Meissner's Mysterious Bodies

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Introduction

Certain three-dimensional convex bodies have a counterintuitive property; they are of constant width. In this particular respect they resemble a sphere without being one. Discovered a century ago, Meissner's bodies have often been conjectured to minimize volume among bodies of given constant width. However, this conjecture is still open. We draw attention to this challenging and beautiful open problem by presenting some of its history and its recent development.

One Century of Bodies of Constant Width

In §32 of their famous book "Geometry and the Imagination", Hilbert and Cohn-Vossen list eleven properties of the sphere and discuss which of these suffices to uniquely determine the shape of the sphere [22]. One of those properties is called *constant width*: if a sphere is squeezed between two parallel (supporting) planes, it can rotate in any direction and always touches both planes. As the reader may suspect, there are many other convex sets with this property of constant width. To indicate this property in common with spheres, such three-dimensional objects are sometimes called *spheroforms* ([8, p. 135], [36], [7, p. 33]).

Some of the three-dimensional convex sets of constant width have a rotational symmetry. They can be generated by rotating plane sets of constant width with a reflection symmetry about their symmetry line. The drawing in Figure 1 is taken from a catalogue of mathematical models produced by the publisher Martin Schilling in 1911 [34, p. 149]. Influenced by mathematicians like Felix Klein, such models were produced for educational purposes, many of which were made by plaster. Figure 1 appears to be the earliest drawing showing a nontrivial three-dimensional body of constant width. This body

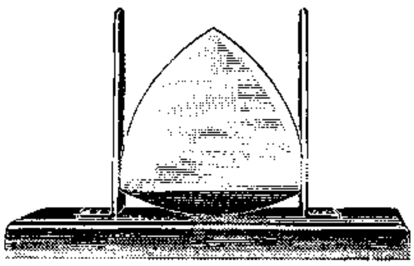

Figure 1: Rotated regular Reuleaux triangle, squeezed between a gauging instrument

is generated by rotating the Reuleaux triangle around its axis of symmetry. The Schilling catalogue also advertises another rotational as well as a nonrotational body of constant width. The author of its mathematical description is Ernst Meissner, with the help of Friedrich Schilling, not to be confused with Martin Schilling, the editor of the catalogue ([34, p. 106f.], and for a slightly expanded version see [28]). Since Meissner seems to have discovered this body, it is called a *Meissner body*. Although it is obvious that its construction can lead to two noncongruent bodies of constant width, Meissner explicitely describes only one of them, M_V (for details we refer to the paragraph headed by "Identifying the Suspect" below). Because their construction follows similar principles, one often speaks of "the" Meissner body.

The earliest printed photograph of a plaster Meissner body, the one described in the Schilling catalogue M_V, can be found in the 1932 German version of "Geometry and the Imagination", shown in Figure 2 [22, p. 216]. Photographs of all three bodies of constant width mentioned by Meissner can be found in more recent publications ([7, p. 64ff.], [16, p. 96–98]). The mathematical models must have been selling well because they can still be found in display cases of many mathematical departments. For instance, they can be found not only at many German universities (for the plaster model of the Meissner body M_V at the Technical University of Halle in Germany see `http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dg-003`) but also at Harvard University in the US and even at the University of Tokyo (`http://math.harvard.edu/~angelavc/models/locations.html`).

Certainly there are many more bodies of constant width than the four mentioned so far. A very nice collection is displayed in the exhibit "Pierres qui roulent" ("Stones that Roll") in the Palais de la Découverte in Paris (see

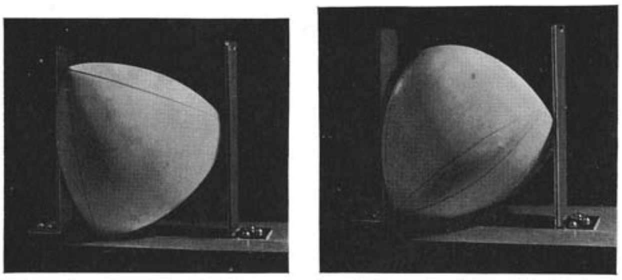

Figure 2: Plaster Model of Meissner body M_V

Figure 8 at the end of the paper). In addition to some rotated Reuleaux polygons (two triangles, four pentagons) it shows two Meissner bodies, both of the same type M_F. The exhibit allows the visitor to obtain a hands-on, tactile experience of the phenomenon of constant width. Sliding a transparent plate over these bodies of the same constant width causes these bodies to roll, while the plate appears to slide as if lying on balls. Of course there are also many other, nonrotational bodies of constant width. For their construction see [36], [24], [31] and [2].

In this article we restrict our attention for the most part to the three-dimensional setting. The reader can find more material in excellent surveys on plane and higher-dimensional sets of constant width, e.g. in Chakerian & Groemer [12], Heil & Martini [21] or Böhm & Quaisser [7, ch. 2].

As already mentioned there are two different types of Meissner bodies M_V and M_F (their construction will be described below). They have not only identical volume and surface area, but are conjectured to minimize volume among all three-dimensional convex bodies of given constant width. As these bodies were discovered one century ago and because the problem is still unsolved, it is appropriate to raise awareness of this challenging and beautiful open problem by presenting some of its history and recent development.

Although we could not find a written record by Meissner himself which explicitly states the conjecture, he seems to have guessed that his bodies are of minimal volume [7, p. 72]. While Hilbert and Cohn-Vossen in their book of 1932 do not comment in this direction, Bonnesen and Fenchel mention the conjecture two years later. In the German edition of their "Theory of Convex Bodies", they write: "es ist anzunehmen" which still reads "it is to be assumed" in the English edition of 1987 [8, p. 144]). Since then the conjecture has been stated again and again. For example, Yaglom and Boltyansky make it in all editions of their book "Convex Figures", from the Russian "predpolagaiut" in 1951, via the German "es ist anzunehmen" in 1956 to the

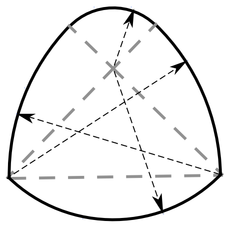

Figure 3: Nonregular Reuleaux tetragon with four circular arcs

English "we shall assume without proof" in 1961 [39, p. 81].

On the other hand, there was the belief that the body which minimizes volume among all three-dimensional bodies of constant width must have the symmetry group of a regular tetrahedron, a property not displayed by the Meissner bodies. This belief was first expressed by Danzer in the 1970's as Danzer has confirmed to us in personal communication ([19, p. 261], [13, p. 34] and [7, p. 72]). In 2009 an attempt was made to arrive at a body of full tetrahedral symmetry and minimal volume via a deformation flow argument [17].

Incidentally, the Minkowski sum $\frac{1}{2}M_V \oplus \frac{1}{2}M_F$, which one obtains half way in the process of morphing M_V into M_F, would render a body with tetrahedal symmetry (see Fig. 7). It actually has the same constant width as M_V and M_F. Its volume, however, is larger than that of the Meissner bodies, due to the *Brunn-Minkowski inequality*. It can be shown that the increase in volume is slightly more than 2 ‰ of the volume of the Meissner bodies [32].

Generating Constant Width Bodies by Rotation

Every two-dimensional convex set can be approximated by convex polygons. Similarly, every two-dimensional convex set of constant width can be approximated by circular arcs and thus by *Reuleaux polygons* of constant width. If the arcs are all of the same length, one has regular Reuleaux triangles, pentagons and so on. But to generate a plane convex set of constant width, it is not necessary that all circular arcs are of the same length. Figure 3 shows a plane set of constant width, a *Reuleaux tetragon*, which is constructed along the lines of [9, p. 192f.]. Note that it is bounded by four circulare arcs.

Whenever a plane set of constant width is reflection symmetric with respect to some axis, it can be rotated around that axis to generate a three-dimensional set of constant width. A regular rotated Reuleaux triangle leads to the body shown in Figure 1, and if an appropriate nonregular Reuleaux

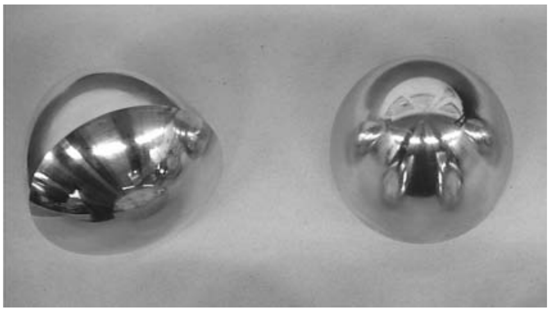

Figure 4: Rotated nonregular Reuleaux tetragon (side and top view)

tetragon or a Reuleaux trapezoid is rotated, one will obtain a body like the one in Figure 4. Both are not only bodies of revolution but three-dimensional sets of constant width [9, p. 196f.].

According to the *theorem of Blaschke-Lebesgue* the Reuleaux triangle minimizes area among all plane convex domains of given width. Thus one could expect that the rotated Reuleaux triangle in Figure 1 would minimize volume among all *rotational* bodies of given width. It was not until recently (1996 and 2009) that this longstanding conjecture was confirmed ([10], [25] and [1]).

Identifying the Suspect: Meissner Bodies

The plane Reuleaux triangle of constant width d is constructed as the intersection of three discs of radius d, each centered at a different corner of an equilateral triangle. In an analogous way a *Reuleaux tetrahedron* R_T can be constructed by intersecting four balls of radius d, each of which is centered at a vertex of a regular tetrahedron with side length d. It consists of four vertices, four pieces of spheres and six curved edges each of which is an intersection of two spheres.

Whenever this Reuleaux tetrahedron is squeezed between two parallel planes with a vertex touching one plane and the corresponding spherical surface touching the other, their distance is d by construction. However, the distance of the planes must be slightly enlarged by a factor of up to $\sqrt{3} - \frac{\sqrt{2}}{2} \approx 1.025$ when the planes touch two opposite edges of R_T. This means that the width of R_T is not constant but varies depending on its direction up to 2.5 %. Incidentally, as Meissner mentioned in [27, p. 49], the ball is the only body of constant width that is bounded only by spherical pieces. Thus R_T is not of constant width, because it is bounded only by spherical pieces and is different from a ball.

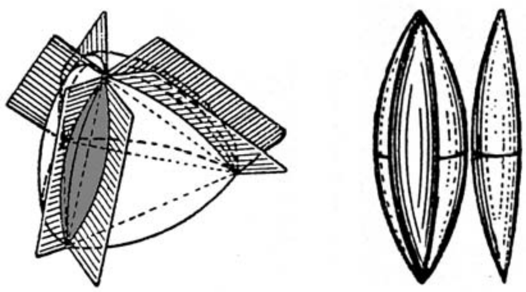

Figure 5: Replacing three wedges (left, one shaded gray) by pieces of spindle tori (right)

Nevertheless, R_T can be used as a starting point for a set of constant width. According to Meissner, some edges must be rounded off by the following procedure ([28], [8, p. 144], [39, p. 81], [6, p. 54f.]):

a) Imagine two planes bounding adjacent facets of the underlying tedrahedron. Remove the wedge located between the two planes and containing the curved edge of R_T of the Reuleaux tetrahedron (see figure 5 from [39, p. 81]).

b) The intersection of the planes with R_T contains two circular arcs that meet in the two ends of the wedge. Rotate one of these arcs around the corresponding edge of the tetrahedron. This generates a spindle-shaped surface, a spindle torus.

c) Notice that now the sharp edge has become a differentiable surface even across the boundary between spindle-torus and spherical piece.

After rounding off three edges of R_T that meet in a vertex, according to this procedure, one obtains the first type of Meissner body, M_V (see Figure 6, left). The second Meissner body M_F is obtained by rounding off three edges surrounding one of faces of R_T (see Figure 6, right). The resulting Meissner body features four vertices, three circular edges, four spherical and three toroidal surfaces. Both bodies have identical volume and surface area, and they are invariant under a rotation of 120° around a suitable axis. A computer animation showing both bodies M_V and M_F from all sides can be watched under [38].

Meissner bodies touch two parallel planes between which they are squeezed always in one of two possible ways: either one contact point is located in a

Figure 6: Meissner body M_V with rounded edges meeting in a vertex (left) and Meissner body M_F with rounded edges surrounding a face (right)

vertex and the antipodal contact point is located on a spherical piece of the body or one contact point is located on a sharp edge and the antipodal contact point is located on a rounded edge of the body.

Their constant width becomes obvious if one intersects a sharp, non-rounded edge opposite the rounded edge with a plane orthogonal to the sharp edge. In this plane the sides of the original tetrahedron form an isosceles triangle like the one in Figure 3. The line segment passing from the sharp edge of R_T through the opposite sharp edge of the regular tetrahedron varies in length and is generally shorter than the width d. If its length is extended to d, one arrives at the boundary of the edge that has been rounded off.

Meissner showed the constant width of his bodies using Fourier series [27, p. 47ff.]. Like Hurwitz, he originally studied convex closed curves inscribed in a regular polygon which remain tangent to all the sides of the polygon during rotations of the curve. Nowadays such curves are called *rotors*. Following Minkowski he characterized the curves by their support functions (length of the polar tangents). These are periodic and thus can be expanded in a Fourier series. Using this technique he finally succeeded in describing all rotors of regular polygons analytically [26]. With the analogous technique in three dimensions, he was able to determine the rotors of the cube as bodies of constant width. He even proved that non-spherical rotors do exist not only for the cube, but also for the regular tetrahedron and octahedron. In contrast, there exist no non-spherical rotors for the regular dodecahedron and icosahedron ([30], for some mechanical adaptions of Meissner's technique see [9, p. 213ff.]).

Volume and Surface Area of the Meissner bodies

In this section we give some numerical results on the volume and surface area of the Meissner body of constant width d. The *volume* V_{M_V} and V_{M_F} of

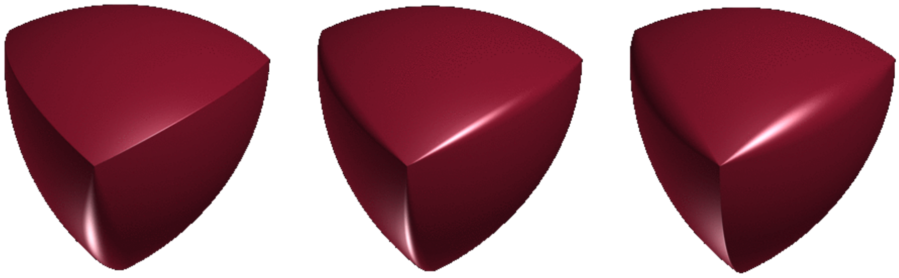

Figure 7: Morphing M_V to M_F including the Minkowski mean (second frame)

the two Meissner bodies is identical and is given by

$$V_{M_V} = V_{M_F} = \left(\frac{2}{3} - \frac{\sqrt{3}}{4} \cdot \arccos\frac{1}{3}\right) \cdot \pi \cdot d^3 \approx 0.419860 \cdot d^3,$$

see [12, p. 68], [7, p. 71], [35, A137615], [31, p. 40–43]. Therefore we will not distinguish between M_V and M_F. The volume of the Meissner body is approximately 80 % of the volume $\pi/6$ of a ball of diameter 1 and it is considerably smaller (by about 6 %) than the volume of the rotated Reuleaux triangle R_3, which is given by

$$V_{R_3} = \left(\frac{2}{3} - \frac{\pi}{6}\right) \cdot \pi \cdot d^3 \approx 0.449461 \cdot d^3.$$

in [10] and [35, A137617]. As far as we know, the highest lower bound for the volume of a body K of constant width 1 is the one given by Chakerian et al. in 1966,

$$V_K \geqslant \frac{\pi}{3} \cdot \left(3\sqrt{6} - 7\right) \cdot d^3 \approx 0.364916 \cdot d^3,$$

see [11] and [24].

The *surface area* S_{M_V} and S_{M_F} of the two Meissner bodies is identical, as well, and is given by

$$S_{M_V} = S_{M_F} = \left(2 - \frac{\sqrt{3}}{2} \cdot \arccos\frac{1}{3}\right) \cdot \pi \cdot d^2 \approx 2.934115 \cdot d^2,$$

see [12, p. 68], [7, p. 71], [35, A137616]. This follows from the remarkable fact that in three dimensions the volume V_K and surface area S_K of a convex body K of constant width d are related through *Blaschke's identity* ([5, p. 294], [12, p. 66])

$$V_K = \frac{1}{2} \cdot d \cdot S_K - \frac{\pi}{3} \cdot d^3.$$

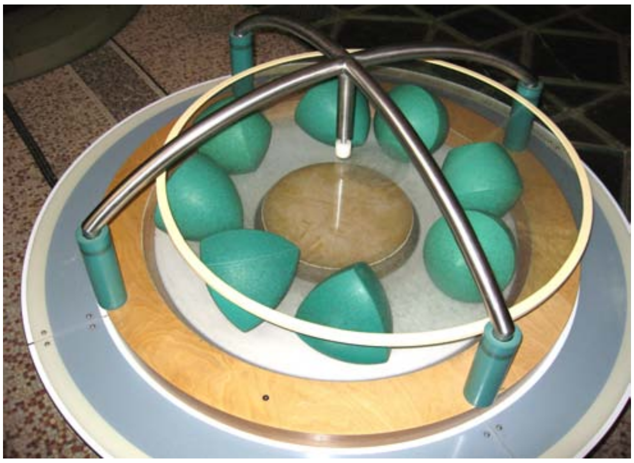

Figure 8: Various Bodies of Constant Width (Palais de la Découverte, Paris)

Since V_K is monotone increasing in S_K, the question of finding the set that minimizes volume is equivalent to finding the set that minimizes surface area (or generalized perimeter) of K among all convex sets of constant width. Incidentally, this is in sharp contrast to the two-dimensional case, in which, according to a *theorem of Barbier*, all sets of constant width d are isoperimetric, that is they have the same perimeter $\pi \cdot d$ ([3], [8, p. 139]).

The major part of the surface of the Meissner body consists of pieces of a sphere of radius d. The rounded edges (or spindle tori) have an angle of rotation of arccos(1/3) and their smaller principal curvature is constant and has the value $1/d$. Their part of the surface area is

$$S_{Sp} = 3 \cdot \frac{\arccos(\frac{1}{3})}{2\pi} \cdot 2\pi \cdot d^2 \cdot \int_0^1 \left(\sqrt{\frac{3}{4} + x - x^2} - \frac{\sqrt{3}}{2} \right) \cdot \mathrm{d}x \approx 0.334523 \cdot d^2.$$

In other words, the nonspherical pieces of the surface of a Meissner body make up about 11 % of the total surface area.

Circumstantial evidence, but no proof

Why do we believe that Meissner bodies minimize volume among all three-dimensional convex bodies of constant width? There are more than a million different reasons for it. Clearly the fact that the conjecture remained unsolved for so long shows that a counterexample is hard to come by. But there is more than this one reason supporting the conjecture. In 2007 Lachand-Robert and Oudet presented a method to construct a large variety of bodies of constant width in any dimension, see [24]. For plane domains this construction boils down to a method of Rademacher and Toeplitz from 1930 [33, p. 175f.]. Their algorithm begins with an arbitrary body of constant width K_{n-1} in $(n-1)$ dimensions and arrives at a body K_n of constant width in n dimensions with K_{n-1} as one of its cross sections. It was used in 2009 to generate randomly one million different three-dimensional bodies of constant width [31]. None of them had a volume as small as that of a Meissner body. It should be noted, however, that while the algorithm can generate every two-dimensional set of constant width from a one-dimensional interval, it cannot generate all three-dimensional sets of constant width, but only those that have a plane cross section with the same constant width. In [14], Danzer describes a set K_3 of constant width d for which each of its cross sections has a width less than d.

Analysts have recently tried to identify the necessary conditions that a convex body M of minimal volume and given constant width must satisfy. The existence of such a body follows from the direct methods in the calculus of variations and the *Blaschke selection theorem*. Let us mention in passing that the boundary of M cannot be differentiable of class C^2. If it were, one could consider $M_\varepsilon := \{x \in M \mid \operatorname{dist}(x, \partial M) > \varepsilon > 0\}$, that is the set M with a sufficiently thin ε-layer peeled off and with a volume less than the one of M. According to the Steiner formula, its volume V_{M_ε} can be expressed in terms of the volume V_M of M, the mean width d_{M_ε} of M_ε and the surface area S_{M_ε} of M as follows

$$V_{M_\varepsilon} = V_M - \varepsilon \cdot S_{M_\varepsilon} - d_{M_\varepsilon} \cdot \varepsilon^2 - \frac{4\pi}{3} \cdot \varepsilon^3.$$

By construction, and because of our smoothness asumption M_ε is a body of constant width $d - 2\varepsilon$. Therefore its mean width is $d_{M_\varepsilon} = d - 2\varepsilon$. If one blows M_ε up by a linear factor of $d/(d-2\varepsilon)$ to a set $\widetilde{M}$, its volume is given by

$$V_{\widetilde{M}} = \left(\frac{d}{d-2\varepsilon}\right)^3 \cdot V_{M_\varepsilon}$$

and $\widetilde{M}$ is of constant width d again. It is now possible to show that $V_{\widetilde{M}} < V_M$, which shows that no body of class C^2 can minimize volume. In fact, $V_{\widetilde{M}} < V_M$, provided $(d-2\varepsilon)^3 \cdot V_{\widetilde{M}} = d^3 \cdot V_{M_\varepsilon} < d^3 \cdot V_M$, or equivalently

$$d^3 \cdot \left(V_M - \varepsilon \cdot S_{M_\varepsilon} - (d - 2\varepsilon) \cdot \varepsilon^2 - \frac{4\pi}{3} \cdot \varepsilon^3 \right) < d^3 \cdot V_M.$$

This shows that for sufficiently small ε the volume $V_{\widetilde{M}}$ stays below the original volume V_M of M and contradicts the minimality of M's volume.

Figure 9: Ernst Meissner (1883–1939), ETH-Bibliothek Zürich, Bildarchiv

In 2007 a stronger result was shown: Any local volume minimizer cannot be simultaneously smooth in any two antipodal (contact) points [4]. In other words, squeezed between two parallel plates, one of the points of contact with the plane must be a vertex or a sharp-edge point. As already pointed out, Meissner bodies have this property. Because rotated Reuleaux polygons satisfy this property as well, this result also supports the conjecture without proving it. Finally, in 2009, it was shown by variational arguments that a volume minimizing body of constant width d has the property that any C^2 part of its surface has its smaller principal curvature constant and equal to $1/d$ [1]. Again, Meissner bodies meet this criterion, as well, because they consist of spherical and toroidal pieces with exactly this smaller principal curvature.

After this paper was accepted for publication we learned from Qi Guo in personal communication, that Qi Guo and and Hailin Jin had just observed another remarkable property of Meissner's bodies. It is well known that the inradius r and circumradius R of a body of constant width add up to d. The ratio R/r of these two radiii is a measure of asymmetry for a set, and according to the observation of Guo and Lin it is maxmized (among all three-dimensional bodies of constat width) by Meissner's bodies. For those bodies $R/r = (3 + 2\sqrt{6})/5) \approx 1.5798$. In fact R is maximized, given d, by a Meissner body, see e.g. [18], and so r is minimized and a fortiori R/r is maximized. It is in this sense that Meisner's bodies are more slender and should have less volume than others of constant width d.

All these results seem to suggest that it will not take another century until the conjecture is confirmed.

Appendix: CV of Ernst Meissner

Who was the man who discovered the body that is presumed to minimize volume? Ernst Meissner was born on 1 September, 1883 as the son of a manufacturer in Zofingen (Switzerland). He attended secondary school in Aarau, where he had the same teacher in mathematics as previously Albert Einstein, Heinrich Ganter. Ganter's style of teaching is described as follows. "He was a good mathematician but not, in his own reckoning good enough to pursue a career in higher mathematics. But he could teach, something that many speculative gentlemen cannot do. [...] Ganter never treated us demeaningly, but taught us as men." Meissner himself described him "as a teacher who, far from transmitting mere information to prepare a pupil for a career, educated the heart and character and truly civilized his charges. If all teachers were like Ganter, [...] there would be no need for school reform." [15, p. 91]

Meissner's own dedication to teaching is evident from a public lecture that he gave on Nov. 18, 1915 in the townhall of Zurich. The renowned newspaper "Neue Zürcher Zeitung" (NZZ) dedicated half a page to his lecture "Why does mathematics appear difficult and boring to some while it does not to others?" [29]. For Meissner grasping a mathematical concept is more than passively understanding its logics. In fact, many people are capable of logical thinking without appreciating mathematics. The deeper understanding of mathematics is rather connected with the creation of one's own mental images and concepts. Meissner promotes the idea that mathematical education should not confront pupils with abstract and fully matured facts. Instead it should enable them to construct and connect mental images in several ways. To a great extent his criticism still applies to contemporary teaching.

After graduating from school Meissner studied from 1902 to 1906 at the Department of Mathematics and Physics of the Swiss Polytechnic, which was later to become the Swiss Federal Institute of Technology (ETH) in Zurich. He was awarded a doctorate there on the basis of a thesis in number theory. After two semesters at the University of Göttingen, where he studied with Klein, Hilbert and Minkowski, he returned to the ETH. There he qualified as a professor (Habilitation) in 1909 in mathematics and mechanics. A year later he was offered the chair of technical mechanics which he held until 1938. Ernst Meissner died on March 17, 1939 in Zollikon (near Zurich).

Meissner's scientific achievements were extraordinarily diverse (for a list of his publications see [23, p. 294f.]). In his earlier works he dealt with questions in pure mathematics (geometry, number theory). Not only his dissertation but also his investigations on sets of constant width fall within this period. During the years between 1910 and 1920 he turned increasingly toward applied mathematics (graphic integration of differential equations, graphic determination of Fourier coefficients) and then to mechanics (geo-

Figure 10: Meissner 1931 teaching students suffering from tuberculosis in the "sanatorium universitaire" in Leysin operated by Swiss universities

physics, seismology, theory of oscillations). It is in these applied papers that Meissner's true scientific achievements lie because, like Franz Reuleaux, the originator of theoretical kinematics, before him, he always sought his models in pure, strict mathematics.

In an obituary from 1939 Meissner is depicted as a person who not only expected much from himself but also from those around him. "Ernst Meissner demanded the most from himself and others. His intense sense of duty and professional ethics made him seem strict and reserved. However, those who knew him better, his nearest friends and his students, were allowed the unforgettable experience of his extraordinarily comprehensive knowledge and his deep perception, a truly classical appreciation of beauty, touching kindness and finely honed wit." [40]

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Figure 11: The authors Weber and Kawohl in action

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Image Sources:

- Fig. 1: [34, p. 149]
- Fig. 2: [22, p. 191]
- Fig. 3: [32]
- Fig. 4: by the authors
- Fig. 5: Models by B. Kawohl
- Fig. 6: [39, p. 81]
- Fig. 7: Models by Chr. Weber
- Fig. 8: Photo taken at the Palais de la Découverte (Paris) by Chr. Weber
- Fig. 9: ETH-Bibliothek Zürich, Bildarchiv
- Fig. 10: [37, p. 459]