# Symmetrization – or how to prove symmetry of solutions to a PDE

by Bernd Kawohl
Mathematical Institute, University of Cologne, D 50923 Cologne, GERMANY
kawohl@mi.uni-koeln.de

Invited Lecture at the ICM Satellite Conference: PDE Prague '98, to appear

**Abstract:** *Many partial differential equations are Euler equations of variational problems, and this circumstance can be used to derive symmetry or monotonicity properties of their solutions. I survey some recent results on symmetrization methods, i.e. on procedures in which a supposedly nonsymmetric minimizer is deformed into a symmetric one with lower energy. I also compare this variational approach with arguments involving the maximum principle. There are specific advantages and disadvantages of both methods. Another symmetry approch uses the unique continuation principle. Finally I address the recently emerging subject of isoparametric surfaces.*

Keywords: symmetrization, symmetry, rearrangement, unique continuation, isoparametric surfaces, Ginzburg-Landau

AMS Subject Classification Number: 35B05, 49K10, 26D10, 35J20

"Nihil est sine ratione" (Nothing is without a reason) was - according to G.Polya [46] - a fundamental idea of W.G.Leibniz. He phrased it in Latin, the lingua franca of the time when Charles University was founded 650 years ago. With my lecture in today's Latin I follow a suggestion of the organizing committee to talk about symmetry. Symmetry of solutions is often taken for granted in symmetric problems, but by no means automatically true [36, 37]. There are many discussions of this topic in the mathematical literature, e.g. [10, 45, 46, 54]. In this lecture I focus on symmetry of elliptic equations, and here in particular on symmetrization methods. For parabolic equations I refer to [31] and the lecture of P.Poláčik. Other methods to prove symmetry, for instance via stability arguments, are listed in [36].

## I. What is symmetrization?

Let me present the geometric idea of symmetrization in one dimension by means of a simple example. Symmetrization maps the nonsymmetric function $u_a(x)$ into a symmetric function $u^*(x) = u_0(x)$.

$$u_a(x) = \begin{cases} \frac{1+x}{1+a} \text{ in } [-1, a] \\ \frac{1-x}{1-a} \text{ in } [a, 1] \end{cases} \quad \mapsto \quad u^*(x) = 1 - |x| \text{ in } [-1, 1]$$

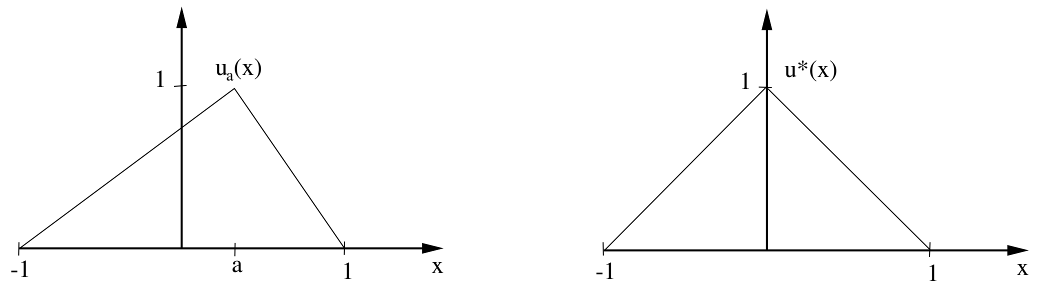


**Figure 1:** The graphs of $u_a$ and $u^*$

**Remark 1:**
One immediately observes the following two properties, which seem to be predestined for use in variational problems.

- $\int_{-1}^{1} F(u_a)\ dx = \int_{-1}^{1} F(u^*)\ dx$ (Cavalieri's principle) for any Borel measurable function $F$.
- $g(a) := \int_{-1}^{1} |du_a/dx|^p\ dx = (1+a)^{1-p} + (1-a)^{1-p} \geq \int_{-1}^{1} |du^*/dx|^p\ dx = g(0)$, because $g$ is even and (even strictly) convex.

**Definitions:**
For $u(x) \in W_0^{1,p}(\Omega, I\!R^+)$ consider the level sets $\Omega_c(u) := \{\ x \in\ I\!R \mid u(x) \geq c\ \}$. Then their rearrangements are defined as

$$\Omega_c^*(u) := \begin{cases} [-|\Omega_c(u)|/2, |\Omega_c(u)|/2] & \text{if } \Omega_c(u) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $u^*(x) :=\ \sup\ \{\ c \in I\!R \mid x \in \Omega_c^*(u)\ \}$, which is defined on $\Omega^* = \Omega_0^*(u)$, is called the **symmetrically decreasing rearrangement** of $u$. Note that the level sets $\Omega_c(u^*)$ of $u^*$ coincide with $\Omega_c^*(u)$ by construction.

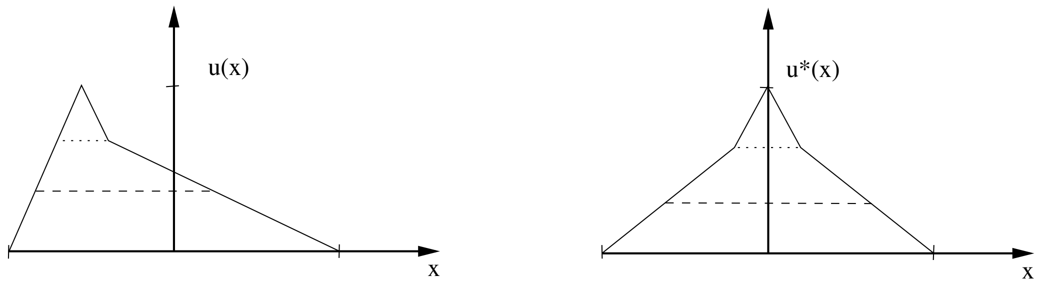


**Figure 2:** The graphs of $u$ and its symmetrically decreasing rearrangement $u^*$

To symmetrize a function of several variables, we have to symmetrize its level sets in a suitable way. One of them is **Steiner symmetrization** with respect to $y = 0$.

For $u(x) \in W_0^{1,p}(\Omega, I\!R^+)$ with $\Omega \subset I\!R^n$ set $x = (x', y)$ with $x' \in I\!R^{n-1}$. Now let us consider the one-dimensional level sets $\Omega_c(u(x', \cdot)) := \{ \; y \in \; I\!R \mid u(x', y) \geq c \; \}$, and define

$$
\begin{aligned}
\Omega_c^*(u(x', \cdot)) &:= \begin{cases} [-|\Omega_c(u(x', \cdot))|/2, |\Omega_c(u(x', \cdot))|/2] & \text{if } \Omega_c(u(x', \cdot)) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \\
\Omega_c^*(u) &:= \bigcup_{x' \in I\!R^{n-1}} \{x'\} \times \Omega_c^*(u(x', \cdot)), \quad \text{the \textit{Steiner symm.} of } \Omega_c(u), \\
u^*(x) &:= \; \sup \; \{ \; c \in I\!R \mid x \in \Omega_c^* \; \}, \qquad \text{the \textit{Steiner symmetrization}}
\end{aligned}
$$

or symmetrically decreasing rearrangement of $u$ in direction $y$.

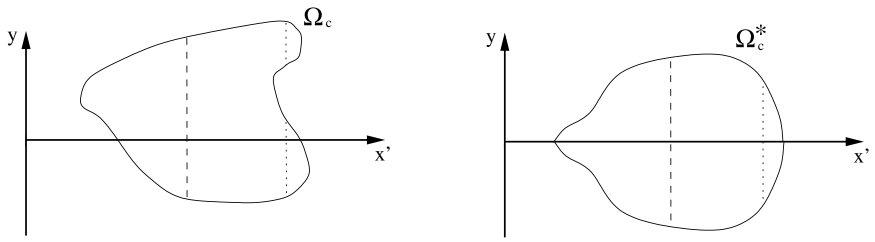


**Figure 3:** A level set $\Omega_c$ and its Steiner symmetrization with respect to $y$

Another suitable way to symmetrize level sets and functions is known as **Schwarz symmetrization** (not Schwartz symmetrization please !), named after Hermann Amandus Schwarz, in which for $u(x) \in W_0^{1,p}(\Omega, I\!R^+)$ with $\Omega \subset I\!R^n$ the level sets $\Omega_c(u) := \{ \; x \in \; I\!R^n \mid u(x) \geq c \; \}$ are replaced by concentric balls centered at zero. Therefore Schwarz symmetrization amounts to defining

$$
\begin{aligned}
\Omega_c^*(u) &:= \begin{cases} \text{ball of same volume as } \Omega_c \text{ centered at } 0 & \text{if } \Omega_c \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \\
u^*(x) &:= \; \sup \; \{ \; c \in I\!R \mid x \in \Omega_c^*(u) \; \}.
\end{aligned}
$$

Let me remark in passing, that Schwarz symmetrization can be obtained as a limit of consecutive Steiner symmetrizations (similar to making dumplings or snowballs).

**Remark 2:**
The following inequalities are common to all these rearrangements, see for instance [33].

- $\int_\Omega F(u) \; dx = \int_{\Omega^*} F(u^*) \; dx$ (Cavalieri's principle)
- $\int_\Omega |\nabla u|^p \; dx \geq \int_{\Omega^*} |\nabla u^*|^p \; dx$ (Polya-Szegö inequality) for $p \in (0, \infty)$
- $\int_\Omega u^* \; v^* \; dx \geq \int_\Omega \; u \; v \; dx$ (Hardy-Littlewood inequality)

**Application 1:**

*The Krahn Faber inequality* $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ is a typical example for the usefulness of Remark 2. Recall that the first Dirichlet eigenvalue for the Laplace operator on any domain $\Omega$ is characterized as $\lambda_1(\Omega) = \min\,\{R(v;\Omega)\ ;\ 0 \not\equiv v \in H_0^{1,2}(\Omega)\}$ and $R(v;\Omega) = ||\nabla v||_2/||v||_2$, where $||\cdot||_2$ denotes the norm in $L^2(\Omega)$. Therefore

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_1|^2\ dx}{\int_\Omega |u_1|^2\ dx} \geq \frac{\int_{\Omega^*} |\nabla u_1^*|^2\ dx}{\int_{\Omega^*} |u_1^*|^2\ dx} = R(u_1^*, \Omega^*) \geq \min_v R(v, \Omega^*) = \lambda_1(\Omega^*),$$

and **equality** holds **only if** $\Omega$ is a ball. This last observation about equality was (as I learned in 1993) first made by F.Schulz in his diploma thesis [49]. It rests on the observation that sometimes one can discuss the equality sign in the second inequality of Remark 2. In fact Schulz did this 1977 for the first eigenfunction and for the Dirichlet integral $\int_\Omega |\nabla u|^2\ dx$, while I discussed it in 1984 [32] for integrals of the type $\int_\Omega F(x', u)G(|\nabla u|)\ dx$ with positive $F$, strictly convex and strictly increasing $G$ and e.g. analytic functions $u$. Moreover, I showed that for $F \equiv 1$ and $G(t) = t^p$ and for a dense subset of $W^{1,p}(\Omega)$ one could discuss equality and conclude that it implies symmetry of $u$. But be careful. There are also functions, even $C^\infty$ functions, with large sets of critical points for which this conclusion is wrong. As a counterexample, imagine a radially symmetric cutoff function $\eta_1$, centered at zero, with a flat top $B_1(0)$, and add to it another radially symmetric cutoff function, centered at $x_0 \neq 0$, such that supp $\eta_2 \subset B_1(0)$. Then the Dirichlet integrals of $\eta = \eta_1 + \eta_2$ and $\eta^*$ coinincide, but $\eta \not\equiv \eta^*$. As for Schwarz-symmetrization, these results were later refined by J.E.Brothers and W.T.Ziemer [12, 13] and by R.Tahraoui [52]. Incidentally, the result of Brothers and Ziemer was first presented at a winter school 1987 in Srni, organized by Charles University. Brothers and Ziemer considered integrals of type $\int_\Omega A(|\nabla u|)\ dx$ under Schwarz symmetrization, with $A(0) = 0$, $A$ increasing and $A^{1/p}$ convex for some $p \in [1, \infty)$. They discussed the equality sign in $\int_\Omega A(|\nabla u|)\ dx \geq \int_\Omega A(|\nabla u^*|)\ dx$ under the assumptions $p > 1$ and that $\{x \in \Omega^*\ ;\ |\nabla u^*(x)| = 0\} \cap {u^*}^{-1}(0, M)$ is a nullset. Here $M$ =ess sup $u^* \leq \infty$. Fortunately, solutions of elliptic equations are often almost nowhere flat, so that these results are applicable for our purposes. Tahraoui considered more general integrals of type $\int_\Omega F(u, |\nabla u|)\ dx$ under Schwarz symmetrization, where $F$ satisfies natural growth assumptions and is convex in its second argument [52]. Finally, Brock considered integrals of type $\int_\Omega F(x, u, \nabla u)\ dx$ under Steiner symmetrization with structural assumptions on $F$ that recover all previous (weak) inequalities and that are too detailed to be reproduced here, see [7] and also [33, p.83].

**Application 2:**
*Nonlinear ground states* on $I\!R^n$ are in a simple setting characterized as minimal energy solutions of

$$\begin{aligned} -\Delta u = \lambda f(u) \quad \text{and} \quad u > 0 \quad & \text{in } I\!R^n, \\ u(x) \to 0 \qquad\qquad\qquad & \text{as } |x| \to \infty. \end{aligned}$$

Modulo suitable assumptions on $f$ there is a metatheorem which states that any ground state $u$ is radial and radially decreasing. This approach was used for instance in [4] or [25]. Of course there are other ways to get such results via the moving plane method, see [29] and [39]. To be a little more precise, in the recent paper [25] of M.Flucher and S.Müller the ground states are characterized as maximizing $\int_{I\!R^n} F(v)\, dx$ over all functions $v$ with Dirichlet integral not exceeding 1. Here $F' = f$, $0 \le F(t) \le c|t|^{2^*}$ with $2^* = 2n/(n-2)$ and $F$ is upper semicontinuous and $F \not\equiv 0$ in $L^1$-sense.
This includes the case that $f(0) = f'(0) = 0$, which was not covered in [29]. In fact, in [29] the assumption $f(0) = 0$ and $f'(0) < 0$ was used, and only later removed in the paper of Y.Li and W.M.Ni. I refer to the lecture of H.Berestycki during this meeting for further details.

**Application 3:**
*Oddness and monotonicity, even one-dimensionality in cylinders* can be proved via Steiner symmetrization as well. To this end let $\Omega = \omega \times I\!R$ with $\omega \subset I\!R^{n-1}$ and consider minimizers $u$ of the Ginzburg-Landau energy $J(v) = \int_\Omega 2|\nabla v|^2 + (v^2-1)^2\, dx$ among those functions $v$ which go to $\pm 1$ as $x_n \to \pm\infty$.

**Claim 1:**

*The nodal surface of $u$ is a plane, without loss of generality it is the plane $\{x_n = 0\}$, so that $u(x', 0) = 0$, and $u$ is odd in $x_n$, i.e. $u(x', x_n) = -u(x', -x_n)$ in $\Omega$.*

Proof: Substitute $w := (1-u^2)$, then $w \ge 0$ and $w \to 0$ as $|x_n| \to \infty$. For every $x' \in \omega$ the function $w(x', \cdot)$ has a maximum of height 1 somewhere. Under this tranformation the energy is converted into a **convex functional**

$$\tilde{J}(w) := \int_\Omega \frac{|\nabla w|^2}{2(1-w)} + w^2\, dx.$$

Incidentally, the convexity of this transformed functional $\tilde{J}$ might explain the somewhat unexpected uniqueness-, symmetry- and Liouville-theorems for Ginzburg-Landau type problems. To prove the oddness of $u$, Steiner-symmetrize $w$ and set $\tilde{u}(x', x_n) := \operatorname{sign}(x_n)\sqrt{1-w^*}$. Then it is easy to see that $J(\tilde{u}) \le J(u)$ and equality holds only if $u$ is odd and monotone. More can be shown, however.

**Claim 2:**

*Not only the nodal surface, but every level surface of $u$ is a plane, and therefore $u$ depends only on $x_n$.*

Proof: For each $c \in (-1,1)$ the level surface $\{(x', x_n); u(x) = c\}$ can be parametrized by $x'$ as $x_n = f_c(x')$. Replace it by its average $x_n = \oint_\omega f_c(x')\,dx'$, i.e. replace the level sets of $w = 1 - u^2$ by cylinders with cross section $\omega$ and with plane ends. It can be shown that this decreases energy.

This approach can be found in a nice paper of G.Carbou [17]. It was recently generalized by F.Brock [8] to apply to travelling waves in cylinders, generalizing previous results of H.Berestycki & L.Nirenberg [3].

What about boundary conditions other than Dirichlet? We just saw an example with zero Neumann data on the lateral boundary $\partial\omega \times I\!R$. Here are two more.

**Application 4:**

*Steiner symmetry for nonlinear boundary conditions* can sometimes be achieved without symmetrization arguments. The following theorem is new, and its assumptions are far from optimal. Rather than specialize in generalizations, I just want to expose the idea of proof.

**Theorem 1:**

*Let $\Omega \subset I\!R^n$ be Steiner symmetric in $x_n$ and consider minimizers $u$ of the functional $J(v) = \int_\Omega \{|\nabla v|^2 - f(x)v\}\,dx + \int_{\partial\Omega} j(v)\,d\sigma$ with $f \in C^1(\Omega)$ nonnegative and Steiner-symmetric and $j \in C^1(I\!R)$ convex and even. Then $u$ is Steiner symmetric, i.e. symmetrically decreasing in $x_n$.*

Note that $u$ satisfies the nonlinear boundary condition $-\partial u/\partial\nu = \partial j(u)$ on $\partial\Omega$.

Proof of Theorem 1: By strict convexity a minimizer $u$ is unique, thus symmetric. Moreover it is nonnegative, because otherwise we can replace it by $|u|$. It remains to show that $u$ is symmetrically decreasing in $x_n$ or equivalently, that $u$ decreases in $x_n$ on $\Omega_> := \Omega \cap \{x_n > 0\}$. Clearly $u_{x_n} = 0$ on the hyperplane $\Omega \cap \{x_n = 0\}$ because of the symmetry, and $u_{x_n} \leq 0$ on $\partial\Omega \cap \{x_n > 0\}$ due to the boundary condition, thus $w = u_{x_n} \leq 0$ on $\partial\Omega_>$. But now we can apply the maximum principle to $w$. In fact $u$ satisfies the Euler equation $-\Delta u = f$ in $\Omega$. Therefore differentiation with respect to $x_n$ yields $-\Delta w = f_{x_n} \leq 0$ in $\Omega_>$. Thus $w < 0$ in $\Omega_>$ and $u$ is Steiner symmetric.

**Remark 3:**

It seems to be difficult to prove Theorem 1 via Steiner symmetrization. It can be shown that the boundary integral in $J$ decreases under symmetrization. The $L^2$-product of $u$ and $f$ increases under symmetrization due to the Hardy-Littlewood inequality in Remark 2. Thus the second term in $J$ decreases. What about the

Dirichlet integral? It can in fact increase under Steiner symmetrization, unless we know already that for a.e. $x'$ the minimum of $u(x', x_n)$ is attained on the boundary points. Otherwise the obstruction from [33, p.56] causes problems. An example (due to my student Miroslaw Michaelowski) for which the Dirichlet integral increases under Steiner symmetrization is provided by the following function. Let $\Omega := (0,1] \times (-2,2) \cup \{(x,y) | \, x \in [1,2), \ |y| < 3-x\}$ and $u(x,y) := 0$ if $|y| < 1$ and $u(x,y) := |y| - 1$ otherwise. Then a simple calculation shows that $\int_\Omega |\nabla u|^2 dx = 3$, while $\int_\Omega |\nabla u^*|^2 dx = 4$.

I wish to point out, however, that Steiner symmetrization can be sucessfully applied to functions which are periodic in $x_n$. This observation is used for instance in circular symmetrization, where a disk in $I\!R^2$ is interpreted as a rectangle in polar coordinates, by identyfying $x' = r$ and $x_n = \phi$, see [33, Ch.II.9] or [21].

**Application 5:**
*Optimal insulation* of bodies against heat loss was recently studied in [21]. Let $\Omega$ be the unit disk, and let $u$ be a stationary temperature in $\Omega$, subject to uniform heating, i.e. $-\Delta u = 1$ in $\Omega$. On the boundary $\partial\Omega$ we have Newton's law of cooling $-(\partial u/\partial \nu) = hu$, where $h$ represents conductivity. We want to maximize insulation in the sense that the average temperature should be maximized among admissible variations of $h$. Depending on the material which bounds $\partial\Omega$, the conductivity constant $h$ is allowed to attain two values $h_1$ and $h_2$ with $0 < h_1 < h_2 < \infty$, but the proportion of $\partial\Omega$ on which $h = h_1$ is prescribed. Its location is free. Mathematically this leads to the problem of minimizing

$$J(v,h) := \int_\Omega \{|\nabla v|^2 - 2v\} \ dx + \int_{\partial\Omega} hv^2 \ ds$$

over $v \in W^{1,2}(\Omega)$; $h : \partial\Omega \to \{h_1, h_2\}$ and $\int_{\partial\Omega} h \ ds = c \in (2\pi h_1, 2\pi h_2)$.
One can think of the subset $\partial\Omega_1$ of $\partial\Omega$ on which $h = h_1$ as a wall, and of $\partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1$ as windows in a room. Then one of the the results in [21] states that having a single large window is preferable to having many small ones. This result was obtained by means of circular symmetrization. A similar problem was studied in [24], where the first eigenvalue was minimized under mixed Dirichlet-Neumann boundary conditions. These occur as limits $h_2 \to \infty$ and $h_1 \to 0$ in our formulation.

## II. Continuous symmetrization

Until now my **message** has been: Global minimizers of variational problems are symmetric or monotone. An apparent advantage of symmetrization lies in the fact that often it requires less regularity than for the moving plane method. Disadvantages of symmetrization until now seemed to be a) that the method did not apply to nonvariational problems and b) that it did not yield results for local minimizers or critical points.

At least the second one of these difficulties has recently been overcome. **F.Brock** has been able to apply **continuous symmetrization** methods to local minimizers of variational problems and to critical points. For functions of a single variable this concept had been developed by H.Matano and myself over ten years ago [42, 34], and for functions of several variables with convex level sets it goes back about 50 years to G.Polya and G.Szegö [47, pp.201–204]. At the beginning of my lecture I presented an example of continuous symmetrization. Remember the function $u_a$ from Figure 1? Consider $a$ as a parameter for a family $\{u_a\}$ of equimeasurable functions and send $a$ to zero. Then $u_a$ tends to $u_0 = u^*$. It is easy to extend this approach to functions with convex level sets, and this was done in [47].
In continuous symmetrization a more general function $u$ is continuously deformed into a family of equimeasurable functions $u^t$ and, as $t \to \infty$, into $u^*$. To this end the level sets of $u$ are shifted towards their final location with a speed that depends on their shape and their distance to the final location. The details of the definition are too long to be reproduced here, see [5, 6]. However, for $u_a$ from Figure 1 above, Brock's $u^t$ corresponds to our $u_b$ with $b = a\ \mathrm{e}^{-t}$. To give you a flavor of Brock's results I list one of them.

**Theorem 2: (Brock [6, Theorem 9.1])**
*Let $\Omega$ be Steiner-symmetric in $x_n$, $p \in (1, \infty)$, $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ a nonnegative solution of*

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x', x_n, u) \quad \text{in } \Omega,$$

*and let $f$ be symmetrically strictly decreasing in $x_n$ and continuous. Then $u = u^*$.*

The idea of the proof proceeds as follows. Test the differential equation with $u^t - u$. Then by the convexity of the function $s \mapsto s^p$ and since $u_t$ is close to $u$ for small $t$

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u^t - u)dx &= \int_\Omega f(x,u)(u^t-u)dx \qquad (\ = o(t)\ ) \\ &\le \frac{1}{p}\int_\Omega (|\nabla u^t|^p - |\nabla u|^p)\ dx \ \le\ 0\end{aligned}$$

Thus

$$\int_\Omega (|\nabla u^t|^p - |\nabla u|^p)\ dx = o(t)$$

and this implies via a delicate quantitative analysis that $u$ is symmetric. To get a feeling for this qualitative analysis consider the function $g(a)$ defined in Remark 1 and note that $g(a+t) - g(a) = o(t)$ if and only if $a = 0$.

F.Brock has many other results, for instance on mean curvature type equations, on equations in $I\!R^n$ et cetera. Further applications of continuous symmetrization have recently emerged, namely

• Applications to overdetermined boundary value problems. More precisely, if $u$ solves $-\Delta_p u = f(u)$ in $D$, $\quad u = 0$ and $\partial u/\partial\nu = c$ on $\partial D$, where $D \subset I\!R^n$ is simply connected and convex, and where $f$ is continuous and nonincreasing, then $D$ is a ball. This result in [9] is a considerable generalization of [26], where $f \equiv 1$. Another variant is given in [19, Sect. 4], this time for solutions of $-\Delta u = \lambda_1 u$ under the same boundary conditions, and again assuming convexity of $\Omega$. Here continuous Steiner symmetrization leads to the well known conclusion that $\Omega$ is a ball.

• Results on the attainability of eigenvalues by D.Bucur, G.Buttazzo, I.Figueiredo [13]. The Krahn Faber inequality states that given the volume of $\Omega$ the range of $\lambda_1(\Omega)$ is a half-infinite interval. In [13] the authors look for the range of the vector $(\lambda_1(\Omega), \lambda_2(\Omega))$, given the volume of $\Omega$, and prove via continuous rearrangement that it is a closed set $E$ in $I\!R^2$ which is convex in both cartesian directions. This shows in particular, that given $\lambda_1(\Omega)$ one can maximize or minimize $\lambda_2(\Omega)$ and vice versa.

• More general shape optimization problems require the convergence $W_0^{1,p}(\Omega^t) \to W_0^{1,p}(\Omega^s)$ as $t \to s^-$ in a suitable sense (of Mosco) in order to prove existence results. Such a convergence has been derived by D.Bucur and A.Henrot via continuous symmetrization in [14].

### III. Symmetry without symmetrization

Let me now come to a somewhat different subject. There are domains with reflection symmetries which are not Steiner symmetric, for instance spherical annuli in $I\!R^n$, i.e. $\Omega = B_{R_2}(0) \setminus B_{R_1}(0)$ with $0 < R_1 < R_2 < \infty$. How about symmetry of solutions on those domains?

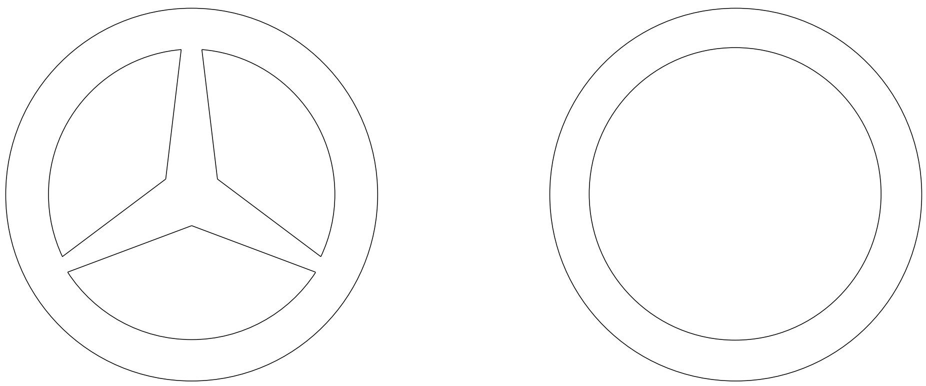

**Figure 4:** Domains without Steiner symmetry

One succesful approach to proving symmetry is based on the **unique continuation principle**.

**Theorem 3: (Lopez [41])**
*Let $u$ minimize $J(v) = \int_\Omega \{|\nabla v|^2 + F(v)\}\ dx$ and let $\Omega$ be symmetric in $x_1$. Then $u$ is symmetric in $x_1$.*

For the proof cut $\Omega$ in half, into $\Omega_r = \Omega \cap \{x_1 > 0\}$ and its reflection $\Omega_l$. Then the energy $J(u)$ is equidistributed,

$$\int_{\Omega_l} \{|\nabla u|^2 + F(u)\}\ dx = \int_{\Omega_r} \{|\nabla u|^2 + F(u)\}\ dx.$$

Otherwise, if say the left hand side is smaller than the right hand side, we could define a function

$$\tilde{u}(x_1, \ldots x_n) = \begin{cases} u(x_1, \ldots x_n) & \text{if } x \in \Omega_l, \\ u(-x_1, \ldots x_n) & \text{if } x \in \Omega_r, \end{cases}$$

such that $J(\tilde{u}) < J(u)$; contradicting the minimality of $u$.

Therefore we see that if $u$ minimizes $J$, so does $\tilde{u}$. But $u = \tilde{u}$ in $\Omega_l$. Consequently $u \equiv \tilde{u}$ on all of $\Omega$, provided the solutions of the associated Euler-equation have the unique continuation property, i.e. if $f$ is for instance Lipschitz continuous.

This trick was used by O.Lopez, and it works even for some elliptic systems such as the Lamé system with variable Lamé constants see [1], but it is not entirely new. M.E.Gurtin and H.Matano used it for instance 1988 to prove monotonicity results in [30, p.308].

On **annuli** there can be symmetry breaking, but then the solutions still show some symmetry under certain group actions.
This effect was shown in [18, Sec.3], [22], [25, p.148 &163], [33, pp.95–99], [38] and [40] for positive solutions of

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and for suitable $f$ with subcritical growth. In these papers prototypes for $f$ are $f(u) = u^p - u$ or $f(u) = u^p$.

## IV. Isoparametric surfaces

are level sets of solutions to the overdetermined system

$$-\Delta u = f(u) \quad \text{in } \Omega \quad \text{and} \quad |\nabla u| = g(u) > 0 \quad \text{in } \Omega. \tag{$*$}$$

The following result is well known among differential geometers, but maybe not so well-known in the PDE-community. It was found around 1920 in attempts to describe the propagation of wave fronts in geometrical optics.

**Theorem 4:**
*If $u$ solves system $(*)$ above in a connected domain $\Omega \subset I\!R^n$, if $f$ is $C^0$ and $g$ is $C^1$, then the level surfaces of $u$ are either planes, spheres or cylinders, i.e. cartesian products of spheres and planes.*

For a recent proof in the three-dimensional case, see [48]; for the general history of Theorem 4 and its generalizations to noneuclidean geometry I refer to [53]. Before giving a generalization of this result, let me demonstrate its usefulness.

**Application 6:**
In 1971 H.Weinberger published an elegant approach to a special overdetermined problem

$$-\Delta u = 1 \qquad\qquad \text{in } D \subset I\!R^n$$
$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = c \neq 0 \qquad \text{on } \partial D$$

for a bounded domain $D$, and showed that $D$ is a ball. This way he recovered a special case of J.Serrin's seminal results [50]. His strategy of proof went through first showing

$$|\nabla u|^2 + \frac{2}{n} u = c^2 \quad \text{in } D$$

and to derive radial symmetry from this [55]. If we set $g(u) = \sqrt{c^2 - \frac{2}{n}u^2}$, and identify $\Omega$ with $D \setminus \{\text{set of critical points of } u\}$, we can apply Theorem 4 to see that $D$ must be a ball.
It seems only natural that Weinberger's trick might extend to more general semilinear equations $-\Delta u = f(u)$, and that then $P(x) = |\nabla u|^2 + \frac{2}{n}F(u)$ should be constant. Let me dismiss any hope about this by pointing out that $P$ is not even constant for the linear function $f(u) = \lambda_1(D)u$ and if $D$ is a ball.

**Application 7:**
An open problem of DeGiorgi, that was already presented in H.Berestycki's lecture, reads precisely as follows: "Consider a solution $u \in C^2(I\!R^n)$ of $\Delta u = u^3 - u$ in $I\!R^n$ such that $|u| \leq 1$, $\frac{\partial u}{\partial x_n} > 0$ in the whole $I\!R^n$. Is it true that for every $\lambda \in I\!R$ the sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$ ?"

In [36] I gave an explanation why the dimension $n \leq 8$ makes things easier. L.Caffarelli, N.Garofalo & F.Segala [16] and later X.Chen [20] showed that the auxiliary function $P(u, x) := |\nabla u(x)|^2 - 2F(u(x))$ is nonpositive in $I\!R^n$. Here $F(u) = (u^2 - 1)^2/4$. This tells us that the "kinetic energy" or Dirichlet integral in the corresponding functional is dominated by the "potential energy" term, except that both terms are infinite. Moreover, under the further assumption, that there is an $x_0 \in I\!R^n$ such that $P(u, x_0) = 0$ they were able to derive a Liouville-type result, namely $P \equiv 0$. Once this result is known one can see that $P_\nu$, the partial derivative of $u$ in direction $-\nabla u$ must vanish. This leads to the observation, see [36], that the $(n-1)$ dimensional level surfaces of $u$ are minimal surfaces, and then a famous theorem of Bernstein applies, provided $n-1 \leq 7$ or $n \leq 8$. Both in [16] and [20] the one-dimensionality of $u$ is derived for any $n$, provided $P$ is constant. But in this case we can cut their proof short by observing that one can identify $g(u)$ with $\sqrt{2F(u)}$ and apply Theorem 4 to reach the conclusion. Incidentally, N.Ghoussoub and C.Gui have recently found another proof of DeGiorgi's conjecture without assuming anything about $P$, but unfortunately only under the restriction $n = 2$ or 3, see [27].

**Application 8:**
O.Mendez & W.Reichel looked recently at the overdetermined exterior boundary value problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } I\!R^n \setminus D \\ u = c > 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} &= -1 && \text{on } \partial D \end{aligned}$$

with $D$ a bounded convex domain $D \subset I\!R^n$ and $n \geq 3$, and showed that $D$ is a ball [43].
They showed first that the auxiliary function

$$P(x) = |\nabla u(x)|^2 \; u(x)^{(2-2n)/(n-2)}$$

equals a constant $C$ in $I\!R^n \setminus D$. If we set $g(u) = \sqrt{C} u(x)^{(n-1)/(n-2)}$ and apply Theorem 4, we can recover their result.

After these applications you might bear with me, if I present another result in this spirit. It is not essential that $(*)$ contains the Laplace operator. To fix ideas let $\Omega \subset I\!R^n$ be a connected domain and suppose that $u : \Omega \to I\!R$ solves the following system of differential equations

$$-\text{div} \; \left( \, a(u, |Du|^2) \; Du \, \right) = f(u, |Du|^2) \quad \text{in } \Omega, \tag{1}$$

$$|Du| = g(u) > 0 \quad \text{in } \Omega. \tag{2}$$

Here $a$ and $g$ are assumed to be of class $C^1$ and $f \in C^0$. For $a \equiv 1$, i.e. if (1) has the special structure $-\Delta u = f(u)$, we recover Theorem 4. However, not even the ellipticity of the operator in (1) is needed to derive its conclusion.

**Theorem 5:**
*Suppose that $a > 0$, then the level surfaces $\Omega_d(u) := \{ \ x \in \Omega \mid u(x) = d \ \}$ of solutions $u$ to* (1), (2) *are subsets of planes, cylinders or spheres.*

Proof: Without loss of generality we may assume that $g \equiv 1$, otherwise we substitute

$$\tilde{u}(x) := \int_0^{u(x)} \frac{1}{g(t)} \ dt$$

for $u$. If we denote differentiation with respect to $x_j$ by subscript $_j$, and let $p$ stand for $|\nabla u|^2$, we can rewrite (1) using Einstein's summation convention as

$$-a(u,p)\Delta u - 2a_p(u,p)u_i u_j u_{ij} - a_u(u,p)p = f(u,p). \tag{3}$$

Consequently, observing that $p = g \equiv 1$

$$-\Delta u = \frac{2a_p(u,1)u_i u_j u_{ij} + a_u(u,1) + f(u,1)}{a(u,1)}. \tag{4}$$

But the right hand side of (4) looks almost as if it depends only on $u$. Let $\nu(x) = -\nabla u(x)/|\nabla u(x)|$ be the direction of steepest descent of $u$ in $x$. Then the term $u_i u_j u_{ij}$ can be expressed as $u_\nu^2 u_{\nu\nu}$, but $g_\nu = (|\nabla u|^2)_\nu = 2u_\nu u_{\nu\nu} \equiv 0$. Therefore (4) degenerates to

$$-\Delta u = \tilde{f}(u) = [a_u(u,1) + f(u,1)]/a(u,1), \tag{5}$$

a variant of (1). But now the proof of Theorem 5 is reduced to Theorem 4.

**Remark 3:**
Let $H(x)$ denote the mean curvature of the level surface through $x$. One can then rewrite $\Delta u$ as $u_{\nu\nu} + (n-1)H(x)u_\nu$ and (3) as (see for example [35])

$$-(n-1)\ a(u,p)\ H(x)\ u_\nu - [a(u,p) + 2pa_p(u,p)]\, u_{\nu\nu} = a_u(u,p)p + f(u,p),$$

and, using $u_{\nu\nu} \equiv 0$, arrive at

$$H(x) = \frac{a_u(u,1) + f(u,1)}{(n-1)a(u,1)} =: h(u) \qquad \text{in } \Omega,$$

i.e. at the fact that the mean curvature of each level surface is constant and the constant depends only on the level $u$. Other equations that $u$ satisfies, once (1) (respectively (5)) and $u_\nu = -1$ hold, are

$$\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u) = |\nabla u|^{q-2}\,[\Delta u + (q-2)u_{\nu\nu}] = \Delta u = \tilde{f}(u)$$

for **any** $q \in (1, \infty)$, or

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{\tilde{f}(u)}{\sqrt{2}} \qquad \text{in } \Omega,$$

i.e, the mean curvature of the graph of $u$ depends only on its level $u$. (Incidentally, if we know in addition that the level surfaces of $u$ are compact, a famous result of Alexandrov implies that they are spheres.) This shows how overdetermined the system (1), (2) is.

**Remark 4:**
Let me conclude with an open problem. In [2] the authors look for a variant of Serrin's result, i.e. for special symmetries of solutions to overdetermined boundary value problems on unbounded domains, and state: "It is tempting to conjecture that a much more general result is true. Assuming that $\Omega$ is a smooth domain with $\Omega^c$ connected and that there exists a positive bounded solution of"

$$\begin{aligned} \Delta u + f(u) = 0 \qquad &\text{and} \qquad u > 0 \qquad\qquad &\text{in } \Omega, \\ u = 0 \qquad &\text{and} \qquad \frac{\partial u}{\partial n} = c \neq 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

"for some Lipschitz function $f$, then $\Omega$ is either a half-space, a ball, the complement of a ball or a circular-cylinder type of domain: $\mathbb{R}^j \times B$ with $B$ a ball."
But such a problem seems to be amenable via a $P$-function, and maybe one can solve it by a combination of results in [51] with Theorem 4 above.

This concludes my little survey. I hope to have demonstrated that there is more than one way to prove symmetry. I learned Liouville theorems in Prague during my time 1979 as a postdoctoral student of J.Nečas. Therefore I feel entitled to say: Universitas Carolina vivat crescat et floreat!

## References

[1] Ang,D.D., M.Ikehata, D.D.Trong & M.Yamamoto, Unique continuation for a stationary isotropic Lamé system with variable coefficients. Comm. Partial Differ Eqs., **23** (1998) pp.371–385.

[2] Berestycki,H., L.Caffarelli & L.Nirenberg: Monotonicity for elliptic equations in unbounded Lipschitz domains. Comm. Pure Appl. Math **50** (1997) pp. 1089-1111.

[3] Berestycki,H. & L.Nirenberg, Travelling fronts in cylinders. Ann. Inst. H. Poincaré, Anal. Non Linéaire **9** (1992) pp.497–572.

[4] Berestycki,H. & P.L.Lions, Nonlinear scalar field equations, I. Existence of a ground state. Arch. Ration. Mech. Anal. **82** (1983) pp.313–345.

[5] Brock,F., Continuous Steiner symmetrization, Math. Nachr. **1722** (1995) pp. 25–48.

[6] Brock,F., Continuous Rearrangement and Symmetry of Solutions of Elliptic Problems. Habilitationsschrift, Leipzig (1998), pp. 1–124.

[7] Brock,F., Weighted Dirichlet-type inequalities for Steiner-symmetrization, Calc. Var. and PDE, to appear.

[8] Brock,F., Averaging rearrangement and one-dimensionality of solutions of some boundary value problems in cylinders, manuscript (1998).

[9] Brock,F. & A.Henrot, A symmetry result for an overdetermined elliptic problem using continuous rearrangement and domain derivative, manuscript (1998).

[10] Brothers,J.E., Invariance of solutions to invariant nonparametric problems. Trans. Amer. Math. Soc. **264** (1981) pp. 91–111.

[11] Brothers,J.E. & W.P.Ziemer: Minimal rearrangements of Sobolev functions. 15th winter school in abstract Analysis, Srni, Acta Univ. Carol. Math. Phys. **28** (1987) pp.13–24.

[12] Brothers,J.E. & W.P.Ziemer: Minimal rearrangements of Sobolev functions. J. reine angew. Math. **384** (1988) pp.153–179.

[13] Bucur,D., Buttazzo,G. & I.Figueiredo, On the attainable eigenvalues of the Laplace operator. Preprint 9, Besancon (1997), to appear in SIAM J. Math. Anal..

[14] Bucur,D. & A.Henrot, Continuous Steiner symmetrization and stability for the Dirichlet problem. Preprint 16, Besancon (1997), to appear in Potential Analysis.

[15] Byeon,J. Existence of many nonequivalent nonradial positive solutions of semilinear elliptic equations on three-dimensional annuli. J. Differ. Equations **136** (1997) pp. 136–165.

[16] Caffarelli,L., N.Garofalo & F.Segala, A gradient bound for entire solutions of quasilinear equations and its consequences. Comm. Pure Appl. Math. **47** (1994) pp. 1457-1473.

[17] Carbou,G., Unicité et minimalité des solutions d'une équation de Ginzburg-Landau. Ann. Inst. H. Poincaré, Anal. Non Linéaire **12** (1995) pp.305–318.

[18] Catrina,F. & Z.Q.Wang, Nonlinear elliptic equations on expanding symmetric domains. manuscript (1998)

[19] Chatelain,T., M.Choulli & A.Henrot, Some new ideas for Schiffer's conjecture, in: *Modelling and Optimization of distributed parameter systems*, Eds: K.Malanowski, Z.Nahorski & M.Peszinska, Chapman & Hall, London (1996) pp. 90–97

[20] Chen,X., Global asymptotic limit of solutions to the Cahn Hilliard equation, J. Differ. Geom. **44** (1996) pp. 262–311.

[21] Cox,S. & B.Kawohl, Circular symmetrization and extremal Robin conditions, Zeitschrift Angew. Math. Phys. (ZAMP) **50** (1999), to appear.

[22] Coffman,C.V., A nonlinear boundary value problem with many positive solutions, J. Differ. Equations **54** (1984) pp.429–437.

[23] DeGiorgi,E., Convergence problems for functions and operators. in:*Recent Methods in Nonlinear Analysis*, Eds: E.De Giorgi, E.Magenes & U.Mosco; Pitagora Editrice, Bologna (1979) pp. 131–188.

[24] Denzler,J., Windows of given area with minimal heat diffusion. Trans. Amer. Math. Soc., to appear.

[25] Flucher,M. & S.Müller, Radial symmetry and decay rate of variational ground states in the zero mass case. SIAM J. Math. Anal. **29** (1997) pp. 712–719.

[26] Garofalo,N. & J.L.Lewis, A symmetry result related to some overdetermined boundary value problems, Amer. J. Math. **111** (1989) pp. 9–33.

[27] Ghoussoub,N. & C.Gui, On a conjecture of De Giorgi and some related problems. Math. Ann. **311** (1998) pp. 481–491.

[28] Gidas,B., W.M.Ni & L.Nirenberg, Symmetry and related properties via the maximum principle. Comm. Math. Phys. **68** (1979) pp. 209–243.

[29] Gidas,B., W.M.Ni & L.Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Math. Anal. and Applications Part A*, Ed. L.Nachbin, Advances in Math. Suppl. Studies 7A, Academic Press (1981) pp. 369–402.

[30] Gurtin,M.E. & H.Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids. Quart. Appl. Math. **46** (1988) pp. 301–317.

[31] Hess,P. & P.Poláčik, Symmetry and convergence properties for non-negative solutions of nonautonomous reaction-diffusion problems. Proc. Roy. Soc. Edinb. Sec A **124** (1994) pp. 573–587.

[32] Kawohl,B., On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems, Arch. Ration. Mech. Anal. **94** (1986) pp. 227–243.

[33] Kawohl,B., *Rearrangements and convexity of level sets in PDE*. Springer Lecture Notes in Math. **1150** Springer Verlag, Heidelberg (1985), pp.1–136.

[34] Kawohl,B., Continuous symmetrization and related problems, in: *Differential Equations*, Eds: C. Dafermos, G.Ladas, G.C.Papanicolaou. Marcel Dekker, New York (1989) pp. 353–360.

[35] Kawohl,B., Comparison results between linear and quasilinear elliptic boundary value problems, J. Math. Anal. Appl. **142** (1989) pp. 162–169.

[36] Kawohl,B., Symmetry or not? Mathematical Intelligencer **20** (1998) pp. 16–22.

[37] Kawohl,B., Was ist eine runde Sache? GAMM Mitteilungen **21** (1998) pp . 43–56.

[38] Li,Y.Y., Existence of many positive solutions of semilinear equations on annulus. J.Differ Equations **83** (1990) pp. 348–367.

[39] Li,Y. & W.M.Ni, Radial symmetry of positive solutions of nonlinear equations in $\mathbb{R}^n$, Comm. Partial. Differ. Eqs. **18** (1993) pp. 1043–1054.

[40] Lin,S.S., Existence of many positive nonradial solutions for nonlinear elliptic equations on an annulus, J. Differ. Equations **103** (1993) pp. 338–349.

[41] Lopez,O.: Radial and nonradial minimizers for some radially symmetric functionals. Electronic Journal of Differential equations **1996** (1996) No. 3, 1–14. On WWW under URL:http//ejde.math.swt.edu

[42] Matano,H. & B.Kawohl, Continuous equimeasurable rearrangement and applications to variational problems, unpublished manuscript (1988).

[43] Mendez,O. & W.Reichel, Electrostatic characterization of spheres, manuscript (1998).

[44] Modica,L. & S.Mortola, Some entire solutions in the plane of nonlinear Poisson equations. Boll. Un. Mat. Ital. **17 B** (1980) pp. 614–622.

[45] Palais,R.S., The principle of symmetric criticality, Commun. Math. Phys. **69** (1979) pp. 19–30.

[46] Polya,G., Inequalities and the principle of nonsufficient reason. in: *Inequalities* Ed: O.Shisha, Acad Press, New York (1967) pp. 1–15.

[47] Polya,G. & G.Szegö, *Isoperimetric inequalities in mathematical physics*. Ann. Math. Studies **27** Princeton Univ. Press, Princeton (1952)

[48] Pucci,P., An overdetermined system, Quarterly Appl. Math. **41** (1983) pp. 365–367.

[49] Schulz,F., Der Faber-Krahnsche Satz für Dirichlet-Gebiete in $n$-Dimensionen, Diplomarbeit, Göttingen (1977).

[50] Serrin,J., A symmetry problem in potential theory, Arch. Ration. Mech. Anal.**43** (1971) pp. 304–318.

[51] Sperb, R., *Maximum principles and their applications*, Academic Press, New York (1981).

[52] Tahraoui,R., Symmetrization inequalities, Nonlin. Anal. Theory Meth. Appl. **27** (1996) pp. 933–955.

[53] Thorbergsson,G., A survey on isoparametric hypersurfaces and their generalizations. In: *Handbook of Differential Geometry*, Eds: F.Dillen, L.Verstraelen, Elsevier Science, to appear.

[54] Waterhouse,W.C., Do symmetric problems have symmetric solutions? American Math. Monthly **90** (1983) pp.378–388.

[55] Weinberger,H., Remark on the preceding paper of Serrin, Arch. Ration. Mech. Anal. **43** (1971) pp. 319–320.