Positive eigenfunctions for the p-Laplace operator revisited

B. Kawohl & P. Lindqvist

Sept. 2006

Abstract: *We give a short proof that positive eigenfunctions for the p-Laplacian are necessarily associated with the first eigenvalue and that they are unique modulo scaling.*

Mathematics Subject Classification (2000). 35P30, 49R05

Keywords. p-Laplacian, eigenvalue, eigenfunction, uniqueness

1 Introduction

In a given bounded domain Ω in $\mathbb{R}^n$ the problem of minimizing the Rayleigh quotient

$$\frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx} \qquad (1 < p < \infty) \tag{1.1}$$

among functions $v \in W_0^{1,p}(\Omega)$ with boundary values zero leads to the Euler-Lagrange equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = 0 \quad . \tag{1.2}$$

Weak solutions of this equation are called eigenfunctions. This nonlinear eigenvalue problem was introduced by E. Lieb in 1983, cf. [9], and independently by F. de Thélin [16]. Restricted to a nodal domain, any eigenfunction is a first one there. This conclusion requires the following theorem.

Theorem 1.1 *Suppose that Ω is an arbitrary bounded domain. A positive eigenfunction in Ω is unique (up to multiplication with constants). Therefore $\lambda = \lambda_1$, the infimum of the Rayleigh quotient* (1.1).

We have stated the theorem for arbitrary domains for the following reason. Although the original domain might be smooth, this is not always inherited by the nodal domains of higher eigenvalues, to which the theorem is often applied. Therefore it is expedient to allow for irregular domains. The theorem is well-known but the hitherto available proofs are rather complicated. The objective of our note is to provide a simple proof, valid in arbitrary domains.

M. Belloni and B. Kawohl have found a short variational proof of the simplicity of the first eigenvalue (principal frequency). In [5] they established the uniqueness of minimizers (which are positive by nature). However, their proof does not exclude the possibility of positive eigenfunctions that are not minimizers of (1.1). We will now complement their proof by showing that such functions cannot exist. In smooth domains a simple and appealing proof of M. Ôtani and T. Teshima excludes such eigenfunctions, cf. [13]. Our contribution is to adapt their reasoning to irregular domains.

Taken as a whole, this yields a simpler proof than the older proofs by A. Anane [2], P. Lindqvist [10] and W. Allegretto & Y. Huang [1].

Remarks.

Our results remain true (and the proofs are literally identical) if the Dirichlet condition $u = 0$ is only assumed on a part Γ_1 of the boundary, where Γ_1 has positive $(n-1)$-dimensional measure. In that case the natural boundary condition on $\Gamma_2 := \partial\Omega \setminus \Gamma_1$ is Neumann's boundary condition $\frac{\partial u}{\partial \nu} = 0$.

The simplicity of the first eigenvalue has been much studied. After a partial result in [17], the case of a ball was proved in [7] and [3]. A. Anane proved the theorem in smooth domains, cf. [2]. C^2-domains were considered in [14] and [4]. In 1990 P. Lindqvist proved the theorem in arbitrary domains, essentially by improving Anane's test function, cf. [10]. Another proof, based on an identity of Picone's type, was constructed by W. Allegretto & Y. Huang in [1]. The variational proof of M. Belloni and B. Kawohl is by far the simplest, though restricted to true minimizers of the Rayleigh quotient, cf. [5]. For the one-dimensional case, we refer to [12], [11] or [15].

Acknowledgement.

This research is part of the ESF program "Global and geometricalk aspects of nonlinear partial differential equations (GLOBAL)".

2 Preliminaries

Let Ω denote an arbitrary bounded domain in $\mathbb{R}^n$. Fix an exponent p in the range $1 < p < \infty$. An elementary inequality shows that

$$\lambda_1 = \inf_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla \varphi|^p dx}{\int_\Omega |\varphi|^p dx} > 0 \quad .$$

The infimum is attained for a function v_1 in the Sobolev space $W_0^{1,p}(\Omega)$, and without loss of generality $v_1 > 0$. To be on the safe side, we give the interpretation of the Euler-Lagrange equation (1.2) in the weak sense.

Definition 2.1 We say that $v \in C(\Omega) \cap W_0^{1,p}(\Omega)$, $v \not\equiv 0$, is an *eigenfunction,* if

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle dx = \lambda \int_\Omega |v|^{p-2} v\varphi dx \tag{2.1}$$

whenever $\varphi \in C_0^\infty(\Omega)$. The corresponding real number λ is called an *eigenvalue.*

By standard elliptic regularity theory the continuity requirement on v is redundant in the definition. *If Ω is smooth, one even has continuity up to the boundary: $v \in C(\bar{\Omega})$ and $v|\partial\Omega = 0$.* If $v \geq 0$, then $v > 0$ because of the *Harnack inequality* [18]: if v is a non-negative eigenfunction, then

$$\max_{\bar{B}_r} v \leq C_{n,p} \cdot \min_{\bar{B}_r} v$$

whenever $B_r = B(x_0, r)$ and $B(x_0, 2r) \subset \Omega$.

In passing, we mention that the existence of a *positive* eigenfunction follows easily. The direct method in the Calculus of Variations yields a minimizer, say v. Then also $|v|$ is minimizing and hence $|v|$ is an eigenfunction. By Harnack's inequality $|v| > 0$ in Ω. By continuity v does not change sign.

Proposition 2.2 *Let*

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega$$

be an exhaustion of $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. *Then*

$$\lim_{j\to\infty} \lambda_1(\Omega_j) = \lambda_1(\Omega) \quad .$$

Proof. Since $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq \ldots \geq \lambda_1(\Omega)$, the limit exists. Given $\varepsilon > 0$, there is a $\varphi \in C_0^\infty(\Omega)$ such that

$$\frac{\int_\Omega |\nabla\varphi|^p dx}{\int_\Omega |\varphi|^p dx} < \lambda_1(\Omega) + \varepsilon \quad ,$$

because $\lambda_1(\Omega)$ is the infimum. For j large enough $\operatorname{supp}(\varphi) \subset \Omega_j$. Hence

$$\lambda_1(\Omega_j) \leq \frac{\int_{\Omega_j} |\nabla\varphi|^p dx}{\int_{\Omega_j} |\varphi|^p dx} = \frac{\int_\Omega |\nabla\varphi|^p dx}{\int_\Omega |\varphi|^p dx} \quad .$$

It follows that $\lambda_1(\Omega) \leq \lambda_1(\Omega_j) \leq \lambda_1(\Omega) + \varepsilon$ for sufficiently large j.

3 The proof

The proof of the lemma below is an adaptation of the method of Ôtani & Teshima in [13]. We avoid their argument about the normal derivatives.

Lemma 3.1 *If $\lambda > \lambda_1$, there are no positive eigenfunctions with eigenvalue λ. In other words, any positive eigenfunction is a minimizer of the Rayleigh quotient.*

Proof By contradiction. Assume that a positive eigenfunction v with eigenvalue $\lambda > \lambda_1(\Omega)$ exists. Using the Proposition and the fact that any bounded domain can be exhausted by a sequence of smooth domains (see for instance ([8], pp.317–319 or [6], p. 124), we can construct a smooth domain $\Omega^* \subset\subset \Omega$ such that also

$$\lambda_1^* = \lambda_1(\Omega^*) < \lambda \quad .$$

Let v_1^* denote the corresponding first eigenfunction in Ω^*. Since $\partial\Omega^*$ is smooth, we have $v_1^* \in C(\overline{\Omega^*})$ and $v_1^* = 0$ on the boundary $\partial\Omega^*$. Because

$$\min_{\overline{\Omega^*}} v > 0 \quad ,$$

we can arrange it so that

$$v_1^* \leq v \;\text{ in }\; \overline{\Omega^*} \tag{3.2}$$

by multiplying v_1^* by a small constant, if needed. We define

$$\kappa := \left(\frac{\lambda_1^*}{\lambda}\right)^{\frac{1}{p-1}} \quad . \tag{3.3}$$

It is decisive that $0 < \kappa < 1$. We claim that

$$-\mathrm{div}(|\nabla v_1^*, |^{p-2} \nabla v_1^*) \leq -\mathrm{div}(|\nabla(\kappa v)|^{p-2} \nabla(\kappa v)) \quad ,$$

from which it follows that

$$v_1^* \leq \kappa v \;\text{ in }\; \Omega^* \quad . \tag{3.4}$$

Indeed, for a test function $\varphi \geq 0$ we can use (3.2) and (3.3) to verify

$$\begin{aligned}
&\int_{\Omega^*} \langle \nabla v_1^* |^{p-2} \nabla v_1^*, \nabla\varphi\rangle dx = \lambda_1^* \int_{\Omega^*} (v_1^*)^{p-1} \varphi dx \\
&\leq \lambda_1^* \int_{\Omega^*} v^{p-1} \varphi dx = \lambda \int_{\Omega^*} (\kappa v)^{p-1} \varphi dx \\
&= \int_{\Omega^*} \langle |\nabla(\kappa v)|^{p-2} \nabla(\kappa v), \nabla\varphi\rangle dx \quad .
\end{aligned}$$

We may take $\varphi = (v_1^* - \kappa v)_+$. It follows that

$$\int_{v_1^* \geq \kappa v} \langle |\nabla v_1^*|^{p-2} \nabla v_1^* - |\nabla(\kappa v)|^{p-2} \nabla(\kappa v), \nabla v_1^* - \nabla(\kappa v)\rangle dx \leq 0$$

and hence $v_1^* \le \kappa v$ by the elementary inequality

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle > 0 \quad \text{for } a \neq b \quad .$$

Thus we have proved (3.4).

Repeat the procedure, now with κv in the place of v. The conclusion is that $v_1^* \le \kappa^2 v$. By iteration,

$$0 \le v_1^* \le \kappa^j v \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty \quad .$$

This yields the contradiction $v_1^* = 0$.

Remark. The zero boundary values of v have no bearing. The proof shows that the equation (1.2) cannot have a positive weak solution for $\lambda > \lambda_1$.

For the benefit of the reader we reproduce the proof from [5] for minimizers.

Lemma 3.2 *The minimizer of the Rayleigh quotient* (1.1) *is unique, except that multiplication with constants is possible.*

Proof. We can assume that the minimizers are positive. If $u_1 > 0$ and $u_2 > 0$ are minimizers, then we consider $v = (u_1^p + u_2^p)^{1/p}$ and observe that for $i = 1, 2$

$$\lambda_1 = \frac{\int_\Omega |\nabla u_i|^p dx}{\int_\Omega |u_i|^p dx} \le \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx} \quad . \tag{3.5}$$

We can write

$$\nabla v = v\left(\frac{u_1^p \nabla \log u_1 + u_2^p \nabla \log u_2}{u_1^p + u_2^p}\right)$$

whence we have a convex combination of $\nabla \log u_i$. By Jensen's inequality for convex functions

$$\begin{aligned} |\nabla v|^p &\le v^p \left(\frac{u_1^p |\nabla \log u_1|^p}{u_1^p + u_2^p} + \frac{u_2^p |\nabla \log u_2|^p}{u_1^p + u_2^p}\right) \\ &= |\nabla u_1|^p + |\nabla u_2|^p \quad . \end{aligned}$$

The inequality is strict where $\nabla \log u_1 \neq \nabla \log u_2$. By integration and (3.5) we get the contradiction

$$\lambda_1 < \frac{\int_\Omega |\nabla u_1|^p dx + \int_\Omega |\nabla u_2|^p dx}{\int_\Omega u_1^p dx + \int_\Omega u_2^p dx} = \lambda_1$$

if $\nabla \log u_1 \neq \nabla \log u_2$ on a set of positive measure. Hence $\nabla \log u_1 = \nabla \log u_2$ a.e. in Ω. It follows that $u_1 = Cu_2$ or $u_2 = Cu_1$ for some positive constant C.

References

[1] Allegretto, W., Huang, Y., A Picone's identity for the p-Laplacian and applications. Nonlinear Analysis **32** (1998), 819–830.

[2] Anane, A., Simplicité et isolation de la première valeur propre du p-laplacien avec poids. C.R. Acad. Sci. Paris, Sér. I Math. **305** (1987), 725–728.

[3] Bhattacharya, T., Radial symmetry of the first eigenfunction for the p-Laplacian in the ball. Proc. AMS **104** (1988), 169–174.

[4] Bhattacharya, T., Some results concerning the eigenvalue problem for the p-Laplacian. Annales Academiae Scientarum Fennicae, Series A.I. Mathematica **14** (1989), 325–343.

[5] Belloni, M., Kawohl, B., A direct uniqueness proof for equations involving the p-Laplace operator. Manuscripta Math. **109** (2002), 229–231.

[6] Heinonen, J., Kilpeläinen, T., Martio, O., *Nonlinear potential theory of degenerate elliptic equations.* Oxford University Press, New York, 1993.

[7] Kawohl, B., Longinetti, M., On radial symmetry and uniqueness of positive solutions of a degenerate elliptic eigenvalue problem. Z. Angew. Math. Mech. **68** (1988), T459–T460.

[8] Kellogg, O.D., *Foundations of Potential Theory*, Die Grundlehren der Mathematischen Wissenschaften **31**, Springer, Berlin 1929.

[9] Lieb, E., On the lowest eigenvalue of the Laplacian for the intersection of two domains. Inventiones Mathematicae **74** (1983), 441–448.

[10] Lindqvist, P., On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. Proc. AMS **109** (1990), 157–164. Addendum, ibid. **116** (1992), 583–584.

[11] Lindqvist, P., Some remarkable sine and cosine functions. Ricerche Mat. **44** (1995), 269–290.

[12] Ôtani, M., A remark on certain nonlinear elliptic equations. Proceedings of the Faculty of Science, Tokai University **19** (1984), 23–28.

[13] Ôtani, M., Teshima, T., On the first eigenvalue of some quasilinear elliptic equations. Proceedings of the Japan Academy, Ser. A Math. Sci. **64** (1988), 8–10.

[14] Sakaguchi, S., Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. Ann. Scuola Nórm. Sup. Pisa IV, **14** (1987), 404–421.

[15] Schmidt, E., Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet. Math. Ann. **117** (1940), 301–326.

[16] de Thélin, F., Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linéaire. C.R. Acad. Sci. Paris, Sér. I Math. **299** (1984), 911–914.

[17] de Thélin, F., Sur l'espace propre associé à la première valeur propre du pseudo-laplacien, C.R. Acad. Sci. Paris Sér. I Math. **303** (1986), 355–358.

[18] Trudinger, N. S., On Harnack type inequalities and their application to quasilinear elliptic equations. Comm. Pure Appl. Math. **20** (1967), 721–747

Author's addresses:

Bernd Kawohl
Mathematisches Institut
Universität zu Köln
D-50923 Köln, Germany
kawohl@mi.uni-koeln.de

and

Peter Lindqvist
Department of Mathematical Sciences
Norwegian University of Science and Technology
N-7491 Trondheim
Norway