Comparison principle for viscosity solutions of fully nonlinear, degenerate elliptic equations

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1 Introduction

It is well known that the classical $C^2(\Omega)\cap C(\bar{\Omega})$ sub- and supersolutions of fully nonlinear elliptic equations

$$F(x,u,Du,D^2u)=0 \ \text{ in } \Omega\subset\mathbb{R}^n \tag{1.1}$$

satisfy the comparison principle when

(i) $F(x,r,p,X)$ is strictly increasing in r and equation (1.1) is degenerate elliptic in the sense of (1.2), or

(ii) $F(x,r,p,X)$ is nondecreasing in r and (1.1) is a uniformly elliptic equation (see [8]).

This simple result is valid for semicontinuous viscosity sub- and supersolutions only in the autonomous case, i.e. when $F(x,r,p,X)$ is independent of x (see [13]). When F depends on x, the nonstrict monotonicity of $F(x,r,p,X)$ with respect to r and the uniform ellipticity of the equation (1.1) are not enough for the validity of the comparison principle for semicontinuous viscosity sub- and supersolutions, as Example 1 at the end of this paper shows. The only case when this is true is if one of the sub- or supersolutions is a C^2 smooth function, or even piecewise C^2-function, i.e. a C^2-function in Ω with possible jumps only of the second normal derivative through a C^1-hypersurface in Ω (see Th. 3 in [12]). In general, some extra structure conditions on F are necessary, as for example condition (3.14) in [6] for degenerate elliptic equations or F^*4 in [17] for uniformly elliptic ones. As it is mentioned in Remark 3.4 in [6] and Example 3.6 there, the important condition (3.14), which naturally appears in the proof of the comparison principle, for first order linear equations is a necessary one as well. For the time being the real role of the important assumptions (3.14) in [6], or F^*4 in [17] is still not well understood. It seems that they are smoothness conditions with respect to x and at the same time growth conditions for

$F(x, r, p, X)$ in the p, X variables. There are many questions about the nature of these conditions, their necessity or how to find some other criteria guaranteeing the validity of the comparison principle for semicontinuous viscosity sub- and supersolutions of proper equations (1.1). Let us recall that F is proper if $F(x, r, p, X)$ is degenerate elliptic and monotone with respect to r, i.e. if for every $x \in \bar{\Omega}$, $p \in \mathbb{R}^n$, $X, Y \in S^n$ (where S^n is the space of all symmetric $n \times n$ matrices) the ellipticity condition (1.2) and the monotonicity condition (1.3) hold:

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ whenever } Y \leq X \ , \tag{1.2}$$

$$F(x, r, p, X) \leq F(x, s, p, X) \text{ whenever } r \leq s \ . \tag{1.3}$$

In the present paper we propose some conditions, which differ from (3.14) in [6] or F^*4 in [17], for the validity of the comparison principle for semicontinuous viscosity sub- and supersolutions. These conditions are almost natural, because they come from the simplest criterion for a priori estimates on the gradient of classical solutions to (1.1), with $F \in C^1$, by means of the Bernstein method, i.e.

$$\sum_{k=1}^{n} p_k F_{x_k}(x, r, p, X) + |p|^2 F_r(x, r, p, X) > 0 \tag{1.4}$$

(see [4] and [5]).

More precisely, for $F \in C^1$ our additional condition for proper equations (1.1), i.e. those satisfying (1.2) and (1.3), is the following one:

$$\sum_{k=1}^{n} p_k F_{x_k}(x, r, p, X) + \frac{1}{4}|p|^2 F_r(x, r, p, X) > 0 \tag{1.5}$$

whenever $F(x, r, p, X) = 0$. In this way we can even cover even equations which are not uniformly monotone with respect to r in the sense of (1.7).

Let us note that in view of (1.3) assumption (1.5) is weaker than (1.4). In this respect our results generalize those in [4] and [5]. In the general case, when F is only a continuous function, (1.5) has to be replaced by

$$F(x - t\frac{p}{|p|}, r - t\frac{|p|}{4}, p, X) < 0 < F(x + t\frac{p}{|p|}, r + t\frac{|p|}{4}, p, X) \tag{1.6}$$

whenever $F(x, r, p, X) = 0$ for $|p| \neq 0$ and for all sufficiently small $t > 0$.

(1.4) and (1.5) shed light on the nature of condition (1.6) as follows. It has a simple geometric interpretation. For $p \neq 0$ the map $t \mapsto F(x + t\frac{p}{|p|}, r + t\frac{|p|}{4}, p, X)$ passes at $t = 0$ through the zeroes of $F(x, r, p, X)$ only from minus to plus. Moreover, the corresponding comparison result in Theorem 2.2 in the present paper is an extension of Theorem 3.3 in [6], because the uniform monotonicity of $F(x, r, p, X)$ with respect to r, namely condition (1.7)

$$\gamma(r - s) \leq F(x, r, p, X) - F(x, s, p, X) \tag{1.7}$$

for every $r \geq s$, $r, s, \in \mathbb{R}$, $x \in \bar{\Omega}$, $p \in \mathbb{R}^n$, $X \in S^n$ and for some positive constant $\gamma > 0$, is now replaced by the weak monotonicity condition (1.3). For example, the entire class of equations (1.1) which are independent of u satisfies (1.3) but not (1.7) and Theorem 2.2 can be applied to them.

We should also point out, that even when (1.7) holds, our results generalize those in [6]. In fact, a careful analysis of the proof of Theorem 2.2 shows that the comparison principle holds under the following generalization of (1.6), which is weaker than (3.14) in [6]:

$$F(x - t\frac{p}{|p|}, r - t\frac{|p|}{4} - M, p, X) < 0 < F(x + t\frac{p}{|p|}, r + t\frac{|p|}{4} + M, p, X) \tag{1.8}$$

whenever $F(x, r, p, X) = 0$ for every positive constant M and for every $t \in (0, t_0]$, where t_0 depends on M, $|p|$ and $||X||$. Condition (1.8) generalizes (3.14) in [6], but it is not as easily checked as (1.6).

Let us mention that there are only few results (see for example. Th. 3.1 in [11], Th. 2.6 in [3], Th. 1.1 in [17]) where the comparison principle was proved under the weaker condition (1.3) instead of (1.7) but only for uniormly elliptic equations (1.1) and continuous or Lipschitz continuous viscosity sub- and supersolutions (see also Th. 3 in [12] where one of the sub- or supersolutions is a piecewise C^2-function). Thus the result in Theorem 2.2 is new for general proper equations (1.1). Incidentally, in [14] the uniform ellipticity condition was replaced by the following strict ellipticity condition: there exists a strictly increasing function $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega_1(0^+) = 0$ such that $F(x, r, p, X + k\ \xi \otimes \xi) \leq F(X) + \omega_1(k)$ for every $k > 0$ and $\xi \in \mathbb{R}^n$. This allowed for applications to fully nonlinear Hessian equations $S(\kappa_1, \dots \kappa_n) = 0$. Here κ_i are the eigenvalues of the Hessian matrix D^2u.

Another advantage of (1.6) is the possibility to replace this condition by a weaker one of the same form when equation (1.1) is a "nondegenerate elliptic equation in at least one direction" at every point $x \in \Omega$, i.e. when

$$F(x, r, p, X) > F(x, r, p, X + \varepsilon Z) \tag{1.9}$$

for every $(x, r, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n$, $p \neq 0$, for all sufficiently small constants $\varepsilon > 0$ and for some $Z \in S^n$, $Z \geq 0$, $Z \neq 0$ (possibly depending on x, r, p and X). A typical example of an equation satisfying (1.9) is the ∞-Laplacian, i.e.

$$-\Delta_\infty u + f(x, u, Du) = -\sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + f(x, u, Du) = 0 \quad \text{in } \Omega. \tag{1.10}$$

This equation is nondegenerate in the nonzero gradient direction p, because (1.9) holds with $Z = p \otimes p$, $p = Du \neq 0$, see also (3) and (4) in [1] for a related condition. In this case a sufficient condition for the validity of the comparison principle is the right inequality in (1.6). As for locally uniformly elliptic equations (1.1) such as mean curvature type equations, it suffices to suppose the weak inequality in (1.6).

Finally let us mention that the comparison principle is the key argument in the Perron method for viscosity solutions ([9], [10], [6]), which guarantees existence of a unique viscosity solution of (1.1) if there are sub- and supersolutions satisfying continuously the (Dirichlet) boundary data. In this way, under the conditions of the comparison principle, we get existence of a unique viscosity solution.

The paper is organized in the following way. In Section 2 the main results and definitions are given. Section 3 deals with the proofs of the main results and in Section 4 some applications to the m-Laplacian, ∞-Laplacian, mean curvature type equations and others are considered. These examples illustrate the advantage of the Perron method for equations in nondivergence form in comparison with some other methods, as for example the fixed point theorem or the method of continuity on parameter.

2 Main results and definitions

Throughout the paper we assume that $F(x, r, p, X)$ is a continuous function of all its variables. For some classes of equations we will suppose extra local Lipschitz regularity of F with respect to p, i.e.

$$|F(x,r,p,X) - F(x,r,q,X)| \le C_0(R)|p-q| \tag{2.1}$$

for every $x \in \bar{\Omega}$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^n$, $X \in S^n$, $|r| + |p| + |q| + \|X\| \le R$. If C_0 is independent of R then F is globally Lipschitz continous in p. Further on we will also use the local uniform ellipticity condition (2.2)

$$C_1(R)\text{trace } (X - Y) \le F(x,r,p,Y) - F(x,r,p,X) \le C_2(R)\text{trace } (X - Y) \tag{2.2}$$

for every $X \ge Y$, $x \in \bar{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X, Y \in S^n$, $|r| + |p| + \|X\| + \|Y\| \le R$, where $C_i(R) > 0$, $i = 1, 2$. When C_i are positive constants independent of R then (2.2) turns into the global uniform ellipticity condition.

Definition 2.1 *Suppose F satisfies (1.2), (1.3). An upper semicontinuous function $u \in USC(\Omega)$ is a viscosity subsolution of (1.1) if $F(x_0, u(x_0), p, X) \le 0$ for every $x_0 \in \Omega$ and for every $(p, X) \in J^{2,+}u(x_0)$.*

Here the second order superjet $J^{2,+}_\Omega u(x_0)$ is defined as the set of those $(p, X) \in \mathbb{R}^n \times S^n$ for which $u(x) \le u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2)$ as $x \to x_0$.

Analogously, the definition of a lower continuous viscosity supersolution $v \in LSC(\Omega)$ is given by means of the opposite inequality $F(y_0, v(y_0), q, Y) \ge 0$ for every $y_0 \in \Omega$ and for every $(q, Y) \in J^{2,-}_\Omega v(y_0)$, where $J^{2,-}_\Omega v(y_0) = J^{2,+}_\Omega(-v(y_0))$.

Finally a continuous function in Ω is a viscosity solution of (1.1) if it is both a viscosity sub- and supersolution.

Throughout the paper by "comparison principle for semicontinuous viscosity sub- and supersolutions" we mean the following statement.

Comparison principle. If $u \in USC(\bar{\Omega})$, $v \in LSC(\bar{\Omega})$ are bounded viscosity sub- and supersolutions of (1.1) and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Theorem 2.2 *Let $u \in USC(\bar\Omega)$, $v \in LSC(\bar\Omega)$ be bounded, $|u|, |v| \le R_0$, viscosity sub- and supersolutions of (1.1). Suppose $F(x, r, p, X)$ satisfies (1.2), (1.3), (1.6) and (2.3)*

$$\forall x \in \Omega \text{ the map } r \mapsto F(x, r, 0, 0) \text{ has at most one zero in } [-R_0, R_0]. \tag{2.3}$$

If $u \le v$ on $\partial\Omega$, then $u \le v$ in $\bar\Omega$.

Remark 2.3 *Note that in view of (1.3) the set of zeroes in the map defined in (2.3) must be an interval. As Example 2 in Section 4 shows, condition (2.3) is necessary for the validity of the comparison principle even for classical solutions of (1.1) under assumptions (1.2), (1.3) and (1.6).*

As for condition (1.6), it cannot be replaced by only one inequality. This follows from Example 3 in Section 4. However, this is true for equations (1.1) which are "nondegenerate in at least one direction" in the sense that (1.9) is satisfied.

Theorem 2.4 *Let $u \in USC(\bar\Omega)$, $v \in LSC(\bar\Omega)$ be bounded, $|u|, |v| \le R_0$, viscosity sub- and supersolutions of (1.1). Suppose $F(x, r, p, X)$ satisfies (1.2), (1.3), (1.9), (2.3) and (2.4)*

$$F(x + t\frac{p}{|p|}, r + t\frac{|p|}{4}, p, X) > 0 \tag{2.4}$$

whenever $F(x, r, p, X) = 0$ for $|p| \ne 0$ and for all sufficiently small positive values of t.

If $u \le v$ on $\partial\Omega$, then $u \le v$ in $\bar\Omega$.

Remark 2.5 *Condition (1.9) in Theorem 2.4 is necessary, see Example 3 in Section 4. In this case all the assumptions of Theorem 2.4 are satisfied except (1.9), but a comparison principle does not hold.*

Further relaxations of the conditions in Theorems 2.2 and 2.4 are possible when the function $F(x, r, p, X)$ is locally uniformly elliptic and a locally Lipschitz continuous function with respect to p, i.e. when (2.1) and (2.2) are satisfied instead of (1.2). In this case (2.3) is superfluous, while (2.4) should be replaced by the similar and slightly more general inequality, i.e.

$$F(x + t\frac{p}{|p|}, r + t\frac{|p|}{2}, p, X) \ge 0 \tag{2.5}$$

whenever $F(x, r, p, X) = 0$ for $|p| \ne 0$ and for all sufficiently small positive values of $t > 0$.

Theorem 2.6 *Let $u \in USC(\bar\Omega)$, $v \in LSC(\bar\Omega)$ be bounded viscosity sub- and supersolutions of (1.1). Suppose $F(x, r, p, X)$ satisfies (1.3), (2.1), (2.2) and (2.5).*

If $u \le v$ on $\partial\Omega$ then $u \le v$ in $\bar\Omega$.

Remark 2.7 *The local Lipschitz continuity of $F(x,r,p,X)$ with respect to p cannot be relaxed even for classical solutions of globally uniformly elliptic equations as a counterexample from [2] shows (see Example 2 in Section 4). The structure condition (2.5) is also necessary for the comparison result even for globally uniformly elliptic equations with analytical coefficients, as illustrated by Example 1 in Section 4.*

Finally, using the comparison principles in Theorems 2.2, 2.4 and 2.6 one can apply Perron's method in order to prove existence of a unique continuous viscosity solution of (1.1), provided there exist a viscosity sub- and supersolution of (1.1).

Theorem 2.8 *Suppose $v \in USC(\bar{\Omega})$, $w \in LSC(\bar{\Omega})$, $v \le w$, are bounded ($|v|, |w| \le R_0$) viscosity sub- and supersolutions of (1.1).*

If the conditions of at least one of the Theorems 2.2, 2.4 or 2.6 are satisfied, then there exist a unique viscosity solution $u \in C(\Omega)$ of (1.1). Moreover, $v \le u \le w$ in Ω.

If additionally $v_*(x) = w^*(x) = \varphi(x)$ on $\partial\Omega$, then $u = \varphi$ on $\partial\Omega$. Here $v_*(x)$ and $w^*(x)$ are the LSC and USC envelopes of v and w, see (4.1) in [6].

3 Proof of the main results

The proof of Theorem 2.2 uses the main idea from [6] of doubling the number of independent variables as in the proof of the comparison principle, Theorem 3.3 in [6], but this time with a different auxiliary function.

Proof of Theorem 2.2. Suppose $\sup_{\overline{\Omega}}(u(x) - v(x)) = M > 0$. From the condition $u(x) - v(x) \le 0$ on $\partial\Omega$ it follows that $(u - v) \in USC(\bar{\Omega})$ attains its positive maximum at some interior point of Ω. Let us consider the auxiliary function $w(x,y) = u(x) - v(y) - \frac{N}{4}|x-y|^4$ in $\Omega \times \Omega$ for all sufficiently large positive constants N. It is clear that $w(x,y)$ has a positive maximum $M_N \ge M$ at some point (x_N, y_N). As in the proof of Theorem 3.1 in [6], it is easy to check that $N|x_N - y_N|^4 \to 0$, $x_N, y_N \to z \in \Omega$, $u(z) - v(z) = M$, $M_N \to M$ when $N \to \infty$ so that x_N, y_N are interior points of Ω for N sufficiently large. Moreover, there exist matrices $X, Y \in S^n$ satisfying the inequality

$$(3.1) \qquad -(N + \|A\|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \frac{1}{N}A^2$$

where $A = \begin{pmatrix} A_1 & -A_1 \\ -A_1 & A_1 \end{pmatrix}$ and $A_1 = N|x_N - y_N|^2 I + 2N\{(x_N - y_N) \otimes (x_N - y_N)\}$ is such that for $p = N|x_N - y_N|^2(x_N - y_N)$

$$(3.2) \qquad (p, X) \in \overline{J^{2,+}}_{\Omega} u(x_N) \quad \text{and} \quad (p, Y) \in \overline{J^{2,-}}_{\Omega} v(y_N).$$

Consequently, see (3.25) in [6],

$$(3.3) \qquad \|X\|, \|Y\| \le 2\|A\|, \qquad X \le Y.$$

If $x_N = y_N$ then $A = 0$ so that from (3.1) we have $X \le 0 \le Y$, $M_N = M$, $x_N = y_N = z$, $u(z) - v(z) = M$ for some $z \in \Omega$. From Definition 2.1 and (1.2), (1.3) we get the inequalities

$$\begin{aligned} F(z, v(z), 0, 0) &\le F(z, u(z), 0, X) \le 0 \\ &\le F(z, v(z), 0, Y) \le F(z, v(z), 0, 0). \end{aligned}$$

Hence $F(z, v(z) + t, 0, 0) \equiv 0$ for every $0 \le t \le M$ which contradicts (2.3).

If $x_N \ne y_N$ then for $p = N|x_N - y_N|^2(x_N - y_N)$, $t = |x_N - y_N|$ we have similar inequalities as above, using (1.3), (3.3) and (1.2), (3.2) etc:

$$\begin{aligned} F(x_N, v(y_N), p, Y) &\le F(x_N, v(y_N) + t\frac{|p|}{4}, p, Y) \\ &\le F(x_N, v(y_N) + M_N + t\frac{|p|}{4}, p, Y) \\ &\le F(x_N, v(y_N) + M_N + t\frac{|p|}{4}, p, X) \\ &= F(x_N, u(x_N), p, X) \\ &\le 0 \le F(y_N, v(y_N), p, Y) \le F(y_N, v(y_N), p, X). \end{aligned} \tag{3.4}$$

In case that $F(y_N, v(y_N), p, Y) = 0$ then from the right inequality in (1.6) it follows that $F(y_N + \tau\frac{p}{|p|}, v(y_N) + \tau\frac{|p|}{4}, p, Y) > 0$ for all sufficiently small positive $\tau > 0$. Since from (3.4) $F(y_N + t\frac{p}{|p|}, v(y_N) + t\frac{|p|}{4}, p, Y) \le 0$, there exist a positive $0 < \tau_0 \le t = |x_N - y_N|$ with the properties $F(y_N + \tau\frac{p}{|p|}, v(y_N) + \tau\frac{|p|}{4}, p, Y) > 0$ for $0 < \tau < \tau_0$ and

$$F(y_N + \tau_0\frac{p}{|p|}, v(y_N) + \tau_0\frac{|p|}{4}, p, Y) = 0.$$

However, this is impossible according to the left inequality (1.6) for $x = y_N + \tau_0\frac{p}{|p|}$, $r = v(y_N) + \tau_0\frac{|p|}{4}$.

If $F(y_N, v(y_N), p, Y) > 0$ then from (3.4)

$$F(y_N + t\frac{p}{|p|}, v(y_N) + t\frac{|p|}{4}, p, Y) = F(x_N, v(y_N) + t\frac{|p|}{4}, p, Y) \le 0$$

so that there exists $0 < \tau_1 \le t = |x_N - y_N|$ with the properties $F(y_N + \tau\frac{p}{|p|}, v(y_N) + \tau\frac{|p|}{4}, p, Y) > 0$ for $0 < \tau < \tau_1$, but $F(y_N + \tau_1\frac{p}{|p|}, v(y_N) + \tau_1\frac{|p|}{4}, p, Y) = 0$. This is impossible according to the left inequality (1.6).

Proof of Theorem 2.4. The proof of Theorem 2.4 is similar to the proof of Theorem 2.2 and we will point out only the differences. Choosing the same auxiliary function as in the proof of Theorem 2.2 and repeating the same arguments, now we are not able to exclude the case $F(y_N + \tau\frac{p}{|p|}, v(y_N) + \tau\frac{|p|}{4}, p, Y) > 0$ for $0 < \tau < \tau_2 \le t = |x_N - y_N|$ but $F(y_N + \tau_2\frac{p}{|p|}, v(y_N) + \tau_2\frac{|p|}{4}, p, Y) = 0$ in (3.4) by means of (2.4).

Therefore let us now fix some τ_0, $0<\tau_0<\tau_2$ for which $F(y_N+\tau_0\frac{p}{|p|}, v(y_N)+\tau_0\frac{|p|}{4}, p, Y)>0$. From (1.9) for the point $(y_N+\tau_2\frac{p}{|p|}, v(y_N)+\tau_2\frac{|p|}{4}, p, Y)$ it follows that there exists a matrix $Z\geq 0$ such that

$$\begin{aligned}&F(y_N+\tau_2\frac{p}{|p|}, v(y_N)+\tau_2\frac{|p|}{4}, p, Y+\varepsilon Z)\\&<F(y_N+\tau_2\frac{p}{|p|}, v(y_N)+\tau_2\frac{|p|}{4}, p, Y)=0\end{aligned}$$

for all sufficiently small positive constants $\varepsilon>0$. If $\varepsilon>0$ is sufficiently small then we can guarantee also the inequality $F(y_N+\tau_0\frac{p}{|p|}, v(y_N)+\tau_0\frac{|p|}{4}, p, Y+\varepsilon Z)>0$ because of the choice of τ_0. Since $F(y_N+\tau_2\frac{p}{|p|}, v(y_N)+\tau_2\frac{|p|}{4}, p, Y+\varepsilon Z)<0<F(y_N+\tau_0\frac{p}{|p|}, v(y_N)+\tau_0\frac{|p|}{4}, p, Y+\varepsilon Z)$ for $\tau_0<\tau_2$, then there exist $\tau_0<\tau_1<\tau_2$ with the properties $F(y_N+\tau\frac{p}{|p|}, v(y_N)+\tau\frac{|p|}{4}, p, Y+\varepsilon Z)<0$ for every $\tau_1<\tau\leq\tau_2$ and $F(y_N+\tau_1\frac{p}{|p|}, v(y_N)+\tau_1\frac{|p|}{4}, p, Y+\varepsilon Z)=0$, which contradicts (2.4).

Proof of Theorem 2.6. Since $F(x,r,p,X)$ does not satisfy (2.3) and since (2.5) is a nonstrict inequality we need some different barrier technique which is inspired by [13].

If $\sup_{\bar\Omega}(u(x)-v(x))=M>0$, we consider the auxiliary function $w(x,y)=u(x)-v(y)-N|x-y|^2/2+\exp(-N_1|x|^2/2)$. Here without loss of generality we assume that $\Omega\subset\{x\in\mathbb{R}^n, r_1<|x|<r_2\}$ and that N, N_1 are sufficiently large positive constants, where $N_1=N_1(N)$ is chosen so that

$$\|e^{-N_1|x|^2/2}\|_{C^2(\bar\Omega)} \leq 1, \tag{3.5}$$

$$r_1C_1N_1 > \max(Nr_1^2C_1, 2C_0r_2), \tag{3.6}$$

$$n+4n^4N_1^3e^{-N_1r_1^2/2} \leq N_1r_1^2/2\ . \tag{3.7}$$

The constants $C_i=C_i(R)>0$, $i=0,1$ in (3.6) are defined in (2.1), (2.2) for $R=(81n^4+2r_2)N+2R_0$ and R_0 is implicitly defined by $|u|,|v|\leq R_0$.

As in the proofs of Theorems 2.2 and 2.4 we conclude that

$$(N(x_N-y_N)+N_1x_Ne^{-N_1|x_N|^2/2}, X)\in\overline{J^{2,+}}_\Omega u(x_N), (N(x_N-y_N), Y)\in\overline{J^{2,-}}_\Omega v(y_N)$$

for some matrices $X,Y\in S^n$ satisfying (3.1) with

$$A=\begin{pmatrix}NI+B & -NI\\ -NI & NI\end{pmatrix},\qquad\text{and}\qquad B=N_1e^{-N_1|x_N|^2/2}\,(I+N_1\{x_N\otimes x_N\}).$$

Simple computations (see (17) in [13] for more details) show that

$$\text{trace }(X-Y)\leq -N_1^2(r_1^2/2)e^{-N_1|x_N|^2/2}. \tag{3.8}$$

Now for $p=N(x_N-y_N)$ from Definition 2.1, (1.3), (2.1), (2.2), (3.5)–(3.7) we get the following chain of inequalities

$$F(y_N, v(y_N), p, Y)\geq 0\geq F(x_N, u(x_N), p+N_1x_N\ e^{-N_1|x_N|^2/2}, X)$$

$$\geq F(x_N, v(y_N) + |x_N - y_N|\frac{|p|}{2} + M, p, Y) + C_1 \text{ trace } (Y - X) - C_0 N_1 |x_N| \; e^{-N_1|x_N|^2/2}$$

$$\geq F(x_N, v(y_N) + |x_N - y_N| \cdot \frac{|p|}{2}, p, Y) + (C_1 N_1^2 r_1^2/2 - C_0 N_1 r_2) \; e^{-N_1|x_N|^2/2}$$

$$> F(x_N, v(y_N) + |x_N - y_N|\frac{|p|}{2}, p, Y)$$

From the above inequalities we have immediately that $x_N \neq y_N$, i.e. $p \neq 0$. The rest of the proof follows from (2.5) in the same way as the proof of Theorems 2.2 and 2.4.

The proof of Theorem 2.8 follows from the Perron method (see [9], [10], [6]) and is left as an exercise to the reader.

4 Some examples and applications

We will start this section with some simple examples which illustrate the sharpness of the conditions in Theorems 2.2, 2.4 and 2.6.

Example 1. This example will illustrate the importance of assumption (2.5) in Theorem 2.6. Consider the following boundary value problem

$$\begin{aligned} &-u''(x) - 18x(u'(x))^4 = 0 \quad \text{in } (-1, 1), \\ &u(-1) = A, \qquad u(1) = B. \end{aligned} \tag{4.1}$$

By separation of variables and as long as $u'(x) \neq 0$ one obtains

$$u'(x) = \frac{K}{[1 - 27K^3(1 - x^2)]^{1/3}} \qquad \text{with } K := u'(-1), \tag{4.2}$$

so that the sign of K determines the sign of u' throughout $(-1, 1)$. Another integration leads to

$$u(x) = A + \int_{-1}^{x} \frac{dt}{[K^{-3} - 27(1 - t^2)]^{1/3}} \tag{4.3}$$

and the requirement $u(1) = B$ implies

$$A + \int_{-1}^{1} \frac{dt}{[K^{-3} - 27(1 - t^2)]^{1/3}} = B. \tag{4.4}$$

The left hand side in (4.4) is monotone decreasing in K and bounded above as $K \to -\infty$. But then for $A > B$

$$\int_{-1}^{1} \frac{dt}{\sqrt[3]{27(1 - t^2)}} \geq A - B \quad . \tag{4.5}$$

is a necessary condition for the existence of a a unique classical solution $u_1 \in C^2(-1, 1) \cap C([-1, 1])$ which satisfies the boundary data in a classical sense, see the plot on top in Figure 1. For large $A - B$ inequality (4.5) is no longer valid.

In that case problem (4.1) has only viscosity solutions with infinite slope $K = -\infty$ on the boundary

$$u_C(x) = C - \int_{-1}^{x} \frac{dt}{\sqrt[3]{27(1-t^2)}}, \quad C = \text{const}, \ A \geq C \geq B.$$

More precisely, $u_C(x) \in C^2(-1,1) \cap C([-1,1])$, but it satisfies the Dirichlet conditions only in a weak (viscosity) sense (see Section 7 in [6]). If $C = A$ then $u_A(x)$ satisfies the Dirichlet data at -1 in a classical sense and at $+1$ in a weak (viscosity) sense (see center plot in Figure 1) while for $C = B + \int_{-1}^{1}[27(1-t^2)]^{-1/3}\, dt$ the solution $u_B(x)$ satisfies Dirichlet data in a classical sense at $x = 1$ and in a weak (viscosity) sense at $x = -1$ (see bottom plot in Figure 1).

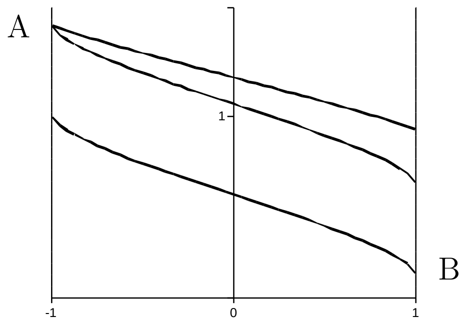

Figure 1: Classical and viscosity solutions of (4.1)

However, the comparison principle still holds since $u'(x) < 0$ for every $x \in (-1,1)$. Moreover, when $-X - 18xp^4 = 0$, $p \neq 0$, we get the inequality $-X - 18(x + \frac{tp}{|p|})p^4 = -18tp^3|p| > 0$ so that condition (2.5) in Theorem 2.6 is fulfilled.

The situation is totally different when $A < B$. In this case $u'(x) > 0$ for every $x \in (-1,1)$ and condition (2.5) in Theorem 2.6 is not satisfied according to the above calculation. For simplicity let us take $A = -B$. One can easily check that for $0 < B < 1$ problem (4.1) still has a unique classical solution $u(x) \in C^2([-1,1])$. However, for $B > 1$ there is no longer a continuous viscosity solution of (4.1), while for $B = 1$ the unique viscosity solution is $u(x) = \sqrt[3]{x}$. The functions $\tilde{u}$ and $\tilde{v}$ given by

$$\tilde{u} = \begin{cases} \sqrt[3]{x} - 1 + B & \text{for } \ 0 \leq x \leq 1, \\ \sqrt[3]{x} + 1 - B & \text{for } \ -1 \leq x < 0, \end{cases}$$

$$\tilde{v} = \begin{cases} \sqrt[3]{x} - 1 + B & \text{for } \ 0 < x \leq 1, \\ \sqrt[3]{x} + 1 - B & \text{for } \ -1 \leq x \leq 0, \end{cases}$$

differ only in the origin. $\tilde{u} \in USC([-1,1])$ and $\tilde{v} \in LSC[-1,1]$ are viscosity sub- and supersolutions of (4.1) for $B > 1$ with identical Dirichlet data, but $\sup_{[-1,1]}(\tilde{u} - \tilde{v}) = \tilde{u}(0) - \tilde{v}(0) = 2(B-1) > 0$ and the comparison principle fails (see Figure 2).

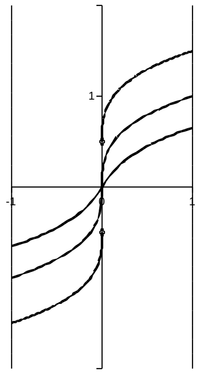

Figure 2: The functions $\tilde{u}$ and $\tilde{v}$ violate the comparison principle

Note that equation (4.1) does not satisfy condition (3.14) in [6], nor does it satisfy the strict monotonicity condition (1.7) with respect to r. Indeed, $F(x,r,p,X) = -X - 18xp^4$ and for

$$-3N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \le \begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix} \le 3N\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}$$

the inequality $F(y,r,N(x-y),Y) - F(x,r,N(x-y),X) = -Y - 18yN^4(x-y)^4 + X + 18xN^4(x-y) = X - Y + 18N^4(x-y)^5 \le \omega(N|x-y|^2 + |x-y|)$ cannot be fulfilled for every $x,y \in (-1,1)$, $N > 0$ and some modulus of continuity $\omega(t), \omega(t) \in C([0,\infty)), \omega(t) > 0$ for $t > 0$, $\omega(0) = 0$. In the same way one can check that condition F^*4 in [17] is not satisfied for (4.1). However, according to Theorem 2.6 the comparison principle holds for all monotone decreasing solutions of (4.1).

Example 2. This example illustrates the necessity of assumption (2.3) in Theorems 2.2 and 2.4.

$$\begin{aligned}&-\Delta u + |Du|^2 + |Du|^m e^{(1-m)u} = 0 \text{ in } B = \{|x| < 1\} \subset \mathbb{R}^n\\ &u = C = \text{const on } \partial B, \text{ where } m \text{ is a constant, } 0 < m < 1.\end{aligned}$$

This equation, after an appropriate change of the variables was considered by G. Barles, G. Diaz, J.I. Diaz in [2]. For $C < -ln\ k(k+n-2)^{\frac{1}{m-1}}$, with $k = (2-m)/(1-m)$, the above problem has two classical solutions,

$$u_1(x) \equiv C \quad \text{ and } \quad u_2(x) = -ln[e^{-C} - k(k+n-2)^{\frac{1}{m-1}}(1 - |x|^k)],$$

so that the comparison principle fails even for classical solutions. All assumptions of Theorems 2.2 and 2.4 are satisfied except (2.3). Since $m \in (0,1)$ the function $F(x,r,p,X) = -\text{trace } X + |p|^2 + |p|^m e^{(1-m)r}$ is monotone with respect to r, uniformly elliptic and $-\text{trace } X + |p|^2 + |p|^m e^{(1-m)(r + \frac{t|p|}{4})} > 0$ whenever $-\text{trace } X + |p|^2 + |p|^m e^{(1-m)r} = 0$ for $|p| \ne 0$. However, $F(x,r,0,0) \equiv 0$ and (2.3) fails.

Example 3. Consider the ordinary differential equation

$$(x^{1/3}u'(x) - 2/3)^2 = 0 \quad \text{in } (-1,1) \quad . \tag{4.6}$$

All conditions of Theorem 2.2 are satisfied except the left inequality in (1.6). Really, $F(x,r,p,X) = (x^{1/3}p - 2/3)^2$ and since $F(x,p,0,0) \neq 0$ (2.3) follows trivially. If $F(x,r,p,X) = 0$, i.e. $x^{1/3}p - 2/3 = 0$ then

$$[(x+t\frac{p}{|p|})^{1/3}p - 2/3]^2 = [(x+t\frac{p}{|p|})^{1/3}p - x^{1/3}p]^2 = p^2[(x+t\frac{p}{|p|})^{1/3} - x^{1/3}]^2 > 0$$

for $|p| \neq 0$. Thus the right inequality in (1.6) is true but the left one fails because

$$[(x-t\frac{p}{|p|})^{1/3}p - 2/3]^2 = p^2[(x-t\frac{p}{|p|})^{1/3} - x^{1/3}]^2 > 0 \text{ for } |p| \neq 0.$$

This example shows that the left and right inequality in (1.6) are necessary for the validity of the comparison principle in the degenerate case. One can easily check that the functions

$$\tilde{u} = \begin{cases} x^{2/3} & \text{for } 0 < x \leq 1 \\ x^{2/3}+1 & \text{for } -1 \leq x \leq 0 \end{cases} \quad , \quad \tilde{v} = \begin{cases} x^{2/3} & \text{for } 0 \leq x \leq 1 \\ x^{2/3}+1 & \text{for } -1 \leq x < 0 \end{cases}$$

satisfy $u \in USC([-1,1])$, $\tilde{v} \in LSC([-1,1])$ and are viscosity sub- and supersolutions of (4.6) which violate the comparison principle, because $\tilde{u}(0) - \tilde{v}(0) = 1 > 0$ (see Figure 3).

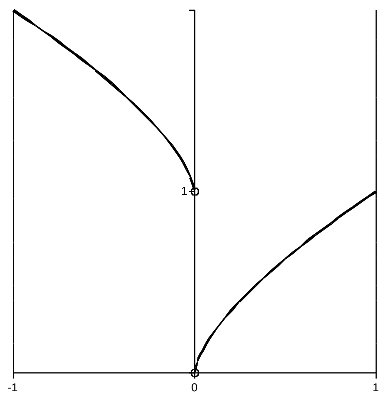

Figure 3: The functions $\tilde{u}$ and $\tilde{v}$ violate the comparison principle

Example 4. We modify Example 3 and consider

$$x^{\frac{1}{3}}u'(x) - 2/3 = 0 \quad \text{in } (-1,1) \quad . \tag{4.7}$$

This equation is almost the same as (4.6), however all conditions of Theorem 2.2 are fulfilled for (4.7) and the comparison principle holds for (4.7). Since $F(x,r,p,X) =$

$x^{1/3}p-2/3$, F is nonstrictly monotone with respect to r, degenerate elliptic, $F(x,r,0,0) = -2/3 \not\equiv 0$ with respect to r and $(x - t\frac{p}{|p|})^{1/3}p - 2/3 < 0 < (x + t\frac{p}{|p|})^{1/3}p - 2/3$ for $p \neq 0$, whenever $x^{1/3}p - 2/3 = 0$. Really,

$$(x \pm t\frac{p}{|p|})^{1/3}p-2/3 = [(x \pm t\frac{p}{|p|})^{1/3} - x^{1/3}]p = \pm t|p|/[(x \pm t\frac{p}{|p|})^{2/3} + (x \pm t\frac{p}{|p|})^{1/3}x^{1/3} + x^{2/3}]$$

has the right sign. Note that condition (3.14) in [6] is also satisfied because

$$\begin{aligned}&F(y,r,N(x-y),Y) - F(x,r,N(x-y),X = -N(x^{1/3} - y^{1/3})(x-y)\\&= -N(x^{1/3} - y^{1/3})^2(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}) \le 0 \le \omega(N|x-y|^2 + |x-y|)\end{aligned}$$

for every modulas of continuity $\omega(t)$. However the Dirichlet problem $u(-1) = A$, $u(1) = B$, $A, B = \text{const}$ has no viscosity solution. We will check this statement only for $A = B = 1$. Since the function $v = L(1-x^2)+1$ is a classical subsolution (and hence a viscosity subsolution of (4.7) for all positive constants L, satisfying the boundary data, from the comparison principle we get the estimate $u(x) \ge v(x)$ in $(-1,1)$ for the viscosity solution $u(x) \in C([-1,1])$ of (4.7). Thus $u(0) \ge v(0) = L$ for every $L > 0$ which is impossible.

This example shows that the assumptions in Theorem 2.8 for existence of viscosity sub- and supersolutions are necessary for the Perron method.

Example 5. The following example of a mean curvature type equation illustrates the importance of condition (2.5).

$$-[(1+|Du|^2)I - Du \otimes Du]D^2u + H(x,u,Du)(1+|Du|^2)^{3/2} = 0. \tag{4.8}$$

If $H(x,r,p)$ is locally Lipschitz continuous with respect to p, then according to Theorem 2.6 the comparison principle holds provided

$$\begin{aligned}&H(x + t\frac{p}{|p|}, r + t\frac{|p|}{2}, p) \ge H(x,r,p)\\&\text{for every } x \in \Omega,\ |r| \le R_0,\ p \neq 0,\ p \in \mathbb{R}^n.\end{aligned} \tag{4.9}$$

If H is of class C^1 with respect to the variables x and r then (4.9) can be rewritten in the following form:

$$\frac{p_i}{|p|}H_{x_i}(x,r,p) + \frac{1}{2}|p|H_u(x,r,p) \ge 0 \text{ for every } x \in \Omega,\ |r| \le R_0,\ p \neq 0,\ p \in \mathbb{R}^n.$$

Note that for functions $H(x,r,p)$ which are strictly monotone in r, condition (3.14) in [6] requires

$$\begin{aligned}&F(y,r,N(x-y),Y) - F(x,r,N(x-y),X)\\&= [H(y,r,N(x-y)) - H(x,r,N(x-y))](1+N^2|x-y|^2)^{3/2}\\&\le \omega(N|x-y|)^2 + |x-y|)\end{aligned}$$

for some modulus of continuity $\omega(t)$ which is a very restrictive assumption. However, for equation (4.8) divided by the positive term $(1+|Du|^2)^{3/2}$ condition (3.14) becomes

$$H(y,r,N(x-y)) - H(x,r,N(x-y)) \le \omega(N|x-y|^2 + |x-y|) \quad .$$

Another mean-curvature type equation was considered by J. Serrin in [16]. If the first variation of a suitable functional vanishes, then

$$\delta \int \phi(x)\sqrt{1+|Du|^2}dx = 0 \quad , \quad \phi > 0,\ \phi \in C^2 \quad .$$

The Euler-Lagrange equation for this variational problem is

$$-[(1+|Du|^2)I - Du \otimes Du]D^2u - \frac{D\phi.Du}{\phi}(1+|Du|^2) = 0 \quad . \tag{4.10}$$

According to Theorem 2.6, the comparison principle holds for (4.10) in the class of viscosity sub- and supersolutions if $(-\ln\ \phi)_{x_i x_k} p_i p_k \ge 0$, i.e. when $\ln\ \phi$ is a concave function. For example, the functions $\phi = e^{-|x|^2}$ or $\phi \equiv 1$ are log concave functions.

Example 6. Consider the following equation involving the m-Laplacian operator for $2 < m \le \infty$

$$-\Delta_m u + f(x,u,Du) = 0 \ \text{ in } \Omega \subset \mathbb{R}^n \quad , \tag{4.11}$$

where $\Delta_m u = \text{div }(|Duu|^{m-2}Du)$ for finite m and where Δ_∞ is given in (1.10).

We consider only the degenerate m-Laplacian, $m > 2$, but the result in Proposition 4.1 is true also for the singular case, $1 < m < 2$. The interested reader should consult [15] or [7] for this case.

Since equation (4.11) is strictly elliptic for $p \ne 0$ we can apply the result in Theorem 2.4.

Corollary 4.1 *Suppose $f(x,r,p)$ is a monotone function with respect to r, i.e.*

$$f(x,s,p) \le f(x,r,p) \text{ for every } r \ge s,\ r,s, \in \mathbb{R},\ x \in \bar{\Omega},\ p \in \mathbb{R}^n, \tag{4.12}$$

$$\forall x \in \Omega \text{ the map } r \mapsto f(x,r,0) \text{ has at most countably many zeroes in } [-R_0,R_0], \tag{4.13}$$

and

$$f(x + t\frac{p}{|p|}, r + t\frac{|p|}{4}, p) > f(x,r,p) \tag{4.14}$$

for every $x \in \Omega$, $|r| < R_0$, $p \in \mathbb{R}^n \setminus 0$ and for all sufficiently small positive values of t.

If $u \in USC(\bar{\Omega})$, $v \in LSC(\bar{\Omega})$ are bounded, $|u|, |v| < R_0$, sub- and supersolutions of (4.11) such that $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Moreover, if $\underline{u} \in USC(\bar{\Omega})$, $\bar{u} \in LSC(\bar{\Omega})$, $\underline{u} \le \bar{u}$ are bounded viscosity sub- and supersolutions of (4.11) then there exist a viscosity solution $u \in C(\Omega)$ and u satisfies $\underline{u} \le u \le \bar{u}$ in Ω.

Example 7. This example deals with the pseudo–m–Laplacian for $2 < m < \infty$.

$$-\text{div}\left(\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{m-2}\frac{\partial u}{\partial x_i}\right) + f(x,u,Du) = 0 \text{ in } \Omega \subset \mathbb{R}^n. \tag{4.15}$$

According to Theorem 2.4, for the degenerate pseudo-m-Laplacian (4.15) we can literally derive the same result, Corollary 4.1, as for the m-Laplacian (with (4.11) replaced by (4.15)).

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