies $Q_t \geq 0$ for $t > 0$.

If the space dimension n is equal to 1, this is in fact true and follows from our results on starshapedness. Starshaped level sets in $I\!R$ are intervals, and intervals are convex. For higher dimensions, however, the situation is more complicated. In fact, if Q_t is not nonnegative, then there exists a finite $T > 0$, a time $t_0 \in (0, T]$ and a pair of points $(y_1, y_2) \in \Omega$ such that Q_t attains its minimum over $[0,T] \times \bar{\Omega}$ for $t = t_0$ and in y_1, y_2. It is easy to prove analogues of Lemma 2, but in order to get the parabolic differential equation into play, one needs a relation between the time derivatives of u. Unfortunately the information that

$$Q_{t_0}(y_1, y_2) \leq \min\{Q_T(x_1, x_2) | t \in (0, T]; x_1, x_2 \in \Omega\},$$

only leads to

$$u_t(t_0, (y_1 + y_2)/2) \leq \min\{u_t(t_0, y_1), u_t(t_0, y_2)\}. \tag{3.16}$$

But (1.1)(3.18)(3.13) and (3.14) do not contradict each other.

Nevertheless it is desirable to show that $Q_0 \geq 0$ implies $Q_t \geq 0$ for $t > 0$, because this result could be applied to some elliptic problems as well. In fact if $u(t,x)$ approaches an equilibrium solution as $t \to \infty$, then this solution is quasiconcave in space. This way we can recover and sometimes improve the results of Kawohl [30, 31] and Vogel [42] for the obstacle problem and of Lewis [37], Caffarelli and Spruck [14] and Kawohl [30, 31] for the exterior problem.

Remark 10.

We want to point out a simple looking related **open problem**: Suppose that $\Omega \subset I\!R^n$ is bounded and convex and $v(t,x)$ is a solution of the linear heat equation

$$\left.\begin{aligned} v_t - \Delta v &= 0 && \text{in } (0,T)\times\Omega \\ v &= 0 && \text{on } (0,T)\times\partial\Omega \\ v(0,x) &= v_0(x) && \text{in } \Omega \end{aligned}\right\} \tag{3.17}$$

with nonnegative and quasiconcave initial data v_o.

A natural conjecture would be:

$$v(t,x) \quad \text{is quasiconcave in } x \text{ for every } t. \tag{3.18}$$

In fact for $n = 1$ we already know (3.17) $\Rightarrow$ (3.18) from a result of Matano [39], see also [$39a, 39b, 41a$]. For $n \geq 2$ one can prove such a result if Ω and u_0 has additional symmetry properties by methods due to Gidas, Ni and Nirenberg [28], but for general convex Ω and quasiconcave u_0 the problem if (3.17) $\Rightarrow$ (3.18) appears to be open.

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