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Maximum and comparison principle for one-dimensional anisotropic diffusion

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1. Introduction and motivation

Consider the initial boundary value problem

$$u_t - \operatorname{div}(a(|\nabla u|^2)\nabla u) = 0 \qquad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$a(|\nabla u|^2)\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \qquad \text{on} \quad \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^1 and ν is the exterior normal to $\partial\Omega$. The structural assumptions on a are

$$\begin{aligned} &a \in C^1([0,\infty)), \qquad\qquad a(s) > 0,\\ &\text{the ellipticity function} \qquad b(s) := a(s) \ + \ 2\ s\ a'(s)\\ &\text{is positive for } s \text{ near 0 and changes sign exactly once at } s_0^2 > 0. \end{aligned} \tag{1.4}$$

Moreover, since we consider C^1 solutions, we implicitly assume the compatibility condition

$$\frac{\partial u_0}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{1.5}$$

Typical examples of such diffusion functions a are $a(s) = \mathrm{e}^{-s}$ or $a(s) = (1+s)^{-1}$. They are used in image enhancement processes, see [24]. The function $u_0(x)$ represents the brightness of a picture which one wants to denoise. Numerical computations have shown that equation (1.1) can produce the desired effect that $u(x,T)$ provides a sharper image than $u(x,0)$. Let us first understand why this is the case. For convenience let us assume for a moment that Ω is a plane domain, that u is continuously differentiable and that $|\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2 \not\equiv 0$ a.e. in $\Omega \times (0, T)$. Then we can rewrite partial derivatives in terms of directional derivatives $\nu = -\nabla u/|\nabla u|$ (in direction of steepest descent of u), and τ in direction tangent to ∇u. Under such a change of coordinates one knows that $\Delta u = u_{\nu\nu} + u_{\tau\tau}$. If we rewrite equation (1.1), it reads

$$u_t - a(s)\Delta u - 2a'(s)su_{\nu\nu} = 0$$

or

$$u_t - b(|\nabla u|^2)u_{\nu\nu} - a(|\nabla u|^2)u_{\tau\tau} = 0. \tag{1.6}$$

For n dimensional domains Ω the term $u_{\tau\tau}$ has to be replaced by the $(n-1)$-dimensional Laplace–operator on the plane tangent to the level surface of u.

Notice that the diffusion coefficient in direction τ differs from the one in direction ν. That is why we speak of **anisotropic diffusion**. As a matter of fact, b switches sign according to (1.4), while a remains positive. So the diffusion in direction ν can be thought of as a backward diffusion for large values of $|\nabla u|^2$, while the diffusion in direction τ is always smoothing.

The backward heat equation violates even the mildest form of regularization in the sense that small solutions (in $L^\infty(\Omega)$) can increase in time. Therefore according to [1] "no uniqueness of the solution and no stability of the process can be expected", see also [2]. Uniqueness is usually obtained via comparison or maximum principles; and we shall derive such principles as well as a uniqueness result below. These comparison and uniqueness results are delicate, because they need special assumptions. We refer to Sects. 3 and 4. In any case we shall show in Sect. 2 that the $L^\infty(\Omega)$–norm of solutions is preserved forward in time. One can imagine that the forward diffusion along level surfaces is responsible for this. At the same time the backward nature of diffusion steepens variations in brightness once they have exceeded a certain threshhold s_0. This leads to a sharpening of the image. So we have the interesting effect that the $L^\infty(\Omega)$–norm of u is preserved but the $W^{1,\infty}(\Omega)$–norm increases. In short: the gradient of u blows up, but u does not.

How can we classify equation (1.1) or (1.6)?

- For small values of $|\nabla u|$ it is a regular parabolic diffusion equation.
- If the first term were missing equation (1.6) would resemble the classical Tricomi problem.
- If the middle term were missing equation (1.6) were a regular diffusion problem.
- If the last term is missing, it looks like a forward-backward diffusion equation.

For small values of $|\nabla u|$ the contours of a brightness distribution are just diffused. That is desirable to fade out flat noise, but it misses the interesting situation that large brightness variations are enhanced. So studying equation (1.6) under the convenient technical assumption that $|\nabla u| < s_0$ would be dishonest.

We cannot neglect the first or second term in the equation either. But we can force the last term to disappear by assuming that $n = 1$. This simplifies the treatment of (1.6) considerably, because now diffusion takes place only in one spatial direction ν. This is in fact what we shall do in Sects. 3 to 6. Before proceeding let us remark, that due to the backward nature of the equation, solutions can only exist if the initial data are smooth, even analytic, in those points where the equation is backward in time. This has been observed by many people and put into writing by Kichenassamy [16].

Section 3 adresses the nonexistence of global C^1 solutions, Sect. 4 provides comparison results under special assumptions on the initial data. These results are optimal in the sense that counterexamples show the necessity of our assumptions. In Sect. 5 we prove uniqueness of C^1 solutions. This is in contrast to a result of Höllig [12], who proved nonuniqueness, however for Lipschitz solutions of equations with piecewise linear diffusion coefficient, which lead to an equation that was forward in time for large values of $|\nabla u|$ and backward for small values. Finally, Sect. 6 adresses local existence questions.

Let us close the introduction with some bibliographical remarks on papers which are not directly related to ours. The more relevant ones are quoted in context in the subsequent sections. The study of forward-backward equations was apparently initiated by Kepinsky and Gevrey [15, 10, 11] in the beginning of this century and had a revival more than 50 years later in the papers [4, 5, 22, 23, 18, 26, 27, 9]. In contrast to their analytical approach, [13, 29, 6, 19] and [20] contain numerical results.

2. L^∞ estimates

Theorem 2.1. Maximum principle
Suppose that u is a Lipschitz continuous (weak) solution to (1.1) (1.2) (1.3). *Then for every* $p \in [2, \infty]$ *the following inequality holds*

$$||u(x,t)||_{L^p(\Omega)} \leq ||u_0(x)||_{L^p(\Omega)}.$$

Proof. For $p < \infty$ multiply (1) with $|u|^{p-2}u$ and integrate over Ω. Integration by parts and boundary condition (2) give

$$\begin{aligned} \tfrac{d}{dt}\int_\Omega |u(x,t)|^p \; dx \quad &= p\int_\Omega |u|^{p-2} \; u \; u_t \; dx = p\int_\Omega |u|^{p-2} \; u \; \operatorname{div}(a(|\nabla u|^2)\nabla u) \; dx \\ &= -p(p-1)\int_\Omega |u|^{p-2} \; a(|\nabla u|^2) \; |\nabla u|^2 \; dx \; \leq \; 0, \end{aligned}$$

from which Theorem 2.1 follows for $p < \infty$. Send $p \to \infty$.

Theorem 2.1 was also found by Weickert [28] independently from us.

3. Nonexistence of global C^1 solutions

From now on we shall restrict ourselves to one space dimension. Then Problem (1.1)(1.2)(1.3) is rewritten as

$$u_t - (a(u_x^2)\, u_x)_x = 0 \qquad \text{in } \; Q := (-1,1)\times\mathbb{R}^+, \tag{3.1}$$

$$u_x(\pm 1, t) = 0 \qquad \text{for } t \in \mathbb{R}^+, \tag{3.2}$$

$$u(x,0) = u_0(x) \text{ on } (-1,1) \qquad \text{and } u_0'(\pm 1) = 0. \tag{3.3}$$

We expect solutions to develop a larger and larger gradient; and we shall try to analyze this behaviour locally by assuming that the initial data have their slope above the threshold s_0 in precisely one compact subinterval of $(-1,1)$, i.e.

$$\begin{gathered} u_0'(x) > s_0 \text{ in } (x_0,y_0) \subset\subset (-1,1)\ , \quad |u_0'(x)| < s_0 \text{ in } (-1,x_0)\cup(y_0,1) \\ \text{and } u_0 \text{ is strictly convex (resp. concave)} \\ \text{in a small symmetric punctured neighbourhood of } x_0 \text{ (resp. } y_0). \end{gathered} \tag{3.4}$$

Remark 3.1. Having a single subinterval on which $|u_0'(x)| > s_0$ is not essential. With cumbersome notation one can extend our analysis to the case of finitely many subintervals of this type. We spare the reader from this generalization, because it will not require any new ideas.

Remark 3.2. Sufficient for the convexity near x_0 is the analyticity of u_0. In fact if $u_0(x)$ is analytic and increasing near x_0 we may assume that it is strictly convex in a small neighborhood (x^*, x_0). Otherwise u_0'' would have infinitely many zeroes in (x^*, x_0), and those zeroes would have an accumulation point. Consequently $u_0'' \equiv 0$ and u_0 would be linear, a contradiction to the first part of (3.4).

To simplify matters even further, let us assume for the moment that $u_0' \geq 0$ and that u_0 is an odd function and that $u_0(x)$ is strictly concave for $x \in (0,1)$. Then $u_0'(x) = s_0 + \varepsilon$ in exactly two points $\pm x_1 \in (-1,1)$ where $\varepsilon > 0$ is suitably small. Then we can expect the central difference quotient $(u(x_1,t) - u(-x_1,t))/2x_1$ to increase in time. Consequently the $L^\infty(\Omega)$-norm of u_x should increase (in contrast to the $L^\infty(\Omega)$-norm of u). In fact due to the choice of x_1 we have $u_t(x_1,t) = b(u_x^2(x_1,t))u_{xx}(x_1,t) > 0$ and $u_t(-x_1,t) = b(u_x^2(-x_1,t))u_{xx}(-x_1,t) < 0$, at least as long as the solution is smooth and t not too large. This heuristic argument proves the increase of the symmetric difference quotient for u_x in the origin.

One can give another reason why the solutions of (3.1) (3.2) (3.3) should not be globally of class C^1. To this end we introduce $v = u_x$ as new function and consider the transformed problem

$$\begin{aligned} v_t - (b(v^2)v_x)_x &= 0 \qquad \text{in} \quad Q := (-1,1)\times\mathbb{R}^+, && (3.5)\\ v(\pm 1, t) &= 0 \qquad \text{for} \quad t \in \mathbb{R}^+ && (3.6)\\ v(x,0) &= {u_0}'(x) \quad \text{on} \quad (-1,1)\ . && (3.7) \end{aligned}$$

We recall that b changes sign if v exceeds the threshold s_0. Therefore, once v becomes large it satisfies a backward equation and we might expect problems with regularity from analogy to similar problems in the literature. We do not know if the porous medium community has studied (3.5) with sign-changing b yet, but there is a statement of nonexistence of global C^2-solutions in [21] for a related problem. Novikov studied

$$\begin{aligned} w_t - (a(w_x^2)w_x)_x &= 0 && \text{in } (0,1)\times\mathbb{R}^+ , \\ w_x(0,t) &= 0 && \text{for } t>0 , \\ w_x(1,t) &= (|\nu_0|/3\nu_2)^{1/2} && \text{for } t>0 , \end{aligned}$$

where $a(s) = \nu_0 + \nu_2 s$, $\nu_2 > 0$, $\nu_0 < 0$. The function $U = w_x$ satisfies

$$\begin{aligned} U_t - (b(U^2)U_x)_x &= 0 && \text{in } (0,1)\times\mathbb{R}^+ , \\ U(0,t) &= 0 && \text{for } t>0 , \\ b(U^2(1,t))U_x(1,t) &= 0 && \text{for } t>0 . \end{aligned}$$

In contrast to our problem, Novikov's equation for U is backward for small values of U and forward for large values of U.

Conjecture 3.1.
In view of Theorem 3.2 below we believe that sign changing solutions of (3.5) (3.6) (3.7) will develop discontinuities along their nodal sets, if b is of porous medium type, e.g. if $b(v^2) = v^2$.

Before stating the result on nonexistence of global C^1-solutions let us state an important qualitative property of solutions. Assuming that a global C^1–solution exists we denote with $Q^+ := \{(x,t) \in Q \text{ with } |u_x(x,t)| < s_0\}$ the forward (subsonic) set of equation (3.1), with $Q^- := \{(x,t) \in Q \text{ with } |u_x(x,t)| > s_0\}$ the backward (supersonic) regime and with $Q^0 := \{(x,t) \in Q \text{ with } |u_x(x,t)| = s_0\}$ the (sonic) set where (3.1) degenerates to a first order equation. This notion is borrowed from the theory of transonic flow for an ideal gas, which is modelled by an elliptic-hyperbolic operator of divergence type, see [8]. Since u satisfies (3.1) in a weak sense, it follows from classical regularity results [17, Ch. V, Theorems 5.2 and 5.3] that u is infinitely differentiable in $Q^+ \cup Q^-$, provided a is of class C^∞. In the following Theorem 3.1 we show that a supersonic regime persists in time and that the level sets of u_x are time-like curves.

Theorem 3.1.
Suppose $u(x,t)$ is a weak C^1 solution of (3.1)(3.2)(3.3) *and that* (1.4) (3.4) *hold. Then the level lines $\Gamma(t,z) := \{\, x \in Q \mid u_x(x,t) = u_0'(z) \,\}$ emanating from z (with $z \neq x_0, y_0$) in some small neighborhood of x_0 and y_0 are well-defined and smooth for every $t > 0$ for which the solution exists.*

The proof will proceed in two steps. First we show that the supersonic regime Q^- does not disappear in time. Otherwise there is a bounded component $G \subset Q^-$ starting at time zero with $u_x = s_0$ on its parabolic (backward) boundary. In G the function $v = u_x(t,x)$ satisfies (3.5). Thus by the classical maximum principle $v \equiv s_0$ in $G \cap \{t = 0\}$, a contradiction to assumption (3.4). To be precise, one has to consider (3.5) in the slightly smaller domain $G_\varepsilon := \{(x,t) \in G \mid$

$u_x(x,t) > s_0 + \varepsilon\}$, where (3.5) is uniformly parabolic. Later on we shall use this approximation without mentioning it explicitly.

Now suppose that u_0 is strictly convex in (x^*, x_0) and that $(x_1, x_2) \subset (x^*, x_0)$ is chosen so that $u_0'(x_1) = s_1 < u_0'(x_2) = s_2$ and $u_0'(x) < s_1$ for $x \in [-1, x_1)$. Let us denote with $G_1 := \{\, (x,t) \in Q^+ \mid u_x(x,t) < s_1 \,\}$ the connected component of Q^+ which contains the segment $[-1, x_1)$ and with $G_2 := \{\, (x,t) \in Q^+ \mid u_x(x,t) < s_2 \,\}$ the component containing $[-1, x_2)$. If Γ_1 and Γ_2 are those parts of the boundaries of G_1 and G_2 where $u_x = s_1$ and $u_x = s_2$ then Γ_1 and Γ_2 are time like curves.

In fact, they cannot have points of self-intersection because then by the classical maximum principle u_x would have to be constant inside the loops, a contradiction. Notice that $v = u_x$ satisfies (3.5) in Q and that such a loop would lie in Q^+. Moreover, Γ_i have no return segments as depicted in Fig. 3.1, since in this case there exist rectangles $K, \tilde{K}$ such that $v = u_x$ satisfies (3.5) in $K, \tilde{K}$, and v has an extremum at the top of K or $\tilde{K}$. Due to the choice of s_1 and s_2 and the first step of this proof, the curves Γ_i cannot return to the lower base of Q, while due to the boundary condition (3.2) they cannot touch the lateral boundary of Q either. Hence Γ_i are defined for every $t > 0$ for which the weak C^1 solution exists.

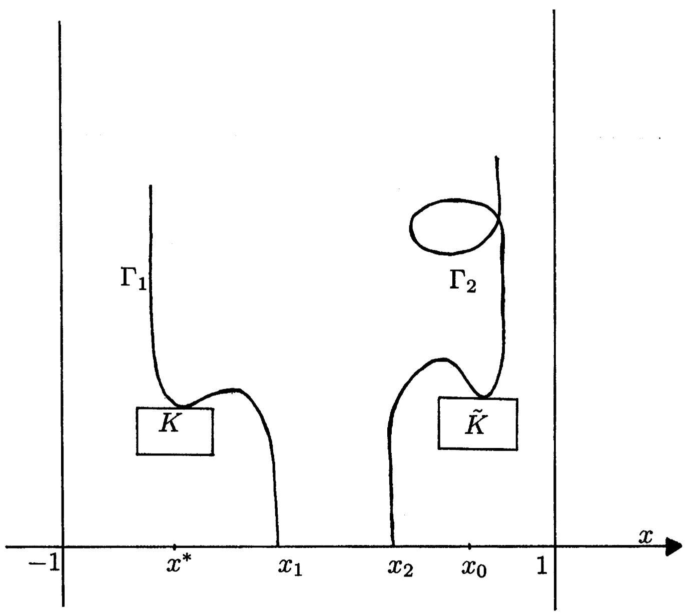

Fig. 3.1. An impossible situation

From the weak maximum principle for u_x in G_i it follows that $u_x < s_i$ in G_i, and since $u_x = s_i$ on Γ_i we have $u_{xx} \geq 0$ on Γ_i because the x-direction on Γ_i points out of G_i. Hence in $G := G_2 \setminus G_1$ the function $w = u_t(x,t)$ satisfies

$$
\begin{aligned}
w_t - (b(u_x^2)w_x)_x &= 0 && \text{in } G,\\
w(x,t) = b(u_x^2)u_{xx} &\geq 0 && \text{on } \Gamma_1 \cup \Gamma_2 \cup [x_1, x_2].
\end{aligned}
\tag{3.8}
$$

From the strong maximum principle we may conclude that $w(x,t) = u_t(x,t) > 0$ in G, i.e. $u_{xx}(x,t) = u_t(x,t)/b(u_x^2) > 0$ in G. The strict convexity of u is enough to define the level lines $\Gamma(t) := \{(x,t) \in G \mid u_x(x,t) = s\}$ emanating from some $\Gamma(0) \in (x_1, x_2)$ for every fixed $s \in (s_1, s_2)$. Choose such an s. Then $\Gamma(t)$ is a C^1 curve for $t > 0$ and u is strictly convex in x and monotone increasing in t along this line.

Theorem 3.2. Nonexistence of global C^1 solutions
Suppose that $a(s)$ and $u_0(x)$ are functions satisfying (1.4) *and* (3.4). *Then Problem* (3.1)(3.2)(3.3) *has no global weak solution in $C^1(Q)$.*

For the proof we use the same notation as in the proof of Theorem 3.1. $\Gamma(t) := \{(x,t) \in G \mid u_x(x,t) = s,\ s < s_0\}$ is a level line of u_x emanating from $\Gamma(0) < x_0$ close to x_0. If we suppose that $u(x,t)$ is a global weak C^1 solution of (3.1)(3.2)(3.3), then, as we have shown in Theorem 3.1, $\Gamma(t)$ is defined for every $t > 0$.

Differentiation of $u(\Gamma(t),t)$ with respect to t $(d/dt)u(\Gamma(t),t) = u_x\Gamma'(t) + u_t = s\Gamma'(t) + u_t(\Gamma(t),t)$, so that after an integration along the curve from $(\Gamma(0),0)$ to $(\Gamma(t),t)$ we obtain

$$
u(\Gamma(t),t) = u_0(\Gamma(0)) + s[\Gamma(t) - \Gamma(0)] + \int_0^t u_t(\Gamma(\tau),\tau)\, d\tau. \tag{3.9}
$$

Let us now calculate the expression $I = \int_{-1}^{\Gamma(t)} u(x,t)\, dx$. Due to Theorem 2.1 this expression is bounded by $2||u_0||_\infty$. On the other hand we intend to show that it is unbounded as $t \to \infty$. Using (3.2) and (3.9) we obtain

$$
\begin{aligned}
\tfrac{d}{dt} I &= \Gamma'(t)u(\Gamma(t),t) + a(s^2)s\\
&= \Gamma'(t)\,[u_0(\Gamma(0)) - s\Gamma(0) + s\Gamma(t)] + \Gamma'(t)\int_0^t u_t(\Gamma(\tau),\tau)\, d\tau + a(s^2)s,
\end{aligned}
$$

and hence

$$
\begin{aligned}
I(t) &= I(0) + \int_0^t \dot{I}(\tau)\, d\tau \\
&= \int_{-1}^{\Gamma(0)} u_0(x)\, dx + a(s^2)st + [\Gamma(t) - \Gamma(0)]\,[u_0(\Gamma(0)) - s\Gamma(0)]\\
&\quad + \frac{s}{2}[\Gamma^2(t) - \Gamma^2(0)] + \int_0^t \Gamma'(\sigma)\int_0^\sigma u_t(\Gamma(\tau),\tau)\, d\tau\, d\sigma.
\end{aligned}
\tag{3.10}
$$

Notice the term $a(s^2)st$ on the right hand side of (3.10). It grows linearly in t. To show the blow up of I we have to control the last term in (3.10). We integrate it by parts.

$$\begin{aligned} J: &= \int_0^t \Gamma'(\sigma) \int_0^\sigma u_t(\Gamma(\tau),\tau)\, d\tau \quad d\sigma \\ &= \Gamma(t) \int_0^t u_t(\Gamma(\tau),\tau)\, d\tau - \int_0^t \Gamma(\sigma)\, u_t(\Gamma(\sigma),\sigma)\, d\sigma \\ &= \int_0^t [\Gamma(t) - \Gamma(\sigma)]\, u_t(\Gamma(\sigma),\sigma)\, d\sigma \end{aligned}$$

Recalling that $u_t > 0$ in G and thus on the curve $(\Gamma(\sigma),\sigma)$ for $\sigma \in (0,t)$ and using the trivial estimate $\Gamma(t) - \Gamma(\sigma) \geq -2$ as well as (3.9) we obtain a lower bound for J

$$\begin{aligned} J \quad &\geq -2 \int_0^t u_t(\Gamma(\sigma),\sigma)\, d\sigma \\ &= -2[u(\Gamma(t),t) - u_0(\Gamma(0))] + 2s[\Gamma(t) - \Gamma(0)] \end{aligned}$$

which is bounded from below in t. Thus the contradiction is reached and the proof of Theorem 3.2 is complete.

4. Comparison principle and counterexample

In this section we shall prove a comparison principle for weak C^1 solutions of (3.1)(3.2)(3.3) under at least one of two special assumptions on the initial data $u_0(x)$ and $v_0(x)$. One of them says that not only are the initial data ordered, i.e. $u_0(x) \leq v_0(x)$ in $(-1,1)$, but they can be separated by a subsonic profile $w_0(x)$:

$$\begin{aligned} \exists \quad & w_0(x) \in C^{2,\alpha} \ \text{ such that } \ u_0(x) \leq w_0(x) \leq v_0(x), \\ & |w_0'(x)| < s_0 \ \text{ in } (-1,1) \quad \text{ and } \quad w_0'(\pm 1) = 0. \end{aligned} \tag{4.1}$$

For convenience we call functions $f(x)$ defined on $[-1,1]$ sub- resp. supersonic if they satisfy $|f'(x)| < s_0$ resp. $|f'(x)| > s_0$. For subsonic initial data equation (3.1) is a forward diffusion equation, while for supersonic data it is backward in time. The other assumption on the initial data under which we can state a comparison result is that

$$\{x \in (-1,1) \text{ with } |u_0'(x)| \geq s_0\} \cap \{x \in (-1,1) \text{ with } |v_0'(x)| \geq s_0\} = \emptyset. \tag{4.2}$$

Loosely speaking (4.2) says that the closures of the supersonic regimes of u_0 and v_0 do not intersect.

Our comparison result will be shown to be sharp, because without our special assumption on the initial data we can construct a counterexample of two solutions whose difference changes sign as t increases.

Therefore we cannot derive a uniqueness result from the comparison result. Instead we shall derive uniqueness directly in Sect. 5 by a different method.

Theorem 4.1. Comparison principle
Suppose that $u(x,t)$ *and* $v(x,t)$ *are weak* C^1 *solutions of* (3.1)(3.2)(3.3) *in* $Q_T := (-1,1) \times (0,T)$ *with* C^1 *initial data* $u_0(x) \le v_0(x)$ *satisfying* (4.1) *or* (3.4)(4.2) *and with* a *satisfying* (1.4)*. Then* $u(x,t) \le v(x,t)$ *in* Q_T*.*

Proof. Notice that case (4.1) can be reduced to (4.2) once we know that (3.1)(3.2) has a C^1 solution $w(x,t)$ with initial data w_0. This existence problem is positively answered by Theorem 6.1 in Sect. 6. However, since there is an ad-hoc proof in case (4.1), let us present it for the reader's convenience.
Case (4.1): It suffices to show that $u(x,t) \le w(x,t)$ in Q_T. Otherwise the weighted difference $z(x,t) = \exp(-t)(w(x,t) - u(x,t))$ has a negative minimum in an "interior" point (x_0,t_0) with $x_0 \in (-1,1)$ and $0 < t_0 \le T$ or on the lateral boundary of Q_T. In the first case, since $z_x(x_0,t_0) = 0$ we have $u_x(x_0,t_0) = w_x(x_0,t_0)$. Recalling that w is everywhere subsonic (from the classical maximum principle) we conclude that (x_0,t_0) belongs to the subsonic regime of u, too, i.e. $(x_0,t_0) \in Q_T^+(u) := \{(x,t) \in Q_T$ with $|u_x(x,t)| < s_0\}$. In $Q_T^+(u)$, though, we can use the classical comparison result for quasilinear parabolic equations. In fact, $z_t(x_0,t_0) = \exp(-t_0)\,(-[w-u] + [w-u]_t)\,(x_0,t_0) \le 0$, so that $[w-u]_t(x_0,t_0) \le [w-u](x_0,t_0) < 0$. Noting that C^1 solutions are even smoother in Q_T^+, this contradicts $[w-u]_t(x_0,t_0) = (b(w_x^2)w_{xx} - b(u_x^2)u_{xx})(x_0,t_0) \ge 0$. In the second case, if $x_0 = \pm 1$ Hopf's Lemma and (3.2) imply that z is constant in a neighborhood of (x_0,t_0). Thus z attains its negative minimum in an interior point, a situation that was just ruled out. This proves Theorem 4.1 under assumption (4.1).
Case (4.2): This time we want to show that $y(x,t) = \exp(-t)(v(x,t) - u(x,t))$ cannot have a negative minimum in some "interior" point (x_0,t_0) with $x_0 \in (-1,1)$ and $0 < t_0 \le T$ or on the lateral boundary. The second case can be dealt with as under (4.1). In the first case $u_x(x_0,t_0) = v_x(x_0,t_0)$ and thus (x_0,t_0) is either in the subsonic regime of both solutions, or in the supersonic regime of both u and v, or in both sonic sets $Q_T^0(u) = \{(x,t) \in Q_T$ with $|u_x(x,t)| = s_0\}$ and $Q_T^0(v)$. In the first case that $(x_0,t_0) \in Q_T^+(u) \cap Q_T^+(v)$, we can argue as under case (4.1) above and reach a contradiction. The other two cases can be ruled out once we show that $[Q_T^-(u) \cup Q_T^0(u)] \cap [Q_T^-(v) \cup Q_T^0(v)] = \emptyset$. This is guaranteed by Theorem 4.2 below, which states that the supersonic regimes of a C^1 solution shrink in time, and by assumption (4.2).

Theorem 4.2. Shrinking of supersonic regimes
Suppose that $u(x,t)$ *is a weak* C^1 *solution of* (3.1)(3.2)(3.3) *and that* (1.4) *and* (3.4) *hold. Then the union of supersonic and sonic regimes of* $u(x,t)$ *shrinks in time, i.e.* $Q_\tau^- \cup Q_\tau^0 := (Q^- \cup Q^0) \cap \{\ t = \tau\ \}$ *satisfies the inclusion* $(Q_\tau^- \cup Q_\tau^0) \subset (Q^- s \cup Q_s^0)$ *for every* $0 \le s \le \tau$*.*

The proof of Theorem 4.2 would be simpler if u were a classical C^2 (or at least a $C^{1,1}$) solution, because then we could argue from the vanishing of b on Q^0 that $u_t = 0$ on Q^0. However, we do not know if u is C^2 across the sonic set and we know little about Q^0, either. Therefore we shall first state and prove a qualitative auxiliary result.

Theorem 4.3. $u_t = 0$ **on boundary of sub- and supersonic set**
Suppose that $u(x,t)$ *is a weak* C^1 *solution of* (3.1)(3.2)(3.3)*, that* (1.4) *and* (3.4) *hold.*

Then $u_t = 0$ *on* $\partial Q^+ \cap Q_T$ *and* $\partial Q^- \cap Q_T$*, for all connected components of* Q^+ *or* Q^- *which contain a segment from* $(-1,1)\times\{0\}$.

Proof. Let G be a connected component of Q^+ containing $(-1,x_0)$ and let γ be the boundary of $G \cap Q_T$. By definition $u_x = s_0$ on γ, and according to Theorem 3.1 γ is a time like curve. Let us suppose that contrary to the claim of Theorem 4.3 $u_t(\tilde{x},\tilde{t}) \not\equiv 0$ at some point $(\tilde{x},\tilde{t}) \in \gamma$. It follows from (3.8) in the proof of Theorem 3.1 and the maximum principle that $u_t(\tilde{x},\tilde{t}) > 0$, and from the continuity of u_t the same inequality holds in a small disc B centered at $(\tilde{x},\tilde{t})$, see Fig. 4.1. Without loss of generality we may assume that $\gamma \cap B$ has no horizontal segments. Otherwise $u_{xx} = 0$ and due to (3.1) $u_t = 0$ on those segments, a contradiction.

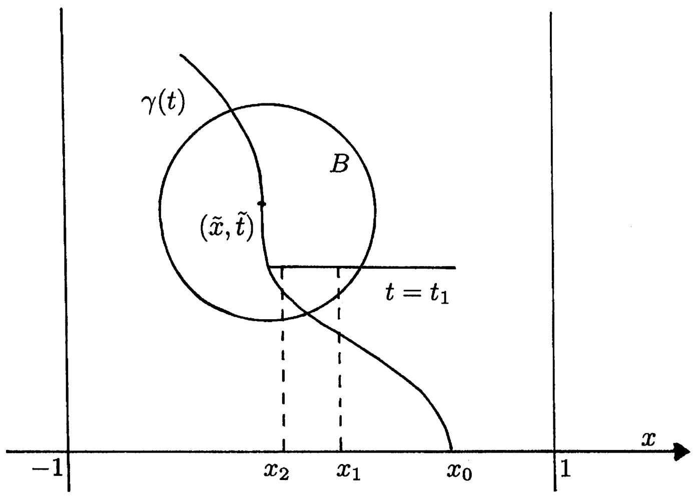

Fig. 4.1. An impossible situation

We will show that in a neighborhood of $(\tilde{x},\tilde{t})$ in $B \setminus \overline{G}$ there are no points from $Q^+ \cup Q^-$. In fact, otherwise there is a point $(x_1,t_1) \in B \setminus \overline{G}$ such that $(x_1,t_1) \in Q^-$. From (3.1) we have that $u_{xx}(x_1,t_1) < 0$ and hence $u_{xx}(x,t_1) < 0$ in some maximal interval $(x_2,x_1) \subset Q^- \cap (B \setminus \overline{G})$. The case $(x_2,t_1) \in Q^-$ but $u_{xx}(x_2,t_1) = 0$ implies $u_t(x_2,t_1) = 0$ and contradicts the choice of B. The case $(x_2,t_1) \in \partial Q^-$, i.e. $(x_2,t_1) \in Q^0$, is also impossible since then $u_x(x_2,t_1) = s_0$ and the strict concavity of u in x on (x_2,x_1) implies $s_0 = u_x(x_2,t_1) > u_x(x_1,t_1)$, a contradiction to $(x_1,t_1) \in Q^-$. Finally, by choosing t_1 sufficiently close to $\tilde{t}$ the line (x,t_1), $x_2 \le x < x_1$ will necessarily intersect γ and not ∂B, so that only the above cases are possible.

In the same way one can prove that in a small neighborhood of $(\tilde{x},\tilde{t})$ in $B \setminus \overline{G}$ there are no points from Q^+. Hence the points in a neigborhood of $(\tilde{x},\tilde{t})$ in $B \setminus \overline{G}$ are from Q^0, i.e. $u_x \equiv s_0$, $u_{xx} \equiv 0$ and from (3.1) $u_t(\tilde{x},\tilde{t}) = 0$, a contradiction. This proves Theorem 4.3.

Proof of Theorem 4.2. We use the notations from the proof of Theorem 4.3. If $G \subset Q^+$ is a connected component containing $(-1, x_0)$ and γ is the boundary of $G \cap Q_T$ then we claim that for every $(y, \lambda), (z, \tau) \in \gamma$ with $0 \le \lambda < \tau$ the inequality $y \le z$ holds. Otherwise $y > z$ and we can apply formula (3.9) with initial time $t = \lambda$ and initial point $y = \gamma(\lambda)$ to the generalized curve $\gamma(t)$. Notice that as a limit of time like curves γ can possibly have horizontal segments. Since $u_t = 0$ on $\gamma(t)$, (3.9) becomes

$$u(z, \tau) = u(y, \lambda) + s_0(z - y). \tag{4.3}$$

Recall that u_t and u_{xx} are positive in G near γ. Therefore $u(z, \lambda) < u(z, \tau)$. This and (4.3) contradict the convexity of u, since

$$u(z, \lambda) < u(y, \lambda) + s_0(z - y).$$

This shows that γ moves to the right and proves Theorem 4.2.

The following example shows that without assumptions of type (4.1) or (4.2) on the initial data one should not expect a comparison principle.

Example 4.4. Suppose that there exist weak C^1 solutions $u(x, t)$ and $v(x, t)$ of (3.1)(3.2)(3.3) with initial data

$$\begin{aligned} u_0(x) &= 3bx + 2cx^2 - bx^3 - cx^4, \\ v_0(x) &= 3bx - 2dx^2 - bx^3 + dx^4, \end{aligned}$$

where b, c, d with $b > s_0/3$ are arbitrary positive constants. Then u and v violate the comparison principle, because $u_0 \ge v_0$ but for every $t > 0$ we have $u(0, t) < v(0, t)$. This can be seen as follows.

Since $u_0'(0) = v_0'(0) = 3b > s_0$, the origin is in the supersonic regime of both u and v. Hence u and v are classical solutions near the origin $(0, 0)$. Now we note that $u_0''(0) > 0$ and $v_0''(0) < 0$. From (3.1) it follows that $u_t(0, 0) < 0$ and $v_t(0, 0) > 0$. Since $u_0(0) = v_0(0)$ the claim follows.

5. Uniqueness of C^1 solutions

In this section we give a direct proof of the uniqueness of weak C^1 solutions to (3.1)(3.2)(3.3). For this purpose we shall need an additional result about qualitative properties of solutions, see Theorem 5.2.

Theorem 5.1. Uniqueness
Suppose that u and v are weak C^1 solutions of (3.1)(3.2)(3.3) *in $Q_T := (-1, 1) \times (0, T)$ with identical and analytic initial data u_0. Moreover assume that $(u_0)_x^2 - s_0^2$ has only simple zeroes and that the diffusion coefficient a is analytic and satisfies* (1.4)*. Then $u(x, t) \equiv v(x, t)$ in Q_T.*

Proof. Consider the difference $w := u - v$. We note that w solves the problem

$$w_t - (B(x,t)w_x)_x = 0 \qquad \text{in} \quad Q_T, \tag{5.1}$$

$$w_x = 0 \qquad \text{on } \{\pm 1\} \times (0,T), \tag{5.2}$$

$$w(x,0) = 0 \qquad \text{on} \quad [-1,1], \tag{5.3}$$

where $B(x,t) = \int_0^1 \ b(|v_x + \theta(u_x - v_x)|^2)\, d\theta$. We want to show that $w \equiv 0$ in Q_T.

Let us first look at w on the set $Q^-(u) \cap Q^-(v)$. If w is nonzero in some point from this set, then there is a thin cylinder $Q^\varepsilon := (x_e - \varepsilon, x_e + \varepsilon) \times (0, t_e)$ in which (5.1) is a backward parabolic equation. Consequently its solution is very regular, even analytic with respect to x in Q^ε. If there exists a time t_0 such that $w(x,t_0) = 0$ for a nontrivial open interval in $(x_e - \varepsilon, x_e + \varepsilon)$ or at infinitely many points, then $w(x,t_0) = 0$ in $(x_e - \varepsilon, x_e + \varepsilon)$ by analytic continuation.

By shifting the initial time, if necessary, we may assume without loss of generality that $w \not\equiv 0$ a. e. in Q^ε. Suppose there exists a nontrivial connected component K of one of the sets $K_\varepsilon^\pm := \{\ (x,t) \in Q^\varepsilon \mid \pm w(x,t) > 0\ \}$ whose common boundary with the initial time $\partial K \cap \{\ t = 0\ \}$ has positive one–dimensional measure. This contradicts the strong interior maximum principle [25] for (5.1) in K, because w attains its zero maximum or minimum at an interior point of K at $t = 0$. Hence all components of $K_\varepsilon^\pm := \{\ (x,t) \in Q^\varepsilon \mid \pm w(x,t) > 0\ \}$ with nonempty intersection with $\{\ t = 0\ \}$ intersect the initial time on a nullset. By construction there are infinitely many such components.

Where do they go as time proceeds? They cannot be compactly embedded in the parabolic interior of Q^ε because of the maximum principle for (5.1). They cannot instantly disappear either. So for some positive time δ the function w has inifinitely many sign changes. But then $w(x,\delta) = 0$ in $(x_e - \varepsilon, x_e + \varepsilon)$, another contradiction. This implies that $Q^-(u) = Q^-(v)$ and $u \equiv v$ in $Q^-(u)$.

Next we investigate the behaviour of w on $Q^0(u) \cap Q^0(v)$. We observe that $w_x \equiv 0$ by definition and $u_t = v_t = w_t \equiv 0$ by Theorem 4.3. Therefore $w = \mathit{const}$. on $Q^0(u) \cap Q^0(v)$.

Theorem 5.2 below implies that each component of $Q^0(u)$ (resp. $Q^0(v)$) touches $Q^-(u)$ (resp. $Q^-(v)$) in at least one point (x,t). From the first step we know that $u \equiv v$ in $Q^-(u)$, so that $w = 0$ everywhere on the joint boundary of Q^- and Q^0. This and the fact that $w = \mathit{const}$. on $Q^0(u) \cap Q^0(v)$ lead to

$$u \equiv v \quad \text{on} \quad Q^0(u) \cap Q^0(v). \tag{5.4}$$

It remains to investigate w on $Q^+(u) \cup Q^+(v)$. Suppose that w has a positive maximum in $(x_e, t_e) \in Q_T$. Three cases are possible. If $(x_e, t_e) \in (Q^+(u) \cup Q^+(v)) \setminus (Q^+(u) \cap Q^+(v))$ then $w_x(x_e, t_e) \not\equiv 0$, a contradiction. The extremum of w will lie elsewhere. If (x_e, t_e) lies on the lateral boundary of Q_T, this contradicts (5.2) unless w is constant in a neighborhood of this maximum point. But then $w > 0$ at some initial point, a contradiction to (5.3).

Finally, if $(x_e, t_e) \in Q^+(u) \cap Q^+(v)$ then by the strong interior maximum principle, the same extremum must occur in a point (y, s) on the parabolic boundary of $Q^+(u) \cap Q^+(v)$, a situation which has already been ruled out. The case $(y, s) \in Q^0(u) \cap Q^0(v)$ is ruled out by (5.4). This proves Theorem 5.1.

Remark 5.1. Let us remark in passing that Theorem 5.1 is well-known if the initial datum u_0 is subsonic, i.e. $|u_0'| < s_0$ in $(-1, 1)$. The degenerate situation that $|u_0'| \leq s_0$ in $(-1, 1)$ and $|u_0'(x)| = s_0$ for some $x \in (-1, 1)$ cannot occur due to our assumptions, but the transsonic case is covered.

In image enhancement we want to preserve edges and shapes. No new edges should be generated as the image evolves, nor do we want edges to disappear. If we call each component of $Q^-(u) \cap \{t = t_0\}$ an edge at time t_0, then we can classify shapes of u according to their number of edges. The following theorem states roughly speaking, that this number is invariant in time. The first step in the proof of Theorem 3.1 consisted in showing that a single edge cannot disappear and gives a special case of Theorem 5.2.

Theorem 5.2. Preservation of Shapes
Under the same assumptions as in Theorem 5.1
i) the number of the connected components of $Q^+(u) \cap \{\ t = s\ \}$ *and of* $\big(Q^-(u) \cup Q^0(u)\big) \cap \{\ t = s\ \}$ *is invariant in* s*, and*
ii) in particular, no component of $Q^0(u)$ *can originate in* $Q^+(u)$*.*

Proof. From the proof of Theorem 3.1 we know that the supersonic regimes Q^- starting from some interval at the initial time zero do not disappear for $t > 0$ as long as the solution exists.

In order to prove that the subsonic regimes Q^+ do not disappear either, suppose that there is an interval $(y, z) \subset (-1, 1)$ contained in $Q^+ \cap \{\ t = 0\ \}$. If the component of Q^+ containing this line segment disappears after some finite time $t_1 > 0$ we will apply formula (3.9) to the left and right boundaries $\mu(t)$ and $\gamma(t)$, resp. of this component of Q^+, where $\mu(0) = y$ and $\gamma(0) = z$. Hence we have

$$\begin{aligned} u(\mu(t_1), t_1) &= u_0(y) \pm s_0[\mu(t_1) - y] \\ u(\gamma(t_1), t_1) &= u_0(z) \pm s_0[\gamma(t_1) - z]. \end{aligned} \tag{5.5}$$

Without loss of generality we assume the + sign in (5.5). Recall that from the reasoning in the proof of Theorem 3.1 we know that these curves are time-like and $\mu(t) \leq \gamma(t)$ for $t \in [0, t_1]$. We distinguish the cases $\mu(t_1) = \gamma(t_1)$ and $\mu(t_1) < \gamma(t_1)$. In the first case the left hand sides of (5.5) coincide. Therefore the difference of the right hand sides $u_0(y) - u_0(z) + s_0[z - y]$ should vanish, or equivalently

$$\frac{u_0(y) - u_0(z)}{y - z} = s_0. \tag{5.6}$$

But this contradicts the assumption that the (modulus of the) slope of u_0 is strictly less than s_0 in (y, z).

In the second case $u(\mu(t_1), t_1) = u(\gamma(t_1) \pm s_0[\mu(t_1) - \gamma(t_1)]$ because the segment $(\mu(t_1), \gamma(t_1)) \times \{ t = t_1 \}$ belongs to Q_0. Again we assume that the + sign holds. Now we arrive again at (5.6), a contradiction.

Thus far we have shown that supersonic Q^- and subsonic Q^+ regimes persist and do not disappear. Next we shall show that the number of supersonic and subsonic regimes does not increase. Such an increase is conceivable for instance if new components of $Q^- \cup Q^0$ appear inside Q^+ at some positive time or if new components of Q^+ appear inside $Q^- \cup Q^0$ at some positive time. Neither one is possible. In fact, one can apply maximum principle arguments to u_x or one can reverse time to see this.

Finally it remains to analyze the interface between adjacent components of Q^+ and Q^-. Without loss of generality, suppose $Q^+(u) \cap \{t = 0\} = (y_1, y_0)$ and $Q^-(u) \cap \{t = 0\} = (y_0, y_2)$. This interface belongs to $Q^0(u)$ and may be of postive measure, in which case new components of Q^+ or Q^- might appear inside Q^0. Let us rule out this final situation with a familiar argument. Denote the right hand boundary of Q^+ by $(\mu(t), t)$ and the left hand boundary of Q_- by $(\gamma(t), t)$ and recall that this is justified because both are time-like curves. As we already proved in the previous paragraph there are no components of Q^+ in the set $I := \{(x, t) \mid x \in (\mu(t), \gamma(t)),\ t > 0\}$. In order to prove that there are no components of Q^- in I, we can apply again formula (3.9), replacing Γ by μ and γ respectively and reach the same contradiction as in (5.5). This concludes the proof of Theorem 5.2.

6. Existence of solutions

Theorem 6.1. Global existence for subsonic initial data
Suppose that a satisfies (1.4) *and that the initial datum* u_0 *in* (1.3) *is in* $C^{2,\alpha}(\Omega)$ *with* $\alpha \in (0, 1)$ *and satisfies the compatibility condition* (1.2) *at time zero. If* u_0 *is subsonic, i.e. if* $|\nabla u_0(x)| < s_0$ *on* $\overline{\Omega}$*, then for every* $T > 0$ *Problem* (1.1)(1.2)(1.3) *has a classical solution in* $C^2(Q_T)$*.*

Proof. Since Ω is bounded $|\nabla u_0(x)| \le s_1 < s_0$. Therefore we can modify the function $a(s)$ outside $[0, s_1]$ in such a way that (1.1) becomes uniformly parabolic. In fact, we choose $s_2 \in (s_1, s_0)$ and set

$$\tilde{a}(s) = \begin{cases} a(s) & \text{for } s \in [0, s_2], \\ a(s_2) + (s - s_2)a'(s_2) & \text{for } s > s_2. \end{cases}$$

Then according to [17, Theorem 7.4, Ch. 5], the modified problem

$$w_t - \operatorname{div}(\tilde{a}(|\nabla w|^2)\nabla w) = 0 \qquad \text{in} \quad \Omega \times (0, T), \tag{6.1}$$

$$\tilde{a}(|\nabla w|^2)\frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{6.2}$$

$$w(x, 0) = u_0(x) \qquad \text{on} \quad \Omega, \tag{6.3}$$

has a classical solution $w \in C^{2,\alpha}(\overline{Q}_T)$ for every $T > 0$. Since (6.1) is a forward parabolic equation, the weak maximum principle holds for the modified version of (3.5)(3.6)(3.7), so that

$$\sup_{Q_T} |\nabla w(x,t)| \leq \sup_{\Omega} |\nabla u_0(x)| \leq s_1 < s_0.$$

Thus w is a solution not only of (6.1)(6.2)(6.3) but of (1.1)(1.2)(1.3). Moreover this solution remains subsonic, and this is in accordance with Theorem 3.1.

For transsonic initial data, the problem of local existence of C^1 solutions appears to be wide open. Various approaches have been suggested, for instance regularization by operators of third and fourth order in x, as described in [3], but they seem to lead to different solutions as the regularization parameter goes to zero. De Giorgio suggested to prove convergence of a numerical difference scheme in [7]. We tried an adaptation of Rothe's method which would take forward resp. backward approximations into account, but could not prove convergence of the method either. We also tried a series expansion of a solution in the analytical case. Formal expansion gave us a good guess on the exact structure of solutions, but we were unable to prove the absolute convergence of our series.

A viscosity solution approach has been dismissed as hopeless by several colleagues, because the divergence type operator in (1.1) has eigenvalues of varying sign and is not even degenerate elliptic. However, one can multiply (1.6) in the one-dimensional situation by b and rewrite it as

$$b(|u_x|^2)u_t - b^2(|u_x|^2)u_{xx} = 0. \tag{6.4}$$

But (6.4) is degenerate elliptic and can be regularized for example to

$$\varepsilon(u^\varepsilon_{tt} + u^\varepsilon_{xx}) + b^2(|u^\varepsilon_x|^2)\ u^\varepsilon_{xx} - b(|u^\varepsilon_x|^2)\ u^\varepsilon_t = 0 \quad \text{in } Q_T. \tag{6.5}$$

Of course (6.5) can be supplemented by Dirichlet data on the top and bottom part of ∂Q_T, i.e. for $t = 0$ and $t = T$, and by homogeneous Neumann data on the lateral boundary of the space-time cylinder, and this elliptic boundary value problem can be solved for every $\varepsilon > 0$. What happens for $\varepsilon \to 0$? As for the ordinary heat equation or in other degenerate parabolic settings such as in [14] we cannot expect the "viscosity limit" to satisfy the given boundary data on the supersonic part of the boundary. This is a well known effect in singular perturbation theory, but one should be aware that the viscosity limit still satisfies the Dirichlet condition in a viscosity sense, see [14]. Instead we should expect the viscosity limit to develop discontinuities, but then we are outside the class of C^1 solutions. One would have to develop a satisfactory theory of discontinuous viscosity solutions for second order equations. This is outside the scope of the present manuscript.

At the risk of confusing the reader, let us finally report on the so-called "staircasing effect" which is sometimes observed by numerical analysts. Staircasing means that spatial profiles of moderate slope develop small step-like oscillations on the scale of the meshsize. This effect is not a numerical artefact introduced by discretization, but there is an analytical interpretation for it. The energy $\int_\Omega A(|\nabla u|^2)\,dx$ associated to (1.1) has a nonconvex integrand A. By developing steps, the energy of u decreases to the one given by the convex relaxation of A. We believe that this happens for the continuous equation after some finite time. Therefore it would not contradict the local existence of C^1 solutions nor our results on shape preservation.

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