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REMARKS ON A FINSLER-LAPLACIAN

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Abstract. We investigate elementary properties of a Finsler-Laplacian operator Q that is associated with functionals containing $(H(\nabla u))^2$. Here H is convex and homogeneous of degree 1, and its polar H^o represents a Finsler metric on $\mathbb{R}^n$. In particular we study the Dirichlet problem $-Qu = 2n$ on a ball $K^o = \{x \in \mathbb{R}^n \ : \ H^o(x) < 1\}$ and present a fundamental solution for Q, suitable maximum and comparison principles, and a mean value property for solutions of $Qu = 0$.

1. Preliminaries

Throughout this paper let $H : \mathbb{R}^n \mapsto \mathbb{R}$ be a nonnegative convex function of class $C^2(\mathbb{R}^n \backslash \{0\})$ which is even and positively homogeneous of degree 1, so that

$$H(t\xi) = |t|H(\xi) \quad \text{for any} \quad t \in \mathbb{R},\ \xi \in \mathbb{R}^n. \tag{1.1}$$

A typical example is $H(\xi) = (\sum_i |\xi_i|^q)^{1/q}$ for $q \in [1, \infty)$.

We shall investigate Euler equations which involve functionals containing the expression

$$\int_\Omega (H(\nabla u))^2 \, dx \ . \tag{1.2}$$

The differential equations contain operators of the form

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i} \left((H(\nabla u))H_{\xi_i}(\nabla u)\right) \ . \tag{1.3}$$

Note that these operators are not linear unless H is the Euclidean norm. In particular, for $H(\xi) = (\sum_k |\xi_k|^q)^{1/q}$ the operator Q becomes

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left(\sum_{k=1}^n \left| \frac{\partial u}{\partial x_k} \right|^q \right)^{(2-q)/q} \left| \frac{\partial u}{\partial x_i} \right|^{q-2} \frac{\partial u}{\partial x_i} \right) \ . \tag{1.4}$$

We set

$$K := \{x \in \mathbb{R}^n : H(x) < 1\}.$$

and $\omega_n = |K|$. Sometimes we will say that H is the gauge of K. If one defines the support function of K as

$$H^o(x) := \sup_{\xi \in K} < x, \xi >,$$

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it is easy to verify that $H^o : \mathbb{R}^n \to [0, +\infty)$ is a convex, homogeneous function and that H, H^o are polar to each other in the sense that

$$H^o(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H(\xi)} \tag{1.5}$$

and

$$H(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H^o(\xi)}. \tag{1.6}$$

For example, it follows that

$$| < x, \xi > | \leq H(x) H^o(\xi).$$

Clearly $H^o(x)$ itself is the gauge of the set

$$K^o := \{x \in \mathbb{R}^n : H^o(x) \leq 1\}.$$

We say that K and K^o are polar to each other. Finally we observe that

$$H(\nabla H^o(x)) = 1 \tag{1.7}$$

and, as a consequence of (1.1), that

$$\sum_{i=1}^{n} H_{\xi_i}(\xi)\ \xi_i = H(\xi). \tag{1.8}$$

2. Constant datum

The simplest case of the function $H(\xi)$ is given by $H(\xi) = |\xi|$, with $H^o(\xi) = |\xi|$. Obviously we have $Qu = \Delta u$, and it is not difficult to show that the function $u(x) = 1 - |x|^2$ solves the problem

$$\begin{cases} -Qu = 2n & \text{in } B = \{x : |x| < 1\}, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{2.1}$$

We want to understand if a similar result holds true for a general operator Qu.

Example. Suppose that

$$H(\xi) = \left(\sum_k |\xi_k|^q \right)^{1/q}, \qquad q > 1.$$

Then it is easy to see that

$$H^o(\xi) = \left(\sum_k |\xi_k|^{q'} \right)^{1/q'}, \qquad q' = \frac{q}{q-1}$$

and

$$\frac{\partial}{\partial x_i} \left((H^o(x))^2 \right) = 2 H^o(x) \frac{\partial}{\partial x_i} (H^o(x)) = 2 (H^o(x))^{2-q'} x_i |x_i|^{q'-2}.$$

If $v(x) = (H^o(x))^2$, then because of (1.7), we have for our example

$$\begin{aligned} Qv &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left((H(\nabla (H^o(x))^2)) H_{\xi_i}(\nabla (H^o(x))^2) \right) \\ &= 2 \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(H^o(x) \frac{x_i}{H^o(x)} \right) = 2n. \end{aligned} \tag{2.2}$$

The calculation that we just did for the special H in the example leads one to believe that the same result must hold for general H. We can in fact prove the following result in two very different ways.

Theorem 2.1. *Consider the problem*

$$\begin{cases} -Qu = 2n & \text{in } K^0, \\ u = 0 & \text{on } \partial K^0. \end{cases} \tag{2.3}$$

The solution to problem (2.3) *is given by*

$$u(x) = 1 - (H^o(x))^2.$$

First proof. Follow the calculations leading up to (2.2) and observe that a) $H(\nabla H^o(x)) = 1$, b) $H_{\xi_i}(\nabla (H^o(x))^2) = H_{\xi_i}(\nabla H^o(x))$, and finally c) $x = H^o(x) H_\xi(\nabla H^o(x))$. The last identity can be found in Lemma 2.2 of [4]. □

Second proof. The solution to problem (2.3) can be found by minimizing the functional

$$J(u) = \int_{K^0} ((H(\nabla v))^2 - 4nv)\ dx\ . \tag{2.4}$$

The minimizer u of the functional J on $H_0^1(K^0)$ is unique, and, because of the Pólya-Szegö inequality, it has to be such that

$$u(x) = u(H^o(x)). \tag{2.5}$$

Indeed, if $u^\#$ denotes the convex symmetrization of u, then the following holds:

$$J(u) \geq J(u^\#).$$

So we need only consider functions of the form

$$v(x) = v(H^o(x)). \tag{2.6}$$

Taking into account (2.5) and (1.7) we have:

$$J(u) = \int_0^1 n\omega_n((v'(r))^2 - 4nv(r)) r^{n-1} dr. \tag{2.7}$$

The corresponding Euler equation of the one-dimensional problem is

$$-(v'(r)r^{n-1})' - 2nr^{n-1} = 0.$$

We immediately have

$$u(r) = 1 - r^2,$$

and then

$$u(x) = 1 - (H^o(x))^2.$$ □

Remark 2.2. If $v(x) = (H^o(x))^2$, then a straightforward calculation gives

$$Qv = 2\left(1 + H^o(x) \sum_{i,j} H_{\xi_i\xi_j}(\nabla H^o(x)) H^o_{x_i x_j}(x)\right).$$

On the other hand, Theorem 2.1 implies $Qv = 2n$. Thus we have shown that

$$H^o(x) \sum_{i,j} H_{\xi_i\xi_j}(\nabla H^o(x)) H^o_{x_i x_j}(x) = n - 1, \tag{2.8}$$

an identity which does not seem to be known.

Remark 2.3. The case of nonconstant right-hand side $-Qu = f(u)$ was treated in [6], in particular the eigenvalue problem $f(u) = -\lambda u$. For $n = 2$ and positive solutions we were able to show that all level sets of u are homothetic to K^o, as in Theorem 2.1. Notice that Theorem 2.1 applies to general n in the present paper. The desire to gain a deeper understanding of the behaviour of Q for general n was the motivation for our present study.

3. Fundamental solution for the operator Q

Our aim is to prove that when the datum of the Poisson equation involving the operator Q is a Dirac delta, then as in Theorem 2.1 the solution can be written in terms of $H^o(x)$.

Theorem 3.1. *The function*

$$u(x) = \begin{cases} \dfrac{1}{\alpha_n} \dfrac{(H^o(x))^{-(n-2)}}{n-2} & \text{if } n > 2, \\ -\frac{1}{\alpha_2} \log(H^o(x)) & \text{if } n = 2 \end{cases} \tag{3.1}$$

solves, in the sense of distribution, the equation

$$-Qu = \delta_0, \tag{3.2}$$

where α_n denotes the perimeter of the unit ball with respect to H^o and δ_0 denotes the Dirac measure at the origin.

Proof. By Theorem 2.1, a direct computation shows that

$$-Qu(x) = 0 \qquad \text{for } x \neq 0. \tag{3.3}$$

For the benefit of the reader let us give the details of this calculation in dimension $n > 2$. If we set $v(x) = (H^o(x))^2$ and $w = \frac{2}{n-2} v^{-(n-2)/2}$, we see that $\nabla w = -v^{-n/2}\nabla v$, so $H(\nabla w) = -v^{-n/2} H(\nabla v)$. Since H_ξ is homogeneous of degree zero, $H_\xi(\nabla w) = H - \xi(\nabla v)$. Thus, using Theorem 2.1,

$$\begin{aligned} -Qw &= \frac{\partial}{\partial x_i}\left(v^{-n/2} H(\nabla v) H_{\xi_i}(\nabla v)\right) \\ &= v^{-n/2} 2n - \frac{n}{2} v^{-(n+2)/2} H(\nabla v) H_{\xi_i}(\nabla v) \\ &= 2n v^{-(n+2)/2}\left(v - \frac{1}{4} H^2(\nabla v)\right). \end{aligned}$$

However, $H(\nabla v) = H(2H^o(x)\nabla H^o(x)) = 2H^o(x)$, so that (3.3) holds.

In view of (3.3), in order to prove the theorem, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \int_{\partial K^0_\varepsilon} H(\nabla u) H_{\xi_i}(\nabla u) \nu_i \varphi \, d\sigma = \varphi(0), \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \tag{3.4}$$

where $K^0_\varepsilon = \{x \in \mathbb{R}^n : H^o(x) \le \varepsilon\}$ and ν is the external normal to ∂K^0_ε. In our situation $\nu = -\nabla u / H(\nabla u)$.

First of all we observe that ∂K_ε^0 is a level set for u given in (3.1). Taking into account the homogeneity of the function H, we see this implies that

$$\begin{aligned} \int_{\partial K_\varepsilon^0} H(\nabla u) H_{\xi_i}(\nabla u)\nu_i \, \varphi \, d\sigma &= -\int_{\partial K_\varepsilon^0} H(\nabla u) H_{\xi_i}(\nabla u) \frac{u_{x_i}}{H(\nabla u)} \, \varphi \, d\sigma \\ &= -\int_{\partial K_\varepsilon^0} H(\nabla u) \, \varphi \, d\sigma. \end{aligned} \tag{3.5}$$

Now we observe that

$$\nabla u = -\frac{1}{\alpha_n} \frac{\nabla H^o(x)}{(H^o(x))^{n-1}}, \tag{3.6}$$

and, using again the homogeneity of H and (1.7), we have

$$H(\nabla u) = -\frac{1}{\alpha_n} H^o(x)^{1-n} = -(\alpha_n \varepsilon^{n-1})^{-1}, \tag{3.7}$$

which proves the assertion for $n > 2$. The proof for $n = 2$ is left as an exercise for the reader. $\square$

Remark 3.2. It is worth noting that the fundamental solution does not in general give rise to a Poisson representation formula, because Q is in general nonlinear. For the same reason we cannot construct Green's functions for Dirichlet problems on bounded domains.

4. Maximum and comparison principles for Q-harmonic functions

We can easily prove the following weak maximum principle for Q-subharmonic functions.

Theorem 4.1. *If $-Qu \le 0$ in Ω and $u = g \le M$ on $\partial\Omega$, then u attains its maximum on the boundary; that is, $u(x) \le M$ a.e. in Ω.*

Proof. We set $M = \max_{x \in \partial\Omega} g(x)$ and $\Omega^+ := \{x \in \Omega \ : u(x) > M\}$. Then we multiply $-Qu \le 0$ by $(u - M)^+$ and integrate over Ω^+ to obtain

$$\begin{aligned} 0 &\ge \int_{\Omega_+} H(\nabla u) H_{\xi_i}(\nabla u) u_{x_i} \, dx + \int_{\partial\Omega^+} H(\nabla u)(u - M)^+ H_{\xi_i}(\nabla u)\nu_i \, d\sigma(x) \\ &= \int_{\Omega_+} (H(\nabla u))^2 \, dx. \end{aligned} \tag{4.1}$$

Therefore Ω^+ has measure zero and $u(x) \le M$ a.e. in Ω. $\square$

In a similarly elegant way one can prove a comparison principle.

Theorem 4.2. *Suppose $-Qu \le -Qv$ in Ω and $u \le v$ on $\partial\Omega$. Then $u \le v$ a.e. in Ω.*

Proof. We assume that the set $\Omega^+ := \{u(x) > v(x)\}$ has positive measure and multiply the differential inequalities by $(u - v)^+$. Then

$$\int_{\Omega^+} [H(\nabla u) H_\xi(\nabla u) - H(\nabla v) H_\xi(\nabla v)] (\nabla u - \nabla v) \, dx \le 0,$$

so that by the strict convexity of H^2 we have $\nabla u = \nabla v$ a.e. in Ω^+. Since $u = v$ on $\partial\Omega^+$ we find that Ω^+ has measure zero. $\square$

5. Mean value property for Q-harmonic functions

Suppose u satisfies $Qu = 0$ in some domain Ω and $0 \in \Omega$. Then for sufficiently small ρ the ball $K^o_\rho = \{x \in \mathbb{R}^n \ : \ H^o(x) < \rho\}$ is contained in Ω. Harmonic functions are known to satisfy mean value properties. For Q-harmonic functions we can prove this only under the following assumption.

$$< H_\xi(a), H^o_\xi(b) >= \frac{< a, b >}{H(a)H^o(b)} \quad \text{for all } a, b \in \mathbb{R}^n. \tag{5.1}$$

This assumption is satisfied for

$$H(\xi) = \left(\sum_i (\beta_i^2 \xi_i^2)\right)^{1/2} \qquad \text{with} \qquad H^o(\eta) = \left(\sum_i (\eta_i^2/\beta_i^2)\right)^{1/2},$$

but violated if $H(\xi) = ||\xi||_p$ and $p \neq 2$.

Theorem 5.1. *Suppose that H and H^o satisfy* (5.1)*. If $Qu = 0$ in Ω and $K^o_\rho \subset \Omega$, then for every ball of radius $r \in (0, \rho)$ the function u satisfies the mean value property on spheres (where α_n denotes $|\partial K^o_1|$),*

$$u(0) = \frac{1}{\alpha_n r^{n-1}} \int_{\partial K^o_r} u(x)\ d\sigma, \tag{5.2}$$

as well as the corresponding mean value property on balls (where k_n denotes $|K^o_1|$),

$$u(0) = \frac{1}{k_n r^n} \int_{K^o_r} u(x)\ dx. \tag{5.3}$$

Proof. We set

$$\Phi(r) := \frac{1}{\alpha_n r^{n-1}} \int_{\partial K^o_r} u(x)\ d\sigma(x) = \frac{1}{\alpha_n} \int_{\partial K^o_1} u(rz)\ d\sigma(z)$$

and show that Φ is in fact constant. Indeed,

$$\Phi'(r) = \frac{1}{\alpha_n} \int_{\partial K^o_1} < \nabla u(rz), z > \ d\sigma(z) = \frac{1}{\alpha_n r^{n-1}} \int_{\partial K^o_r} < \nabla u(x), \frac{x}{r} > \ d\sigma(x),$$

and by (5.1) we have $< \nabla u, x >= H(\nabla u) < H_\xi(\nabla u), H^o_\xi(x) > H^o(x)$. Therefore, since $H^o(x) = r$ and $\nu = H^o_\xi(x)$ on ∂K^o_r, an integration by parts yields

$$\Phi'(r) = \int_{\partial K^o_r} \sum_{i=1}^n H(\nabla u) H_{\xi_i}(\nabla u)\ \nu_i\ d\sigma(z) = \int_{K^o_r} Qu\ d\sigma(x) = 0.$$

This proves the mean value property on spheres. The one for balls follows upon integration with respect to r. $\square$

Remark 5.2. Assumption (5.1) is not only sufficient but also necessary for the mean value property. Since Q is translation-invariant, a counterexample can be constructed from the fundamental solution if one considers for instance $H(\xi) = ||\xi||_p$ for p close to 1, $x_0 = (2, 0) \in \mathbb{R}^2$, and compares $u(x_0)$ to its mean value over the sphere $\partial K_1(x_0)$ of radius 1 centered at x_0.

Remark 5.3. Our notion of Q-harmonic function should not be confused with the “mean value Laplacian” from [7], [8] or the “Laplacian in Minkowski space” from [11]. Contrary to our definition, Centore’s and Thompson and Thompson’s is linear in u. This discrepancy is a common phenomenon in Finsler geometry, where certain notions like volume, which have equivalent definitions in Euclidean space, provide different objects depending on the definition. Our implicit definitions of volume and perimeter follow the concept in [5].

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